REPRESENTATION OF TREE PERMUTATIONS BY WORDS

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ABSTRACT. The problem of solving equations in groups can be stated as follows: given a group G and a free group $F = F(x_1, x_2, \dots)$, for which pairs (w, g) with $w = w(x_1, x_2, \dots) \in F$, $g \in G$, is it possible to find elements $g_i \in G$ such that $w(g_1, g_2, \dots) = g$? We investigate the corresponding question of solving equations in the group $A(\Omega)$ of all automorphisms of a transitive tree Ω . If the tree has isomorphic cones at a branch point, then certain equations of the form $x^n = g$ cannot be solved (Theorem 2.3). If the tree is sufficiently transitive, we find large classes of equations $w = g$ which can be solved (Theorems 2.13, 2.16).

0. INTRODUCTION

A p.o. set $(\Omega, \leq)$ is called a *tree* or *lower semilinear order* if the following three conditions are satisfied:

- (a) for each $\alpha \in \Omega$ the set $\{\tau \in \Omega : \tau \leq \alpha\}$ is totally ordered (a chain);
- (b) for all $\alpha, \beta \in \Omega$ there is $\tau \in \Omega$ with $\tau \leq \alpha$ and $\tau \leq \beta$;
- (c) $(\Omega, \leq)$ contains an infinite subchain.

A tree which is not a chain will be called a *strict tree*. A *tree permutation group* is a pair (G, Ω) , where Ω is a tree and G is a subgroup of the *order-automorphism group* $A(\Omega)$; i.e., $g \in A(\Omega)$ if and only if g is a one-to-one function from Ω onto Ω that satisfies for $\alpha, \beta \in \Omega$, $\alpha g \leq \beta g$ if and only if $\alpha \leq \beta$. Let (G, Ω) be a tree permutation group. If Ω is a chain, we say that (G, Ω) is an *ordered permutation group* and we endow G with the *pointwise ordering*: $f \leq g$ if $\alpha f \leq \alpha g$ for all $\alpha \in \Omega$. If (G, Ω) is an ordered permutation group, and for each $f, g \in G$, the elements $f \vee g$ and $f \wedge g$ of $A(\Omega)$ actually belong to G , we say that (G, Ω) is an ℓ -*permutation group*.

If G is a permutation group and g is an odd permutation, the equation $x^2 = g$ has no solution. Lyndon [6] and Mycielski [9] recently settled a conjecture of Silberger [10] by showing that if G is the full symmetric group on an infinite set, and if w is not a power (greater than 1) of any element of F , then the equation $w = g$ can always be solved in G . Adeleke and Holland [1] have recently proved results for the ordered analogue, i.e., when G is the automorphism

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group of a transitive chain. In this paper we consider the more general case when G is the order-automorphism group of a transitive tree, and we extend many of the results of Adeleke and Holland [1].

Throughout this paper we will use $\mathbf{Z}$ for the integers, and $\mathbf{R}$ for the reals. $\mathbf{Z}^+$ will denote the positive integers, and $\mathbf{Z}^-$ will denote the set of negative integers. Unless otherwise specified, these sets will all possess their natural total order. For any set K , $|K|$ will denote the cardinality of K , and $S(K)$ will denote the full symmetric group on K . We denote by e the identity element of any group.

1. BACKGROUND AND NOTATION

For elements α, β of a partially ordered set Ω , if either $\alpha \leq \beta$ or $\beta < \alpha$ we say that α and β are *comparable*, otherwise we say that α and β are *incomparable* and write $\alpha \parallel \beta$. If $\alpha \leq \beta$ we write $[\alpha, \beta] = \{\tau \in \Omega : \alpha \leq \tau \leq \beta\}$ for the closed interval, and $(\alpha, \beta) = \{\tau \in \Omega : \alpha < \tau < \beta\}$ for the open interval. A subset S of Ω is said to be *convex* in Ω if whenever $\alpha, \beta \in S$ and $\alpha \leq \tau \leq \beta$ ($\tau \in \Omega$), then $\tau \in S$.

We say that a tree permutation group (G, Ω) is *k-homogeneous* if every isomorphism between k -element subsets of Ω extends to an element of G ; *k-transitive* if, whenever two k -element subsets of Ω are isomorphic, there is an element of G taking one subset to the other; *weakly k-transitive* if any isomorphism between subchains of Ω of length k extends to an element of G . Note that weak k -transitivity is weaker than k -transitivity which is weaker than k -homogeneity. However, they all coincide if Ω is a chain or $k = 1$. If (G, Ω) is 1-transitive, we say that (G, Ω) is *transitive*. We also say that Ω is *weakly k-transitive* (*k-transitive*, *k-homogeneous*) if $A(\Omega)$ is ($k \in \mathbf{Z}^+$).

A partially ordered set $(P, \leq)$ is called *Dedekind-complete* if each nonempty subset of P which is bounded below in P has an infimum (greatest lower bound) in P , or equivalently, if each nonempty subset of P which is bounded above in P has a supremum (least upper bound) in P . It is easy to see that a tree $(\Omega, \leq)$ is Dedekind-complete if and only if each maximal subchain of Ω is Dedekind-complete and Ω is a meet-semilattice. The following proposition shows that each tree $(\Omega, \leq)$ has a unique Dedekind-completion.

Proposition 1.1. *Let $(\Omega, \leq)$ be a tree. Then there is a Dedekind-complete tree $(\overline{\Omega}, \leq')$ with the following properties:*

- (i) $\Omega \subseteq \overline{\Omega}$, and $\leq, \leq'$ coincide on Ω ;
- (ii) if $\alpha \in \overline{\Omega}$, then there is a nonempty subset S of Ω , with S bounded below in Ω , such that α is equal to the greatest lower bound of S in $\overline{\Omega}$.

Further, given trees Ω_1, Ω_2 and corresponding Dedekind-complete trees $\overline{\Omega}_1, \overline{\Omega}_2$ satisfying (i) and (ii), each isomorphism $\Phi : \Omega_1 \rightarrow \Omega_2$ extends to a unique isomorphism $\overline{\Phi} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$.

Proof. Call a convex subchain $C \subseteq \Omega$, with C unbounded below in Ω and C not a maximal subchain of Ω , *good* if $C = L(U(C))$ where, for any subset S of Ω , $L(S) = \{\alpha \in \Omega : \text{for all } \tau \in S, \alpha \leq \tau\}$ and $U(S) = \{\alpha \in \Omega : \text{for all } \tau \in S, \tau \leq \alpha\}$. For each $\alpha \in \Omega$, the set $C_\alpha = \{\tau \in \Omega : \tau \leq \alpha\}$ is a good chain. Let $(\overline{\Omega}, \leq')$ be the set of all good chains in Ω , partially ordered by inclusion. Now embed Ω into $\overline{\Omega}$ via the map $\alpha \rightarrow C_\alpha$ ($\alpha \in \Omega$) and identify Ω with its image in $\overline{\Omega}$. Conditions (i) and (ii) follow, and $(\overline{\Omega}, \leq')$ is a Dedekind-complete tree. Finally, if Ω_1 and Ω_2 are trees and $\Phi : \Omega_1 \rightarrow \Omega_2$ is an isomorphism, the mapping $\overline{\Phi} : \overline{\Omega}_1 \rightarrow \overline{\Omega}_2$ defined by $\alpha\overline{\Phi} = \sup\{\tau\Phi : \tau \in \Omega_1, \tau \in \alpha\}$ is an isomorphism extending Φ ; clearly $\overline{\Phi}$ is unique. $\square$

We shall denote by $\overline{\Omega}$ the Dedekind-completion of a tree Ω . Note that each $g \in A(\Omega)$ extends to a unique automorphism of $\overline{\Omega}$, which we will also denote by g . An element of $\overline{\Omega} \setminus \Omega$ will be called a *hole* of Ω .

If A is a subset (usually a chain) of a tree Ω , we define A^{cl} (the closure of A in Ω) by $A^{cl} := \{\tau \in \Omega : \text{there are } \alpha, \beta \in A \text{ such that either } \alpha \leq \tau \leq \beta, \text{ or } \alpha < \tau \text{ and } \tau \parallel \beta\}$. If $\alpha, \beta \in \Omega$ with $\alpha \leq \beta$, we define $\langle \alpha, \beta \rangle := \{\alpha, \beta\}^{cl}$ and call $\langle \alpha, \beta \rangle$ an *interval* of Ω . If Ω has no maximal elements, α will be the smallest element of $\langle \alpha, \beta \rangle$ and β will be the unique maximal element of $\langle \alpha, \beta \rangle$, and hence any isomorphism from $\langle \alpha, \beta \rangle$ to $\langle \tau, \sigma \rangle$ will take α to τ and β to σ .

Recall the terminology from Droste, Holland, and Macpherson [3]. A convex subset S of a tree Ω is said to be *nontrivial orbital* of $g \in A(\Omega)$ if S is an infinite chain, $Sg = S$, and for some (equivalently, all) $\alpha \in S$ the set $\{\alpha g^i : i \in \mathbb{Z}\}$ is unbounded above and below in S . We say that g has *positive* (*negative*) *parity* on S if $\alpha < \alpha g$ ($\alpha g < \alpha$) for some (equivalently, all) $\alpha \in S$.

A subgroup G of $A(\Omega)$ is said to be *depressible* if, whenever $g \in G$, S a nontrivial orbital of g , and $f \in A(\Omega)$ coincides with g on S^{cl} and fixes $\Omega \setminus S^{cl}$ pointwise, we have $f \in G$. Clearly, for any tree Ω , $A(\Omega)$ is depressible.

Next, let (G, Ω) be a tree permutation group and $C \subseteq \Omega$ a convex subchain. We define $G_C := \{g \in G : Cg = C\}$, and $G|_C := G_C/N$, where N is the normal subgroup $\{g \in G_C : g|_C = e\}$ of G_C . If $C = \{\alpha\}$, we will write G_α for $G_{\{\alpha\}}$. Note that $G|_C$ may be considered as a subgroup of $A(C)$. Let $g \in G$; the support of g , denoted by $\text{supp}(g)$, is the set $\{\alpha \in \Omega : \alpha g \neq \alpha\}$.

A transitive ordered permutation group (G, Ω) is said to be *primitive* if it has no nontrivial congruences, where a congruence of (G, Ω) is an equivalence relation on Ω with convex classes which are preserved by each $g \in G$. An ordered permutation group (G, Ω) is said to be *uniquely transitive* if given $\alpha, \beta \in \Omega$, there is a unique $g \in G$ such that $\alpha g = \beta$. Our work is dependent upon the following classification due to McCleary [8].

Theorem 1.2. *Let (G, Ω) be a transitive primitive depressible $\mathcal{L}$ -permutation group. Then either (G, Ω) is doubly transitive, or (G, Ω) is the right regular representation of a subgroup of $\mathbf{R}$ and thus uniquely transitive.*

2. REPRESENTATION OF TREE PERMUTATIONS BY WORDS

We say that a word w in a free group F is *universal* for a group G if for every $g \in G$, the equation $w = g$ can be solved in G . Following Adeleke and Holland [1], we now give an example.

Consider $G = A(\mathbf{Z})$. Now $A(\mathbf{Z})$ is isomorphic to $\mathbf{Z}$, and hence, abelian. Let t_n denote translation of $\mathbf{Z}$ by the integer n . If $w' = \prod x_i^{m_i}$ is the abelianized form of w (e.g., if $w = x^2 y^2 x^{-1} y$, then $w' = xy^3$) and w' is a power, that is, $\text{g.c.d.}(m_i) \neq 1$, then clearly the equation $w = t_1$ cannot be solved in $A(\mathbf{Z})$, so w is not universal for $A(\mathbf{Z})$. On the other hand, if $\text{g.c.d.}(m_i) = 1$, then there are integers k_i such that $\sum k_i m_i = 1$ and then the substitution $g_i = t_{nk_i}$ results in $w(g_1, g_2, \dots) = t_{\sum nk_i m_i} = t_{n'}$, so every nontrivial word w in which w' is not a power is universal for $A(\mathbf{Z})$.

Let (G, Ω) be a transitive ordered permutation group with congruence $\equiv$. Then the quotient $\Omega/\equiv$ is totally ordered in a natural way and (G, Ω) induces $(\hat{G}, \Omega/\equiv)$, where $\hat{G} = G/L$ with L the lazy subgroup of the action $(G, \Omega/\equiv)$. Henceforth, we will denote $\hat{G}$ by G and thus make no notational distinction between them. It will be clear from the context which group we are considering. If $\equiv$ and $\approx$ are congruences of (G, Ω) such that $\approx$ covers $\equiv$ in the sense that $\approx$ contains $\equiv$ and there is no congruence of (G, Ω) between them, and if K is a $\approx$ -class, then (G, Ω) induces the transitive ordered permutation group $(G, K/\equiv)$, called the *component* of (G, Ω) relative to $(\equiv, \approx)$ at K (Lemma 1.7.1 of Glass [4] shows that this component is independent of the particular $\approx$ -class chosen), as a subquotient, and $(G, K/\equiv)$ is primitive (Theorem 1.7.2 of Glass [4]). Now the set of all congruences of a transitive ordered permutation group forms a chain under inclusion (Holland [5]). Hence, the set of components is also naturally totally ordered.

Following Adeleke and Holland [1], if $g \in G$, $\alpha \in \Omega$, and $\alpha g \neq \alpha$, then there is a congruence $\equiv$, maximal with respect to separating αg and α . The intersection of all congruences which identify αg and α is a congruence (Corollary 3B of Glass [4]) $\approx$ covering $\equiv$, and if K is the $\approx$ -class containing α , then K contains the nontrivial orbital of g which contains α . We refer to the primitive component $(G, K/\equiv)$ as *the component determined by g at α* , and note that it is independent of which point we choose in the orbital of g containing α . The ordered set K is isomorphic to the antilexicographically ordered product $\Lambda \overleftarrow{\times} \Gamma$, where Λ is any $\equiv$ -class and $\Gamma = K/\equiv$.

Now let Ω be a tree. The next two lemmas give sufficient conditions under which elements of $A(\Omega)$ are conjugate. We will use the following fact to help simplify the notation: Let Ω be a tree, let S be a nontrivial orbital of some $g \in A(\Omega)$, and let $C \subseteq \Omega$ be a maximal subchain containing S . Then $(A(\Omega)|_C)|_S = A(\Omega)|_S$.

Lemma 2.1 (Generalization of Lemma 5 of Adeleke and Holland [1]). *Let Ω be a tree, let $C \subseteq \Omega$ be a maximal subchain such that $(A(\Omega)|_C, C)$ is transitive,*

and for some $g, h \in A(\Omega)$, let $S \subseteq C$ be a nontrivial orbital of both g and h on which g and h have the same parity. Suppose that the maximal primitive component of $(A(\Omega)|_S, S)$ is 2-transitive. Then there exists $f \in A(\Omega)$ which maps S^{cl} to itself and fixes $\Omega \setminus S^{cl}$ pointwise, such that $\sigma f^{-1} g f = \sigma h$ for all $\sigma \in S^{cl}$.

Proof. Choose $\alpha \in S$. The sets $\{\alpha g^i : i \in \mathbb{Z}\}$ and $\{\alpha h^i : i \in \mathbb{Z}\}$ are both unbounded above and below in S , and $\alpha < \alpha g$ if and only if $\alpha < \alpha h$. Without loss of generality assume that $\alpha < \alpha g$. Now S is isomorphic to $\Lambda \bar{\times} \Gamma$, where the maximal primitive component $(A(\Omega)|_S, \Gamma)$ is 2-transitive. Then from 2-transitivity, the intervals $[\alpha, \alpha g]$ and $[\alpha, \alpha h]$ of Γ are isomorphic by an induced element of $A(\Omega)|_S$. The interval $[\alpha, \alpha g]$ of S is a disjoint union $A \cup (\bigcup B_\tau) \cup D$, where A is isomorphic to a closed upper segment of Λ , τ ranges over the closed interval $[\alpha, \alpha g]$ of Γ , each B_τ is isomorphic to Λ , and D is isomorphic to a closed lower segment of Λ . Now $(A(\Omega)|_S, S)$ is transitive, and thus so is $(A(\Omega)|_S, \Lambda)$. Hence all closed upper and lower segments of Λ are isomorphic by induced elements of $A(\Omega)|_S$. Since $[\alpha, \alpha h]$ is constructed in exactly the same way, it follows that there is an induced isomorphism $f_0 : [\alpha, \alpha g] \rightarrow [\alpha, \alpha h]$ ($f_0 \in A(\Omega)|_S$). This induces an isomorphism $f_0 : \langle \alpha, \alpha g \rangle \rightarrow \langle \alpha, \alpha h \rangle$. Now for each $i \in \mathbb{Z}$, let f_i be the isomorphism $g^{-i} f_0 h^i$ from $\langle \alpha g^i, \alpha g^{i+1} \rangle$ onto $\langle \alpha h^i, \alpha h^{i+1} \rangle$. By “patching” there exists $f \in A(\Omega)$ which maps S^{cl} onto itself, extends each f_i ($i \in \mathbb{Z}$) and fixes $\Omega \setminus S^{cl}$ pointwise. If $\sigma \in S^{cl}$ then $\sigma f^{-1} g f = \sigma f_i^{-1} g f_{i+1} = \sigma h$, as required. $\square$

Lemma 2.2 (Generalization of Lemma 4 of Adeleke and Holland [1]). *Let Ω be a tree, let $C \subseteq \Omega$ be a maximal subchain such that $(A(\Omega)|_C, C)$ is transitive, and for some $g, h \in A(\Omega)$, let $S \subseteq C$ be a nontrivial orbital of both g and h . Suppose that the maximal primitive component of $(A(\Omega)|_S, S)$ is uniquely transitive. Let $\bar{g}$ denote the induced element of this maximal primitive component. If $\bar{g} = \bar{h}$ then there exists $f \in A(\Omega)$ which maps S^{cl} onto itself and fixes $\Omega \setminus S^{cl}$ pointwise, such that $\sigma f^{-1} g f = \sigma h$ for all $\sigma \in S^{cl}$.*

Proof. Choose $\alpha \in S$. The sets $\{\alpha g^i : i \in \mathbb{Z}\}$ and $\{\alpha h^i : i \in \mathbb{Z}\}$ are both unbounded above and below in S , and $\alpha < \alpha g$ if and only if $\alpha < \alpha h$. Without loss of generality assume that $\alpha < \alpha g$. Now S is isomorphic to $\Lambda \bar{\times} \Gamma$, where the maximal primitive component $(A(\Omega)|_S, \Gamma)$ is uniquely transitive. The interval $[\alpha, \alpha g]$ of S is a disjoint union $A \cup (\bigcup B_\tau) \cup D$, where A is isomorphic to a closed upper segment of Λ , τ ranges over the closed interval $[\alpha, \alpha g] = [\alpha, \alpha h]$ of Γ , each B_τ is isomorphic to Λ , and D is isomorphic to a closed lower segment of Λ . The proof then follows as in the proof of Lemma 2.1. $\square$

Let Ω be a tree. Following Droste, Holland, and Macpherson [3], we let $\text{ram}(\Omega) := \{\alpha \in \bar{\Omega} : \alpha = \inf\{\tau, \sigma\} \text{ for some } \tau, \sigma \in \Omega \text{ with } \tau \parallel \sigma\}$, the set of ramification points of Ω . Let Ω be a tree and $\alpha \in \text{ram}(\Omega)$. The maximal

lower directed subsets of $\{\tau \in \Omega : \alpha < \tau\}$ are called the *cones* of α . We let $C(\alpha)$ denote the set of all cones of α . Let κ be a finite or infinite cardinal. If $|C(\alpha)| = \kappa$ for all $\alpha \in \text{ram}(\Omega)$, κ is called the *ramification order* of Ω , and we write $\kappa = \text{r.o.}(\Omega)$.

Theorem 2.3. *Let Ω be a tree. If Ω has at least $n(1 < n \in \mathbb{Z}^+)$ pairwise isomorphic cones at some $\alpha \in \text{ram}(\Omega)$, then the word x^n is not universal for $A(\Omega)$.*

Proof. Choose $A_i \in C(\alpha)$ ($i = 1, \dots, n$) such that $\{A_i\}$ is a collection of pairwise isomorphic cones. Then there exist isomorphisms $h_i : A_i \rightarrow A_{i+1}$ ($i = 1, \dots, n-1$). Define $g \in A(\Omega)$ by

$$g = \begin{cases} h_i \text{ on } A_i & (i = 1, \dots, n-1) \\ (h_1 h_2 \cdots h_{n-1})^{-1} & \text{on } A_n \\ e \text{ on } \Omega \setminus \bigcup \{A_i\}. \end{cases}$$

Then $g^n = e$ and $\text{supp}(g) = \bigcup \{A_i\}$. Now suppose there exists $f \in A(\Omega)$ such that $f^n = g$. We need the following three lemmas.

Lemma 2.4. *If $\tau \in \bigcup \{A_i\}$, then also $\tau f \in \bigcup \{A_i\}$.*

Proof. Suppose $\tau f \notin \bigcup \{A_i\}$. Then $(\tau f)g = \tau f \Rightarrow (\tau f)f^n = \tau f \Rightarrow \tau f^n = \tau \Rightarrow \tau g = \tau$, a contradiction. $\square$

Lemma 2.5. *If $\tau \in \text{supp}(f)$, then $\tau f \parallel \tau$.*

Proof. Suppose $\tau f > \tau$. Then $\tau < \tau f < \tau f^2 < \cdots < \tau f^{n^2} = \tau(f^n)^n = \tau g^n = \tau e = \tau$, a contradiction. Similarly, if $\tau f < \tau$. $\square$

Lemma 2.6. $\alpha f = \alpha$.

Proof. Suppose $\alpha f \neq \alpha$. Then from Lemma 2.5 we have $\alpha f \parallel \alpha$. Let $\tau \in A_1$. Then $\alpha < \tau$ and so $\alpha f < \tau f$. Now from Lemma 2.4, $\tau f \in \bigcup \{A_i\}$ and so $\alpha < \tau f$. Since $\{\sigma \in \Omega : \sigma < \tau f\}$ is a chain we have either $\alpha < \alpha f$ or $\alpha f < \alpha$, a contradiction. $\square$

Now from Lemmas 2.4 and 2.6, we see that f permutes the set $\{A_i\}$, and thus f induces a solution to the equation $x^n = (12 \dots n)$ in S_n (the symmetric group on n symbols), a contradiction since it is well known that the equation x^n equal to an n -cycle ($1 < n \in \mathbb{Z}^+$) has no solution in any symmetric group. $\square$

Corollary 2.7. *Let Ω be a tree. If Ω has infinitely many pairwise isomorphic cones at some $\alpha \in \text{ram}(\Omega)$, then every free group word w , which is a proper power, is not universal for $A(\Omega)$.*

A *partially ordered permutation group* is a pair (G, Ω) , where Ω is a p.o. set and G is a subgroup of the order-automorphism group $A(\Omega)$. We now define the wreath product of two partially ordered permutation groups. The following lemma is easily verified, where we write $\{g_\tau\}$ as a shorthand expression for $\{g_\tau : \tau \in T\}$.

Lemma 2.8 (Modification of Lemma 5.1 of Glass [4]). *Let (G, Ω) and (H, T) be partially ordered permutation groups. Let $\Lambda = \Omega \overleftarrow{\times} T$ and $W = \{(\{g_\tau\}, h) : h \in H \text{ and } g_\tau \in G \text{ for all } \tau \in T\}$. Define a binary operation on W by: $(\{f_\tau\}, h_1)(\{g_\tau\}, h_2) = (\{a_\tau\}, h_1 h_2)$, where $a_\tau = f_\tau g_{\tau h_1}$ ($\tau \in T$). Then W is a group with identity $(\{e_\tau\}, e)$, where $e_\tau = e$ ($\tau \in T$), and $(\{g_\tau\}, h)^{-1} = (\{f_\tau\}, h^{-1})$, where $f_\tau = (g_{\tau h^{-1}})^{-1}$ ($\tau \in T$); and (W, Λ) is a partially ordered permutation group if $(\alpha, \sigma)(\{g_\tau\}, h) := (\alpha g_\sigma, \sigma h)$ ($\alpha \in \Omega, \sigma \in T$).*

(W, Λ) is called the *Wreath product* of (G, Ω) and (H, T) , and is written $(G, \Omega)Wr(H, T)$. If $f = (\{g_\tau\}, h) \in W$, then h is called the *global component* of f and $\{g_\tau\}$ the set of *local components* of f . We will also write $W = GWrH$ if we wish to emphasize the group part of $(G, \Omega)Wr(H, T)$.

Now let Ω be a tree with infinitely many pairwise isomorphic cones at some $\alpha \in \text{ram}(\Omega)$. Choose $B_i \in C(\alpha)$ ($i \in \mathbb{Z}^+$) such that $\{B_i\}$ is a collection of pairwise isomorphic cones, and let $\Gamma = \bigcup \{B_i\}$. Let w be any nontrivial free group word and let $g \in A(\Omega)$ be such that $\text{supp}(g) \subseteq \Gamma$. Then solving the equation $w = g$ in $A(\Omega)$, with elements whose supports are contained in Γ , is equivalent to solving the equation $w = \bar{g}$ (where $\bar{g}$ is the element of $A(\Gamma)$ induced by g) in $(A(\Gamma), \Gamma) = (A(B_1), B_1)Wr(S(\mathbb{Z}^+), \mathbb{Z}^+)$, where we view $\mathbb{Z}^+$ as a totally unordered set. This suggests the following more general problem.

Let (G, Ω) be any partially ordered permutation group. Which nontrivial free group words are universal for $(G, \Omega)Wr(S(V), V)$, where V is any trivially ordered countably infinite set? Clearly proper powers are not since $S(V)$ is a homomorphic image of $GWrS(V)$. What about other words? We now consider an example.

Let $w = x^2 y^2$ and let $g = (\{g_\tau\}, h) \in GWrS(V)$. Suppose there exist $a = (\{a_\tau\}, h_1), b = (\{b_\tau\}, h_2) \in GWrS(V)$ such that $a^2 b^2 = g$. Then for each $(\alpha, \sigma) \in \Omega \overleftarrow{\times} V$ we have $(\alpha, \sigma)a^2 b^2 = (\alpha, \sigma)g$, and carrying out the algebra we see that $\alpha a_\sigma a_{\sigma h_1} b_{\sigma h_1} b_{\sigma h_1^2} h_2^2 = \alpha g_\sigma$ and $\sigma h_1^2 h_2^2 = \sigma h$. This indicates that a solution (if any) of the general equation $w = g$, w a nontrivial free group word which is not a proper power and $g = (\{g_\tau\}, h) \in GWrS(V)$, is dependent on how the global solution to the equation $w = h$ is constructed in $S(V)$. From Lyndon [6] and Mycielski [9] we know that we can solve the equation $w = h$ in $S(V)$. The construction involves graph theory.

We now give some background information concerning graphs. Following Lyndon [6], by a *graph* Γ we mean a 1-complex; to each edge e there corresponds an inverse edge e^{-1} with $(e^{-1})^{-1} = e$ and $e^{-1} \neq e$. We consider only *labelled* graphs: Let B be a basis for some free group F and let $X = B \cup B^{-1}$. A function Φ is given assigning to each edge e its label $\Phi(e) = x$ in X , with $\Phi(e^{-1}) = x^{-1}$. The label on a path $\alpha = e_1 \cdots e_n$ is defined to be $\Phi(\alpha) = \Phi(e_1) \cdots \Phi(e_n)$. If for each vertex P and each x in X , there is exactly one edge at P with label x , then the graph Γ is a *total graph*.

The concept of a total graph with vertex set V is equivalent to that of a homomorphism from F into $S(V)$. Whenever a total graph Γ is mentioned with a word w , we assume that Γ is a total graph with respect to the variables appearing in w .

Definition 2.9. A w -path of a total graph Γ is a path with label w . A w -path is said to be *completed* when each of its edges has been marked. We leave as undefined what it means for an edge to be marked.

Now let (G, Ω) be a partially ordered permutation group, let w be a non-trivial free group word which is not a proper power, and let $g = (\{f_\tau\}, h) \in (G, \Omega)Wr(S(V), V)$, where V is any trivially ordered countably infinite set. Then the equation $w = g$ has a solution in $GW rS(V)$ if there exists a total graph Γ which (1) for w , represents the cycle structure of h , and for which (2) there exists an algorithm to mark all the edges of Γ such that whenever an edge is marked at most one w -path is completed. The result of Lyndon [6] and Mycielski [9] gives the existence of a total graph Γ satisfying (1). Condition (2) would allow us to correctly choose the local components of a solution without having to solve simultaneous systems of equations. This suggests the following strictly graph theoretic question: let V be any countably infinite set. For each pair (w, h) , where w is a nontrivial free group word which is not a proper power and $h \in S(V)$, does there exist a total graph Γ satisfying both (1) and (2)? In particular, do the total graphs constructed in Lyndon [6] satisfy (2)?

Lemma 2.10. Let (G, Ω) be a partially ordered permutation group, and let $(W, \Lambda) = (G, \Omega)Wr(S(V), V)$, where V is any trivially ordered countably infinite set. Let w be a nontrivial free group word which is not a proper power, and let $g = (\{f_\tau\}, h) \in W$ be such that h has one infinite orbit consisting entirely of V . Then the equation $w = g$ has a solution in W .

Proof. From Lyndon [6] and Mycielski [9] we can solve $w = h$ in $S(V)$, and so there exist $h_1, h_2, \dots \in S(V)$ such that $w(h_1, h_2, \dots) = h$. Now let $g_i = (\{k_{i,\tau}\}, h_i) \in W$, where $k_{i,\tau}$ is defined as follows: let $n_{i,\tau}$ be the unique positive integer such that $\tau h^{n_{i,\tau}} = \tau h_i$. We define $k_{i,\tau}$ by $\alpha k_{i,\tau} = \mu$, where $(\mu, \tau h^{n_{i,\tau}}) = (\alpha, \tau)g^{n_{i,\tau}}(\alpha \in \Omega)$. Then $w(g_1, g_2, \dots) = g$. $\square$

The following lemma is due to Adeleke and Holland [1987].

Lemma 2.11 (Lemma 2 of Adeleke and Holland [1]). If Ω is a 2-transitive chain which has a countable subset with no upper bounds and no lower bounds, and w is any nontrivial free group word, then there is a substitution in $A(\Omega)$ such that $w(g_1, g_2, \dots)$ has only one orbital and moves every point of Ω up.

Proof. By hypothesis, we may find a set $\{\sigma(i) : i \in \mathbb{Z}\} \subseteq \Omega$ which has no upper or lower bounds. Let $x_1, x_2, \dots, x_k$ be the variables involved in the reduced nontrivial word w , and let $w = z_1 z_2 \dots z_j$ where for each n , $z_n \in \{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_k, x_k^{-1}\}$. Our goal is to find substitutions $g_r \in A(\Omega)$

such that for each i , $\sigma(i)w(g_1, g_2, \dots) = \sigma(i+1)$. Clearly, 2-transitivity implies that the order of Ω is dense. Choose points $\tau(i, s)$, $i \in \mathbb{Z}$, $s = 1, 2, \dots, j-1$, such that for each i , $\sigma(i) < \tau(i, 1) < \tau(i, 2) < \dots < \tau(i, j-1) < \sigma(i+1)$. Since z_1 is x_r or x_r^{-1} for some r , 2-transitivity together with “patching” implies that we may construct $g_r \in A(\Omega)$ such that for each i , $\sigma(i)z_1 = \tau(i, 1)$. Similarly, we may arrange that for each $n = 2, \dots, j-1$, $\tau(i, n-1)z_n = \tau(i, n)$, and that $\tau(i, j-1)z_j = \sigma(i+1)$. The result is that for each i , $\sigma(i)w(g_1, g_2, \dots) = \sigma(i+1)$. Hence, $w(g_1, g_2, \dots)$ has just one orbital, consisting of all of Ω . Furthermore, if we prefer we may choose our substitution such that $w(g_1, g_2, \dots)$ has only one orbital and moves every point of Ω down. $\square$

We say that a tree Ω is *chain transitive* if for each maximal subchain $C \subseteq \Omega$, $(A(\Omega)|_C, C)$ is a transitive ordered permutation group (i.e., for each $\alpha, \beta \in C$ there exists $g \in A(\Omega)$ such that $\alpha g = \beta$ and $Cg = C$).

Lemma 2.12. *Let Ω be a chain-transitive tree such that for each maximal subchain $C \subseteq \Omega$, no primitive component of $(A(\Omega)|_C, C)$ is uniquely transitive. Let w be a nontrivial free group word, and let S be a nontrivial orbital of some $g \in A(\Omega)$. Then there is a substitution in $A(\Omega)$ such that $w(g_1, g_2, \dots) = g$ on S^{cl} and $\text{supp}(g_i) \subseteq S^{cl}$.*

Proof. Let $C \subseteq \Omega$ be any maximal subchain containing S . Now no primitive component of $(A(\Omega)|_S, S) = ((A(\Omega)|_C)|_S, S)$ is uniquely transitive. Moreover, $(A(\Omega)|_S, S)$ has a maximal primitive component, and hence this maximal primitive component must be 2-transitive (Theorem 1.2, Lemma 3.13 of Maroli [7]). Without loss of generality assume that g has positive parity on S . Now since $(A(\Omega)|_S, S)$ is closed under “patching” (Lemma 3.13 of Maroli [7]), it follows as in the proof of Lemma 2.11 that there are substitutions $g_i \in A(\Omega)|_S$ such that $w(g_1, g_2, \dots)$ has only one positive orbital and moves every point of S up. It follows that the member of the maximal component induced by $w(g_1, g_2, \dots)$ also has only one positive orbital and moves every point up. Extend each g_i to an element of $A(\Omega)$ such that $\text{supp}(g_i) \subseteq S^{cl}$. Then from Lemma 2.1, there exists $f \in A(\Omega)$ such that $f^{-1}w(g_1, g_2, \dots)f = g$ on S^{cl} and $\text{supp}(f) \subseteq S^{cl}$. Let $h_i = f^{-1}g_i f$. Then $w(h_1, h_2, \dots) = g$ on S^{cl} and $\text{supp}(h_i) \subseteq S^{cl}$. $\square$

Let Ω be a tree. For notational purposes we define $G_1(\Omega) := \{g \in A(\Omega) : \text{whenever there exists } \alpha \in \text{ram}(\Omega) \text{ with } \alpha g = \alpha, \text{ then for all } B \in C(\alpha), Bg = B \text{ (i.e., } g \text{ acts like the identity on } C(\alpha))\}$; and $G_2(\Omega) := \{g \in A(\Omega) : \text{whenever there exists } \alpha \in \text{ram}(\Omega) \text{ with } \alpha g = \alpha, \text{ then for all } B \in C(\alpha), \text{ either } Bg = B \text{ or } |B\langle g \rangle| = \aleph_0 \text{ (i.e., all the nontrivial orbits of } g \text{ acting on } C(\alpha) \text{ are infinite)}\}$.

Theorem 2.13. *Let Ω be a chain-transitive tree such that for each maximal subchain $C \subseteq \Omega$, no primitive component of $(A(\Omega)|_C, C)$ is uniquely transitive.*

Then if (i) w a nontrivial free group word and $g \in G_1(\Omega)$, or (ii) w is a nontrivial free group word which is not a proper power and $g \in G_2(\Omega)$, we can solve the equation $w = g$ in $A(\Omega)$.

Proof. (i) Now $\text{supp}(g) = \bigcup \{S^{cl} : S \text{ is a nontrivial orbital of } g\}$, and this union is disjoint. We will construct our substitutions separately on each S^{cl} . Let S be a nontrivial orbital of g . From Lemma 2.12, there is a substitution $g_i \in A(\Omega)$ with $w(g_1, g_2, \dots) = g$ on S^{cl} and $\text{supp}(g_i) \subseteq S^{cl}$. We do this on each S^{cl} , S a nontrivial orbital of g , and then patch together the resulting substitutions.

(ii) Now $\text{supp}(g) = \bigcup \{S^{cl} : S \text{ a nontrivial orbital of } g\} \cup \bigcup \{B\langle g \rangle : \text{there exists } \alpha \in \text{ram}(\Omega) \text{ and } B \in C(\alpha) \text{ such that } \alpha g = \alpha \text{ and } Bg \neq B\}$. The set $B\langle g \rangle$ will be referred to as a *nontrivial cone orbital of } g*. We will construct our substitutions separately on each S^{cl} and $B\langle g \rangle$. Let S be a nontrivial orbital of g . From Lemma 2.12, there is a substitution $g_i \in A(\Omega)$ with $w(g_1, g_2, \dots) = g$ on S^{cl} and $\text{supp}(g_i) \subseteq S^{cl}$. Now let $B\langle g \rangle$ be an arbitrary nontrivial cone orbital of g . Note that $B\langle g \rangle$ is a partially ordered set which is not a tree, and that $(A(B\langle g \rangle), B\langle g \rangle) \cong (A(B), B)Wr(S(V), V)$, where V is a trivially ordered countably infinite set. Thus from Lemma 2.10, there is a substitution $g_i \in A(B\langle g \rangle)$ with $w(g_1, g_2, \dots) = g|_{B\langle g \rangle}$. Now extend each g_i to an element of $A(\Omega)$ such that $\text{supp}(g_i) \subseteq B\langle g \rangle$. We do this on each S^{cl} , S a nontrivial orbital of g , and on each nontrivial cone orbital of g , and then patch together the resulting substitutions. $\square$

Corollary 2.14 (Theorem 7 of Adeleke and Holland [1]). *Let Ω be a transitive chain and suppose Ω has no primitive component which is uniquely transitive. Then every nontrivial free group word w is universal for $A(\Omega)$.*

Proof. If Ω is a chain, then $G_1(\Omega)$ is equal to $A(\Omega)$. $\square$

Corollary 2.15. *Let Ω be a 2-transitive strict tree which is not 2-homogeneous. Then every nontrivial free group word w is universal for $A(\Omega)$.*

Proof. From Proposition 4.2 of Droste, Holland, and Macpherson [3] we have for each maximal subchain $C \subseteq \Omega$, $(A(\Omega)|_C, C)$ is 2-transitive and thus primitive (Corollary 1.6.6 of Glass [4]). Hence Ω satisfies the conditions of Theorem 2.13. Now a 2-transitive strict tree is a tree in the sense of Droste [2] (Definition 5.1), and thus from Theorem 5.17 and Corollaries 5.36 and 5.37 of Droste [2], we see that $\text{r.o.}(\Omega) = 2$ and Ω has two nonisomorphic cones at each ramification point. Therefore, $G_1(\Omega)$ is equal to $A(\Omega)$. $\square$

Theorem 2.16. *If Ω is a chain-transitive tree and w is a nontrivial free group word such that w' (the abelianized form of w) is not a power and $g \in G_2(\Omega)$, then we can solve the equation $w = g$ in $A(\Omega)$.*

Proof. We proceed exactly as in the proof of Theorem 2.13 (ii), except that we may encounter nontrivial orbitals S of g such that $(A(\Omega)|_S, S)$ has a maximal

component which is uniquely transitive. In this case we proceed as follows: let $(A(\Omega)|_S, K)$ denote this maximal primitive component, which we may consider to be a subgroup of the reals, acting on itself by translation. Let $\bar{g}$ denote the induced element of this maximal component, and let t_r denote translation by $\bar{g}$ ($r \in \mathbf{R}$). Let $w' = \prod x_i^{m_i}$. Since w' is not a power, $\text{g.c.d.}(m_i) = 1$, and thus there are integers k_i such that $\sum k_i m_i = 1$. Then the substitution $\bar{g}_i = t_{rk_i}$ results in $w(\bar{g}_1, \bar{g}_2, \dots) = t_{\sum rk_i m_i} = t_r = \bar{g}$. Extend each $\bar{g}_i$ arbitrarily to an element g_i of $A(\Omega)$ such that $\text{supp}(g_i) \subseteq S^{cl}$. Then $w(g_1, g_2, \dots)$ induces $\bar{g}$ on $(A(\Omega)|_S, K)$. Now from Lemma 2.2, there exists $f \in A(\Omega)$ such that $f^{-1}w(g_1, g_2, \dots)f = g$ on S^{cl} and $\text{supp}(f) \subseteq S^{cl}$. Let $h_i = f^{-1}g_i f$. Then $w(h_1, h_2, \dots) = g$ on S^{cl} and $\text{supp}(h_i) \subseteq S^{cl}$. $\square$

Corollary 2.17 (Theorem 8 of Adeleke and Holland [1]). *If Ω is a transitive chain and w is a nontrivial free group word such that w' is not a power, then w is universal for $A(\Omega)$.*

Proof. If Ω is a chain, then $G_2(\Omega)$ is equal to $A(\Omega)$. $\square$

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