ARC-SMOOTHNESS AND CONTRACTIBILITY IN WHITNEY LEVELS

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ABSTRACT. Let X be a continuum. Let 2^X (resp., $C(X)$) be the space of all nonempty closed subsets (resp., subcontinua) of X . In this paper we prove that if X is an arc-smooth continuum, then there exists an admissible Whitney map $\mu: 2^X \rightarrow \mathbf{R}$ such that $\mu|_{C(X)}: C(X) \rightarrow \mathbf{R}$ is admissible and for every $t \in (0, \mu(X))$, $\mu^{-1}(t)$ and $(\mu|_{C(X)})^{-1}(t)$ are arc-smooth. This answers a question by J. T. Goodykoontz, Jr. Also we give an example of a contractible continuum X such that, for every Whitney map $v: C(X) \rightarrow \mathbf{R}$ there exists $t \in (0, v(X))$ such that $v^{-1}(t)$ is not contractible.

0. INTRODUCTION

A *continuum* is a nondegenerate compact connected metric space. The *hyperspaces* of a continuum X are the spaces $2^X = \{A \subset X: A \text{ is nonempty and closed in } X\}$ and $C(X) = \{A \in 2^X: A \text{ is connected}\}$ metrized with the Hausdorff metric H . We identify $\{\{x\}: x \in X\} \subset C(X) \subset 2^X$ with X . A *Whitney map* for a hyperspace $\mathcal{H}$ of X is a continuous function $\mu: \mathcal{H} \rightarrow \mathbf{R}$ ($\mathbf{R}$ is the real line) such that $\mu(\{x\}) = 0$ for each $x \in X$ and if $A \subset B \neq A$, then $\mu(A) < \mu(B)$. If μ is a Whitney map for $C(X)$ and $0 \leq t < \mu(X)$, then $\mu^{-1}(t)$ is called a *Whitney level*; if $0 < t < \mu(X)$, then $\mu^{-1}(t)$ is called a *positive Whitney level*. A topological property P is called a *Whitney property* provided that whenever a continuum X has property P , so does $\mu^{-1}(t)$ for each Whitney map for $C(X)$ and $0 \leq t < \mu(X)$. A Whitney map μ is called an *admissible Whitney map* for $\mathcal{H}$ [4] provided that there exists a continuous homotopy $F: \mathcal{H} \times [0, 1] \rightarrow \mathcal{H}$ satisfying:

- (a) For all $A \in \mathcal{H}$, $F(A, 1) = A$ and $F(A, 0) \in X$.
- (b) If $\mu(F(A, t)) > 0$ for some $A \in \mathcal{H}$ and $t \in [0, 1]$, then $\mu(F(A, s)) < \mu(F(A, t))$ whenever $0 \leq s < t$.

Let $p \in X$. Then X is *arc-smooth* at p ([1] and [2]) provided there exists a continuous function $\alpha: X \rightarrow C(X)$ satisfying:

- (a) $\alpha(p) = \{p\}$.

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- (b) For each $x \in X - \{p\}$, $\alpha(x)$ is an arc from p to x .
- (c) If $x \in \alpha(y)$, then $\alpha(x) \subset \alpha(y)$.

(The arc $\alpha(y)$ will be denoted by $[p, y]$).

In [5], J. Krasinkiewicz and S. B. Nadler, Jr. asked whether contractibility is a Whitney property. This question was answered negatively by A. Petrus in [7]. She showed that if X is a 2-cell, then it is possible to define a Whitney map $\mu: C(X) \rightarrow \mathbf{R}$ such that for certain $t \in (0, \mu(X))$, $\mu^{-1}(t)$ is not contractible. Since arc-smoothness is a special type of contractibility, this example also showed that arc-smoothness is not a Whitney property. In [3], J. T. Goodykoontz, Jr. extended a previous result by Petrus giving sufficient conditions for a Whitney map to have contractible Whitney levels, and he posed the following question: Let X be an arc-smooth continuum. Does there exist some Whitney map μ for $C(X)$ such that for each $t \in (0, \mu(X))$, $\mu^{-1}(t)$ is arc-smooth?

In this paper we answer this question affirmatively by proving that if X is an arc-smooth continuum, then there exists a Whitney map $\mu: 2^X \rightarrow \mathbf{R}$ such that $\mu^{-1}(t)$ and $(\mu|C(X))^{-1}(t)$ are arc-smooth continua for every $t \in (0, \mu(X))$, and μ and $\mu|C(X)$ are admissible. Then arc-smoothness is a sufficient condition for $C(X)$ and 2^X to have admissible Whitney maps (compare with [3, 2.13, 2.14, 2.15, and 2.16]).

In the second part of this paper we show that an analogous construction changing arc-smoothness to contractibility is not possible. We give an example of a contractible continuum X such that if $\nu: C(X) \rightarrow \mathbf{R}$ is a Whitney map, then there exists $t \in (0, \nu(X))$ such that $\nu^{-1}(t)$ is not contractible.

1. CONSTRUCTION OF μ

Throughout this section we suppose that X is arc-smooth at p . Then [2, Theorem 1-4-A] X admits a metric d which is radially convex at p (that means that whenever $x, y \in X$ and $x \in [p, y]$, then $d(x, y) = d(p, x) + d(x, y)$). We suppose that $\sup\{d(p, x): x \in X\} = 1$. Let I denote the unit interval $[0, 1]$.

We consider the map $h_d: X \times I \rightarrow X$ defined in [3, p. 144]. That is,

$$h_d(x, t) = \begin{cases} x & \text{if } d(p, x) \leq 1 - t, \\ \text{the unique } y \in [p, x] & \text{such that } d(p, y) = \\ 1 - t & \text{if } d(p, x) \geq 1 - t. \end{cases}$$

This map has the following properties:

- (1) For each $x \in X$, $h_d(x, 0) = x$.
- (2) For each $x \in X$, $h_d(x, 1) = p$.
- (3) For each $x \in X$ and $s, t \in I$, $h_d(h_d(x, s), t) = h_d(x, \{\max s, t\})$.

Define $G: 2^X \times I \rightarrow 2^X$ by $G(A, t) = \{h_d(a, t): a \in A\} = h_d(A \times \{t\})$.

The following lemma is easy to prove.

Lemma 1.1. *G has the following properties:*

- (a) *G is continuous.*
- (b) *$G|C(X) \times I: C(X) \times I \rightarrow C(X)$ and $g|X \times I: X \times I \rightarrow X$.*
- (c) *$G(A, 0) = A$ and $G(A, 1) = \{p\}$ for every $A \in 2^X$.*
- (d) *$G(G(A, s), t) = G(A, \max\{s, t\})$.*

Define $l: 2^X \rightarrow I$ by $l(A) = \sup\{d(p, a): a \in A\}$.

Lemma 1.2. *l has the following properties:*

- (a) *l is continuous.*
- (b) *$l(G(A, s)) = \min\{1 - s, l(A)\}$ for every $(A, s) \in 2^X \times I$.*
- (c) *$s \leq t$ implies that $l(G(A, s)) \geq l(G(A, t))$.*
- (d) *$G(A, 1 - l(A)) = A$ for every $A \in 2^X$.*

Proof. We only prove (b). Let $(A, s) \in 2^X \times I$. Notice that if $d(p, h_d(a, s)) \leq 1 - s$ for all $a \in A$, then $l(G(A, s)) \leq 1 - s$. If $l(A) \leq 1 - s$, then $h_d(a, s) = a$ for each $a \in A$. So $A = G(A, s)$. And if $l(A) \geq 1 - s$, take $a_0 \in A$ such that $l(A) = d(p, a_0)$. Then $d(p, h_d(a_0, s)) \geq 1 - s$. So $l(G(A, s)) = 1 - s$.

Fix a Whitney map $v: 2^X \rightarrow I$ with $v(X) = 1$, and define $\mu: 2^X \rightarrow I$ by $\mu(A) = \int_{1-l(A)}^1 v(G(A, t)) dt$.

Theorem 1.3.

- (a) *μ is a Whitney map.*
- (b) *$s \leq r$ implies that $\mu(G(A, s)) \geq \mu(G(A, r))$.*
- (c) *$(\mu|C(X))^{-1}(t)$ and $\mu^{-1}(t)$ are arc-smooth continua for all $t \in (0, \mu(X))$.*
- (d) *μ and $\mu|C(X)$ are admissible.*

Proof. (a) To see that μ is continuous, let $\varepsilon > 0$. Take $\delta > 0$ such that $\varepsilon > \delta$ and $H(A, B) < \delta$ implies that $|l(A) - l(B)| < \varepsilon$ and $|v(G(A, t)) - v(G(B, t))| < \varepsilon$ for every $t \in I$. Let $A, B \in 2^X$ be such that $H(A, B) < \delta$. Suppose, for example, that $l(A) \leq l(B)$. Then

$$\begin{aligned} |\mu(A) - \mu(B)| &\leq \int_{1-l(A)}^1 |v(G(A, t)) - v(G(B, t))| dt + \int_{1-l(B)}^{1-l(A)} v(G(B, t)) dt \\ &< \varepsilon l(A) + (l(B) - l(A)) < 2\varepsilon. \end{aligned}$$

Hence μ is continuous.

Now, take $A, B \in 2^X$ such that $A \subset B \neq A$. Then $l(A) \leq l(B)$ and $v(G(A, t)) \leq v(G(B, t))$ for all $t \in I$. If $l(A) = l(B)$, then $v(G(A, 1 - l(A))) = v(A) < v(B) = v(G(B, 1 - l(A)))$. Thus $\mu(A) < \mu(B)$. If $l(A) < l(B)$, since $0 < v(B) = v(G(B, 1 - l(B)))$, we have that $\int_{1-l(B)}^{1-l(A)} v(G(B, t)) dt > 0$. Therefore $\mu(B) > \mu(A)$. Hence μ is a Whitney map.

(b) Let $s \leq r$. Property 1.2(b) implies that $r \leq 1 - l(G(A, s))$. Then Property 1.1(d) implies that $\mu(G(A, r)) = \int_{1-l(G(A, r))}^1 v(G(A, t)) dt$. Thus, by 1.2(c) we have $\mu(G(A, s)) \geq \mu(G(A, r))$.

(c) Since $\mu(A) = \mu(G(A, 0)) \geq \mu(G(A, s))$ for every $s \in I$, $\mu|C(X)$ satisfies the hypothesis of Theorem 5.2 in [3]. Then $(\mu|C(X))^{-1}$ is arc-smooth for all $t \in (0, \mu(X))$. It is easy to check that Theorem 5.2 in [3] also holds, changing $C(X)$ to 2^X . So $\mu^{-1}(t)$ is arc-smooth for all $t \in (0, \mu(X))$.

(d) Define $F: 2^X \times I \rightarrow 2^X$ by $F(A, t) = G(A, tl(A) + 1 - l(A))$. Then F is continuous, $F(A, 0) = A$ and $F(A, 1) = \{p\}$ for each $A \in 2^X$, and $F|(C(X) \times I): C(X) \times I \rightarrow C(X)$. Suppose that $\mu(F(A, t)) > 0$ and take $s \in (t, 1]$. Since $F(A, t) \neq \{p\}$, we have $l(A) > 0$. Set $t_1 = tl(A) + 1 - l(A)$ and $s_1 = sl(A) + 1 - l(A)$. Then $0 \leq l(G(A, s_1)) = 1 - s, < 1 - t_1 = l(G(A, t_1))$. Since $G(A, t_1) = F(A, t)$ has more than one point, it follows that $\mu(F(A, t)) > \mu(F(A, s))$. Hence μ and $\mu|C(X)$ are admissible.

2. THE EXAMPLE

Let $\mathbf{R}^3$ be the Euclidean 3-dimensional space. Given $p, q \in \mathbf{R}^3$, we will denote by $\overline{pq}$ the segment joining p and q . If $A, B \subset \mathbf{R}^3$ and α is a real number, define $A + B = \{p + q \in \mathbf{R}^3: p \in A \text{ and } q \in B\}$, $\alpha A = \{\alpha p \in \mathbf{R}^3: p \in A\}$, $\mathcal{C}(A, B) = \bigcup \{\overline{pq}: p \in A \text{ and } q \in B\}$ and if $\alpha \in I$, $\mathcal{C}(A, B, \alpha) = \{tp + (1 - t)q \in \mathbf{R}^3: p \in A, q \in B, \text{ and } 0 \leq t \leq \alpha\}$. Then $\mathcal{C}(A, B, 1) = \mathcal{C}(A, B)$, $\mathcal{C}(A, B, 0) = B$, and if D is a fixed convex continuum, $D \subset \mathbf{R}^3$, then the function $(A, B, \alpha) \rightarrow \mathcal{C}(A, B, \alpha)$ from $2^D \times 2^D \times I$ into 2^D is continuous.

Let $\mathbf{N}$ denote the set of positive integers. For $n \in \mathbf{N}$, set $a_n = 1/2^n$. Define $J = \{(n, m) \in (\mathbf{N} \cup \{0\}) \times (\mathbf{N} \cup \{0\}): 0 < n < m \text{ or } m = 0 < n \text{ or } m = 0 = n\}$.

Set $S_0 = \{(x, y, 0) \in \mathbf{R}^3: x^2 + y^2 = 1\}$ and $a_0 = 0$. For $n \in \mathbf{N}$, set $S_{2n} = \{(x, y, 0) \in \mathbf{R}^3: x^2 + y^2 = (1 + a_{2n})^2 \text{ and } \|(x, y, 0) - (0, 1 + a_{2n}, 0)\| \geq a_{2n}\}$ and $S_{2n-1} = \{(x, y, 0) \in \mathbf{R}^3: x^2 + y^2 = (1 + a_{2n-1})^2 \text{ and } \|(x, y, 0) - (0, -1 - a_{2n-1}, 0)\| \geq a_{2n-1}\}$. Then $S_0, S_1, S_2, \dots$ are pairwise disjoint continua and $S_n \rightarrow S_0$.

For $(n, m) \in J$, define

$$\begin{aligned} L(n, m) = & \mathcal{C}(\{(3, 0, a_n + a_m)\} + a_n S_m, \{(3, 0, 1 + a_n + a_m)\}) \\ & \cup \mathcal{C}(\{(8, 0, a_n + a_m)\} + a_n S_m, \{(8, 0, 1 + a_n + a_m)\}) \\ & \cup \overline{(3 + a_n(1 + a_m), 0, a_n + a_m)(8 - a_n(1 + a_m), 0, a_n + a_m)} \end{aligned}$$

and

$$K(n, m) = L(n, m) \cup \overline{(0, 0, a_n + a_m)(3 - a_n(1 + a_m), 0, a_n + a_m)}.$$

Then define $X = (\bigcup \{K(n, m): (n, m) \in J\}) \cup \overline{(0, 0, 0)(0, 0, 2)}$. Clearly, X is a contractible continuum.

Theorem 2.1. *If $\mu: C(X) \rightarrow \mathbf{R}$ is a Whitney map, then there exists $t \in (0, \mu(X))$ such that $\mu^{-1}(t)$ is not contractible.*

Proof. Let $\mu: C(X) \rightarrow \mathbf{R}$ be a Whitney map. Set

$$\begin{aligned} D &= \overline{(3, 0, 0)(8, 0, 0)}, \\ E &= D \cup \overline{(8, 0, 0)(8, 0, 1)}, \\ F &= D \cup \overline{(3, 0, 0)(3, 0, 1)}, \end{aligned}$$

and

$$G = \overline{(0, 0, 0)(8, 0, 0)}.$$

Take any $t \in \mathbf{R}$ such that $\mu(D) < t < \mu(E)$, $\mu(F)$ and $\mu(G)$. We will prove that $\mu^{-1}(t)$ is not contractible. Since $D \subset E$, there exists $H \in C(X)$ such that $\mu(H) = t$ and $D \subset H \subset E$. Suppose that $\mu^{-1}(t)$ is contractible. Then there exists a continuous function $f: \mu^{-1}(t) \times I \rightarrow \mu^{-1}(t)$ such that $f(A, 0) = A$ and $f(A, 1) = H$ for every $A \in \mu^{-1}(t)$.

Given $(n, m) \in J$ and $r \in I$, let

$$\begin{aligned} L(n, m, r) &= \mathcal{C}(\{(3, 0, 1 + a_n + a_m)\}, \{(3, 0, a_n + a_m)\} + a_n S_m, r) \\ &\quad \cup \mathcal{C}(\{(8, 0, 1 + a_n + a_m)\}, \{(8, 0, a_n + a_m)\} + a_n S_m, r) \\ &\quad \cup \overline{(3 + a_n(1 + a_m), 0, a_n + a_m)(8 - a_n(1 + a_m), 0, a_n + a_m)}. \end{aligned}$$

Then $L(n, m, 1) = L(n, m)$, $L(n, 0, 0) \rightarrow D$, and $L(n, 0, 1) \rightarrow E \cup F$. So we can choose $n \in \mathbf{N}$ such that $\mu(L(n, 0, 0)) < t < \mu(L(n, 0, 1))$. We may also assume that if $A \in C(X)$ and $(3 + a_n, 0, a_n)$, $(8, 0, 1 + a_n) \in A$ or $(3, 0, 1 + a_n)$, $(8 - a_n, 0, a_n) \in A$ or $(0, 0, a_n)$, $(8 - a_n, 0, a_n) \in A$, then $\mu(A) > t$. Let u_0 be the unique element of the interval $(0, 1)$ such that $\mu(L(n, 0, u_0)) = t$.

Since $L(n, m, 0) \rightarrow L(n, 0, 0)$ and $L(n, m, 1) \rightarrow L(n, 0, 1)$ when $m \rightarrow \infty$, there exists $M > n$ such that, for every $m \geq M$, $\mu(L(n, m, 0)) < t < \mu(L(n, m, 1))$. Then for each $m \geq M$ there exists a unique $u_m \in (0, 1)$ such that $\mu(L(n, m, u_m)) = t$. Set $C_m = L(n, m, u_m)$. The uniqueness of u_0 implies that $u_m \rightarrow u_0$ and then $C_m \rightarrow L(n, 0, u_0)$ when $m \rightarrow \infty$.

Let $C = L(n, 0, u_0)$, $p_0 = (3 + a_n, 0, a_n)$, and $q_0 = (8 - a_n, 0, a_n)$. Then $p_0, q_0 \in f(C, 0) - f(C, 1)$. Let $s_0 = \max\{s \in I: p_0 \text{ and } q_0 \text{ belongs to } f(C, r) \text{ for all } r \in [0, s]\}$. Then $s_0 < 1$. Let $L = L(n, 0, 0)$ ($= (\{(3, 0, a_n)\} + a_n S_0) \cup (\{(8, 0, a_n)\} + a_n S_0) \cup \overline{(3 + a_n, 0, a_n)(8 - a_n, 0, a_n)}$), $R = L \cup \overline{(0, 0, a_n)(3 - a_n, 0, a_n)} \subset K(n, 0)$, $p_1 = (3, 0, 1 + a_n)$, and $q_1 = (8, 0, 1 + a_n)$. Let $h: K(n, 0) - \{p_1, q_1\} \rightarrow R$ be the natural retraction.

The choice of n and s_0 implies that p_1, q_1 , and $(0, 0, a_n) \notin f(C, r)$ for all $r \in [0, s_0]$. Then it is possible to define $g: [0, s_0] \rightarrow C(R)$ by $g(s) = h(f(C, s))$ (the image under h of $f(C, s)$).

Let $r_0 = \max\{r \in [0, s_0]: L \subset g(s) \text{ for all } s \in [0, r]\}$. We will prove that $r_0 = s_0$. Suppose that $r_0 < s_0$. Then there exists $r_1 \in (r_0, s_0)$ such that $L \not\subset g(r_1)$ and $H(g(r_0), g(r_1)) < a_n/4$ (H is the Hausdorff metric for $C(X)$). Since $p_0, q_0 \in g(r_1)$, then $\overline{p_0 q_0} \subset g(r_1)$. So we assume, for example, that

there exists a point $p_2 = (3, 0, a_n) + a_n z_0 \in L - g(r_1)$ with $z_0 = (x_0, y_0, 0) \in S_0 - \{(1, 0, 0)\}$. We analyze the case $y_0 \geq 0$. The case $y_0 \leq 0$ is similar.

Let $l = \overline{p_1 p_2}$. Since $h^{-1}(p_2) = l - \{p_1\}$, we have $f(C, r_1) \cap l = \emptyset$. Given m odd with $m > n$, set $z_m = (1 + a_m)z_0 \in S_m$, $p_m = (3, 0, a_n + a_m) + a_n z_m \in L(n, m)$, and $l_m = \overline{p_m(3, 0, 1 + a_n + a_m)} \subset L(n, m)$. Then $l_m \rightarrow l$ when $m \rightarrow \infty$ (m odd). Since $f(C, r_1) \cap l = \emptyset$, $f(C_m, r_1) \rightarrow f(C, r_1)$, and $(0, 0, a_n) \notin f(C, s)$ for each $s \in [0, r_1]$, there exists $M_1 > M$ such that if $m \geq M_1$ and m is odd, then $f(C_m, r_1) \cap l_m = \emptyset$ and $(0, 0, a_n + a_m) \notin f(C_m, s)$ for each $s \in [0, r_1]$. Since $f(C_m, 0) = C_m \subset K(n, m)$, if $m \geq M_1$ and m is odd, then $f(C_m, r_1) \subset K(n, m) - l_m$. Since $f(C_m, r_1) \rightarrow f(C, r_1)$ and $q_0 \in f(C, r_1)$, there exists $M_2 \geq M_1$ such that if $m \geq M_2$ and m is odd, then $f(C_m, r_1)$ is contained in the component of $K(n, m) - l_m$ which has the point $(8 - a_n(1 + a_m), 0, a_n + a_m)$. This implies that $f(C, r_1)$ is contained in the closure of the component of $K(n, 0) - (l \cup \overline{p_1(3, -a_n, a_n)})$, which has the point $(8 - a_n, 0, a_n)$. Then $g(r_1) = h(f(C, r_1))$ is contained in the closure of the component of $L - \{p_2, (3, -a_n, a_n)\}$, which has the point $(8 - a_n, 0, a_n)$. This is a contradiction since $y_0 \geq 0$, $H(g(r_0), g(r_1)) < a_n/4$, and $L \subset g(r_0)$. Hence $r_0 = s_0$.

In particular, $L \subset g(s_0) \subset K(n, 0) - \{(0, 0, a_n)\}$. Since $s_0 < 1$, it follows that there exists $s' \in (s_0, 1)$ such that $p_0, q_0 \in f(C, s)$ for all $s \in [s_0, s']$. This contradicts the choice of s_0 and ends the proof that $\mu^{-1}(t)$ is not contractible.

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