INDEX AND TOTAL CURVATURE OF SURFACES WITH CONSTANT MEAN CURVATURE

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ABSTRACT. We prove an analogue, for surfaces with constant mean curvature in hyperbolic space, of a theorem of Fischer-Colbrie and Gulliver about minimal surfaces in Euclidean space. That is, for a complete surface M^2 in hyperbolic 3-space with constant mean curvature 1, the (Morse) index of the operator $L = \Delta - 2K$ is finite if and only if the total Gaussian curvature is finite.

1. INTRODUCTION

Let $x: M^2 \rightarrow R^3$ be a minimal immersion into the Euclidean space R^3 , and let K , dA , and Δ respectively be the Gaussian curvature, the area element, and the Laplacian, of the induced metric. Recently, the following result that relates the total curvature $\int_M (-K) dA$ of M with the (Morse) index $\text{Ind}_M L$ for the operator $L = \Delta - 2K$ was proved.

Theorem A (D. Fischer-Colbrie [2], R. Gulliver [3]). *Let $x: M^2 \rightarrow R^3$ be a complete minimal surface. Then*

$$\text{Ind}_M L \text{ is finite} \Leftrightarrow \int_M (-K) dA \text{ is finite.}$$

Here $\text{Ind}_M L$ is defined as follows. Let $D \subset M$ be a domain with compact closure $\overline{D}$ and let $\mathcal{F}(D)$ be the set of piecewise smooth functions in M whose supports are contained in $\overline{D}$. An eigenfunction $g \in \mathcal{F}(D)$ of L satisfies the equation $Lg + \lambda g = 0$ for some (real) λ that is called an eigenvalue of L ; the multiplicity of λ is the (finite) dimension of the vector space of functions that satisfy the above equation. The *index* $\text{Ind}_D L$ is the number (counted with multiplicities) of negative eigenvalues of L in D . By letting M be exhausted by a sequence $D_0 \subset D_1 \subset \cdots \subset D_i \subset M$ of increasing domains, it can be shown that

$$\sup_i (\text{Ind}_{D_i} L) = \text{Ind}_M L$$

is independent of the choices made; it is called the *index* of L in M .

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The geometric significance of the operator L is that minus the quadratic form associated to it

$$I(f) = - \int_M (f \Delta f - 2K f^2) dA$$

is, modulo a constant factor, the second derivative of the area function of a normal variation given by the function f . Thus $\text{Ind}_D L$ measures that “number of variations” in D that decrease area.

We want to extend Theorem A to the following situation. Let $H^3(-1)$ be the three-dimensional hyperbolic space of constant sectional curvature -1 , and let $x: M^2 \rightarrow H^3(-1)$ be a surface in $H^3(-1)$ with constant mean curvature $H = 1$. Such surfaces were extensively studied in a recent paper by R. Bryant [1], and they share many properties with minimal surfaces in R^3 . For instance, the induced metric ds^2 has Gaussian curvature $K \leq 0$, and K vanishes only at isolated points of M . Further, away from points where $K = 0$, the metric $d\tilde{s}^2 = (-K)ds^2$ has curvature $\tilde{K} = 1$, and the second derivative of the area function for a normal variation given by $f \in \mathcal{F}(D)$ is

$$I(f) = - \int_M (f \Delta f - 2K f^2) dA.$$

Thus the operator $L = \Delta - 2K$ on M has the same geometric meaning as before. The main result of this paper is that Theorem A holds in this situation.

Theorem B. *Let $x: M^2 \rightarrow H^3(-1)$ be a complete surface with constant mean curvature $H = 1$. Then*

$$\text{Ind}_M L \text{ is finite} \Leftrightarrow \int_M (-K) dA \text{ is finite.}$$

2. PRELIMINARY CONSIDERATIONS

An interesting remark about Theorem A is that the implication $\Rightarrow$ is actually an intrinsic result. This was observed by Gulliver and Lawson [4] and is essentially contained in the proof of Theorem A given by Fischer-Colbrie in [2]. More specifically, we have the following result:

Theorem C (see, e.g., A. M. da Silveira [6]). *Let M^2 be a surface with a complete Riemannian metric ds^2 . Let $q \geq 0$ be a smooth function on M , and consider the operator $L = \Delta + q - K$, where Δ and K are the Laplacian and the Gaussian curvature of ds^2 . Assume that $\text{Ind}_M L < \infty$. Then M is conformally equivalent to a compact Riemann surface minus finitely many points, and $\int_M q dA < \infty$.*

By setting $q = -K$, we obtain the implication $\Rightarrow$ for both Theorems A and B.

On the other hand, all the available proofs of the implication $\Leftarrow$ in Theorem A make essential use of the Gauss map of the immersion. Since $\int_M (-K) dA < \infty$, it follows from a result of Huber [5] that M is conformally equivalent to a

compact Riemann surface M_c punctured at finitely many points $p_1, \dots, p_k$. The fact that the Gauss map extends across such punctures is then used.

In the present case, $x: M^2 \rightarrow H^3(-1)$, there is a natural Gauss map to be considered that, when x has constant mean curvature $H = 1$, has similar properties to those of the Gauss map of a minimal surface in R^3 (cf. [1]). Since Huber's theorem is intrinsic, the condition that the total curvature is finite once more implies that M is conformally equivalent to $M_c - \{p_1, \dots, p_k\}$. However, as shown by various examples (cf. [1, Example 1]), such a Gauss map may no longer extend across the punctures.

The main point of the proof of Theorem B is to show that the proof of the implication $\Leftarrow$ can be made independent of the Gauss map. The proof will also apply to complete minimal surfaces in R^3 (see Remark 2 below).

3. PROOF OF THEOREM B

Since $\int_M (-K) dA < \infty$, M is conformally equivalent to $M_c - \{p_1, \dots, p_k\}$, where M_c is a compact Riemann surface and $p_i \in M_c$, $i = 1, \dots, k$. We will assume that there is only one $p_i = p$; i.e., that there is a conformal diffeomorphism $F: M_c - \{p\} \rightarrow M$. The general case will be treated similarly.

Let ds^2 be the metric induced by x on M , and let ds_c^2 be a metric on M_c compatible with the conformal structure of M_c . Then $F^* ds^2 = \lambda ds_c^2$, where λ is a real positive function on $M_c - \{p\}$. By the completeness of ds^2 , $\lim \lambda(p_n) = \infty$, for a sequence $\{p_n\} \rightarrow p$, $p_n \in M_c - \{p\}$.

Frequently we will use the notation loosely and denote by the same symbol f both a function defined on M and the composite $f \circ F$ defined on $M_c - \{p\}$.

Consider the operator $L_c = \Delta_c - 2K\lambda$ on $M_c - \{p\}$, where Δ_c is the Laplacian of ds_c^2 and K is considered a function on $M_c - \{p\}$. Let f be a function on M with compact support, and denote by $dA_c = (1/\lambda)F_* dA$ the area element of ds_c^2 . Then

$$\begin{aligned} I_L(f) &= - \int_M (f \Delta f - 2K f^2) dA = \int_M (|\nabla f|^2 + 2K f^2) dA \\ &= \int_{M_c} (|\nabla_c f|^2 + 2K \lambda f^2) dA_c = I_{L_c}(f), \end{aligned}$$

where ∇f and $\nabla_c f$ are the gradients of f in the metrics ds^2 and ds_c^2 , respectively. Thus we must show that the index of L_c in $M_c - \{p\}$ is finite. Notice that, since the total curvature is finite, $\lim K(p_n) = 0$, for a sequence $\{p_n\} \rightarrow p$, $p_n \in M_c - \{p\}$.

Now let

$$d\tilde{s}^2 = (-K) ds^2.$$

Then $d\tilde{s}^2$ is a pseudometric in M (pseudometric here means that $d\tilde{s}^2$ can be zero only at isolated points) that has Gaussian curvature $\tilde{K} \equiv 1$ and finite area. Therefore

$$F^* d\tilde{s}^2 = F^*((-K) ds^2) = -K \lambda ds_c^2$$

is a pseudometric in $M_c - \{p\}$ with the same properties.

We will need the following lemma, due to Robert Bryant.

Lemma [1, Proposition 4]. *Let $d\tilde{s}^2$ be a pseudometric of constant Gaussian curvature $\tilde{K} = 1$ defined in a punctured disk $\Omega^* = \{\omega \in \mathbb{C} - \{0\}; |\omega| < 1\}$ of the complex plane $\mathbb{C}$. Assume that the area of Ω^* in the metric $d\tilde{s}^2$ is finite. Then there exist a number $\varepsilon > 0$, an isothermal parametrization z in $\Omega_\varepsilon = \{\omega \in \mathbb{C}; |\omega| < \varepsilon\}$, with $z(0) = 0$, and a real number $\beta > -1/2$ such that in $\Omega_\varepsilon - \{0\}$,*

$$d\tilde{s}^2 = \frac{4(1+2\beta)^2(z\bar{z})^{2\beta}}{\{1+(z\bar{z})^{2\beta+1}\}^2} dz d\bar{z}.$$

Furthermore, β is unique and z is unique up to a rotation.

Remark 1. The above lemma applies in the case of a complete minimal surface of R^3 with finite total curvature, as can be seen by considering $d\tilde{s}^2 = -Kds^2$, where ds^2 is the induced metric. In this case, β is an integer and $d\tilde{s}^2$ extends to a pseudometric in a neighborhood of the origin (cf. [1]).

It follows from the lemma that, in a neighborhood U of p in M_c , there exists an isothermal parameter z , with $z(p) = 0$, and a number $\beta > -1/2$ such that in $U - \{p\}$

$$F^* d\tilde{s}^2 = -K\lambda ds_c^2 = -K\lambda\nu dz d\bar{z},$$

where, as a function of the parameter z ,

$$-K\lambda\nu = \frac{4(1+2\beta)^2(z\bar{z})^{2\beta}}{\{1+(z\bar{z})^{2\beta+1}\}^2}.$$

We notice that $-K\lambda\nu$ is constant along the circles $z\bar{z} = \text{const.}$

We have two cases to consider:

Case 1. $\beta \geq 0$. In this case, it follows from the above expression that $-K\lambda\nu \rightarrow 0$ as $z \rightarrow 0$. Thus $-K\lambda$ is bounded in M_c . Set

$$C_0 = \sup_{M_c} (-2K\lambda),$$

and consider the operator $L_0 = \Delta_c + C_0$. Since M_c is compact, the index of L_0 in M_c is finite. On the other hand, given $f: M_c - \{p\} \rightarrow \mathbb{R}$ with compact support,

$$I_{L_c}(f) = \int_{M_c} (|\nabla_c f|^2 + 2K\lambda f^2) dA_c \geq \int_{M_c} (|\nabla_c f|^2 - C_0 f^2) dA_c = I_{L_0}(f).$$

Thus, if f is such that $I_{L_c}(f) < 0$, then $I_{L_0}(f) < 0$. It follows that

$$\text{Ind}_{M_c - \{p\}} L_c \leq \text{Ind}_{M_c} L_0 < \infty,$$

and this proves the theorem in this case.

Remark 2. The above proof, together with Theorem C, gives a proof of Theorem A that makes no use of the Gauss map (this corresponds to the case where

β is an integer; cf. Remark 1). The case $\beta \geq 0$ also includes various examples of surfaces with constant mean curvature 1 in $H^3(-1)$, to which the Gauss map does not extend (cf. [1, Example 1]).

Remark 3. The above proof also contains the following intrinsic theorem. Let M^2 be a surface with a complete Riemannian metric ds^2 . Assume that the Gaussian curvature K of ds^2 satisfies $K \leq 0$ and that $\int_M (-K) dA < \infty$. By Huber [5], M is conformally equivalent to $M_c - \{p_1, \dots, p_k\}$, where M_c is a compact Riemann surface. Let ds_c^2 be a Riemannian metric in M_c compatible with its conformal structure, and define $\lambda: M_c - \{p_1, \dots, p_k\} \rightarrow \mathbb{R}$ by $ds^2 = \lambda ds_c^2$. If $K\lambda$ is bounded in M_c , then the index of $L = \Delta - 2K$ in M is finite. Furthermore, an upper bound for this index is the number (with multiplicities) of eigenvalues of Δ_c that are smaller than the supremum of $-2K\lambda$ in M_c .

Case 2. $-1/2 < \beta < 0$. In this case, set

$$0 < 1 + 2\beta = \mu < 1,$$

and assume initially that $\mu = 1/n$, n an integer,

Let S^2 be the unit sphere in $\mathbb{R}^3$, and let ζ be an isothermal parameter in S^2 such that the canonical metric in S^2 is given by

$$\frac{4}{(1 + \zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}.$$

Observe that the metric induced in $U - \{p\}$ by a branch of the map $h: U \rightarrow S^2$ given by

$$\zeta = h(z) = z^{1/n},$$

with $1/n = 1 + 2\beta$, is

$$\frac{4}{(1 + (z\bar{z})^{1/n})^2} |h'(z)|^2 dz d\bar{z} = \frac{4(1 + 2\beta)^2 (z\bar{z})^{2\beta}}{\{1 + (z\bar{z})^{1+2\beta}\}^2} dz d\bar{z}.$$

It follows that $h^{-1}(\zeta) = \zeta^n$ is a branched isometric covering of a domain of S^2 onto U with the metric $-K\lambda ds_c^2$.

Now let $M_0 \subset M_c - \{p\}$ be a domain with compact closure and regular boundary ∂M_0 . We first choose M_0 such that ∂M_0 is a simple closed curve contained in U and that $-2K\lambda\nu = \text{const.} = C_0 > 0$ along ∂M_0 . By hypothesis, $-K\lambda\nu$ is not bounded, so we can choose M_0 sufficiently large so that $C_0 > 4/n$. We will define an operator G in M_c by

$$G = \begin{cases} \frac{1}{n}\Delta_c - 2K\lambda & \text{in } M_0 \\ \frac{1}{2}\Delta_c + 2C_0/\nu & \text{in } M_c - M_0. \end{cases}$$

Clearly, $\text{Ind}_{M_c} G$ is finite. We want to show that

$$\text{Ind}_{M_c - \{p\}} L_c \leq \text{Ind}_{M_c} G < \infty,$$

and this will prove the theorem for $\beta < 0$ and $\mu = 1/n$.

For that purpose, let $f: M_c - \{p\} \rightarrow R$ be a function with compact support. We can assume that the support of f contains M_0 and that the boundary of the support is a simple closed curve where $K\lambda\nu = \text{const}$. Then

$$\begin{aligned} I_H(f) &= \int_{M_0} \left(\frac{1}{n} |\nabla_c f|^2 + 2K\lambda f^2 \right) dA_c + \int_{M_c - M_0} \left(\frac{1}{n} |\nabla_c f|^2 - \frac{2C_0}{\nu} f^2 \right) dA_c \\ &\leq \int_{M_0} (|\nabla_c f|^2 + 2K\lambda f^2) dA_c + \int_{M_c - M_0} \left(\frac{1}{n} |\nabla_0 f|^2 - 2C_0 f^2 \right) dz \wedge d\bar{z}, \end{aligned}$$

where we have used ∇_0 to denote the gradient of the flat metric.

Now let the plane z be stereographically projected into the unit sphere S^2 . Thus, the metric of S^2 is given by $\alpha dz d\bar{z}$, where $\alpha = 4/(1 + z\bar{z})$. Thus the second term of the right-hand side of the above inequality can be estimated by

$$\begin{aligned} \int_{M_c - M_0} \left(\frac{1}{n} |\nabla_0 f|^2 - 2C_0 f^2 \right) dz d\bar{z} &= \int_{M_c - M_0} \left(\frac{1}{n} |\nabla_{S^2} f|^2 - \frac{2C_0 f^2}{\alpha} \right) dA_{S^2} \\ &= n \int_{M_c - M_0} \left(\frac{1}{n} |\nabla_c f|^2 + \frac{2C_0}{\alpha} K\lambda f^2 \right) dA_c \\ &\leq \int_{M_c - M_0} \left(|\nabla_c f|^2 + \frac{2C_0 n}{4} K\lambda f^2 \right) dA_c \\ &\leq \int_{M_c - M_0} (|\nabla_c f|^2 + 2K\lambda f^2) dA_c, \end{aligned}$$

where in the second equality we have used the fact that, locally, the metric of S^2 covers n times the metric $-K\lambda ds_c^2$ of M_c , and in the two last inequalities we used the fact that $\alpha < 4$ and $C_0 > 4/n$.

It follows that

$$\begin{aligned} I_G(f) &\leq \int_{M_0} (|\nabla_c f|^2 + 2K\lambda f^2) dA_c + \int_{M_c - M_0} (|\nabla_c f|^2 + 2K\lambda f^2) dA_c \\ &= I_{L_c}(f), \end{aligned}$$

hence, $\text{Ind}_{M_c - \{p\}} L_c \leq \text{Ind}_{M_c} G$, as we wished.

Finally we consider the case in which $\mu \neq 1/n$ for every integer n . Choose an integer n_0 such that $1/n_0 < \mu$, and define β_0 by

$$1 + 2\beta = \mu > 1/n_0 = 1 + 2\beta_0.$$

Then, in a neighborhood of $z = 0$,

$$q_0 = \frac{4(1 + 2\beta_0)^2 (z\bar{z})^{2\beta_0}}{\{1 + (z\bar{z})^{1+2\beta_0}\}^2} > \frac{4(1 + 2\beta)^2 (z\bar{z})^{2\beta}}{[1 + (z\bar{z})^{1+2\beta}]^2} = q.$$

Thus, if f is a function as above and U is sufficiently small,

$$\int_U (|\nabla_0 f|^2 - 2q_0 f^2) dz \wedge d\bar{z} < \int_U (|\nabla_0 f|^2 - 2q f^2) dz \wedge d\bar{z}.$$

It follows that $\text{Ind}_{M_c - \{p\}} L_c$ computed with $\mu < 1/n_0$ is smaller than or equal to $\text{Ind}_{M_c} \tilde{G}$, where

$$\tilde{G} = \begin{cases} \frac{1}{n_0} \Delta_c - 2K\lambda & \text{in } M_0 \\ \frac{1}{n_0} \Delta_c + 2C_0/\nu & \text{in } M_c - M_0. \end{cases}$$

Since the latter is finite, the theorem is completely proved.

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