WEAK AMENABILITY OF GROUP ALGEBRAS OF CONNECTED COMPLEX SEMISIMPLE LIE GROUPS

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ABSTRACT. We consider the problem of whether every continuous derivation from a group algebra $L^1(G)$ into its dual $L^\infty(G)$ (where the $L^1(G)$ actions on $L^\infty(G)$ are the adjoint of multiplication in $L^1(G)$) is inner, that is, of the form $D(a) = aF - Fa$ for some $F \in L^\infty(G)$. This had previously been established to hold for discrete and amenable groups and is now established for $G = \text{Gl}(n, \mathbb{C})$ and for all connected semisimple complex Lie groups.

1. INTRODUCTION

The dual A^* of a Banach algebra A is an A -bimodule under the operations

$$\begin{aligned}(aF, b) &= (F, ba), \\ (Fa, b) &= (F, ab), \quad a, b \in A, F \in A^*.\end{aligned}$$

The algebra A is *weakly amenable* if every continuous derivation from A into A^* is inner, that is, of the form $a \mapsto aF - Fa$ for some $F \in A^*$. In [3] the author showed that, for a discrete group, and more generally for a SIN group (that is a group for which the group algebra has a central approximate identity), the group algebra $L^1(G)$ is weakly amenable. In that paper the fact that the group algebra of the matrix group $\text{Gl}(n, \mathbb{C})$ is weakly amenable was announced without proof. In this paper we prove the result for $\text{Gl}(n, \mathbb{C})$ (Corollary 2.4) and extend the method to show that the group algebra of any connected semisimple complex Lie group is weakly amenable. Before going into the details it is worthwhile to indicate why the result is rather easy for discrete groups, how the extensions we have made are possible and why the case of a general locally compact group has not been settled. By the well known correspondence between continuous linear maps from A into A^* and elements of $(A \hat{\otimes} A)^*$ and the identity $L^1(G) \hat{\otimes} L^1(G) \simeq L^1(G \times G)$, a derivation D from A into A^* corresponds via the identity

$$(Da, b) = \iint F(g, h) a(g) b(h) dg dh$$

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to an element F of $L^\infty(G \times G)$ with

$$(*) \quad F(gh, k) = F(h, kg) + F(g, hk),$$

for almost all $(g, h, k) \in G \times G \times G$. Our method depends on being able to say that there is a bounded measurable function F' on $G \times G$ which is equal to F almost everywhere for which the relationship $*$ holds for all $g, h, k \in G$. Techniques exist for this but they all involve some degree of commutativity between the approximate identities used and the group operations. For the case of $G = \text{Gl}(n, \mathbb{C})$ the crucial feature is that, for the action of G on G by conjugation, a G invariant subset, namely, the set of matrices with distinct eigenvalues, has complement of measure zero and is the product of a G space X on which G acts trivially and the G space G/H of right cosets of the subgroup H of diagonal matrices (and G acts by right translation). For any bounded measure μ on X , integration of functions on this product space with respect to μ gives a map from $L^\infty(G)$ onto $L^\infty(G/H)$ which commutes with the G action.

The body of the paper is divided into two sections, one dealing with our general method and obtaining the result for $\text{Gl}(n, \mathbb{C})$ and the other with general semisimple Lie groups.

2. THE GENERAL METHOD

Theorem 2.1. *Let G be a locally compact group with a D' sequence $\{U_n\}$ (see [1, p. 637]), and let $\mathbf{X}$ be a Banach space. On $L^1(G) \hat{\otimes} \mathbf{X}$ define the structure of an $L^1(G)$ -bimodule by*

$$\begin{aligned} b(a \otimes x) &= \lambda(b)a \otimes x, \\ (a \otimes x)b &= ab \otimes x, \quad a, b \in L^1(G), x \in \mathbf{X}, \end{aligned}$$

where λ is a left Haar measure on G . Then every continuous derivation from $L^1(G)$ into $(L^1(G) \hat{\otimes} \mathbf{X})^*$ is inner.

Proof. Let D be a continuous derivation from $L^1(G)$ into $(L^1(G) \hat{\otimes} \mathbf{X})^*$. Then there is a trilinear functional S with $S(a, b, x) = (D(a), b \otimes x)$ ($a, b \in L^1(G)$, $x \in \mathbf{X}$). Let LIM denote a Banach limit on $\mathbb{N}$ and define

$$\tilde{S}(g, h, x) = \text{LIM}_m \text{LIM}_n S(\delta_g e_m, \delta_h e_n, x), \quad g, h \in G, x \in \mathbf{X},$$

where $e_m = \lambda(U_m)^{-1} \times$ characteristic function of U_m and δ_g is unit mass situated at g . $\tilde{S}$ is linear in x and $|\tilde{S}(g, h, x)| \leq \|D\| \|x\|$.

The derivation law for D gives

$$\begin{aligned} S(\delta_g e_m \delta_h e_n, \delta_k e_p, x) &= S(\delta_g e_m, \delta_k e_p, x) + S(\delta_h e_n, \delta_k e_p \delta_g e_m, x), \\ g, h, k \in G, \quad m, n, p \in \mathbb{N}, \end{aligned}$$

because $\lambda(\delta_h e_n) = 1$. We now apply LIM to each side of this equation, first on p , then on m and finally on n . Because $e_m \delta_h e_n \rightarrow \delta_h e_n$ in $L^1(G)$ norm

as $m \rightarrow \infty$, we get

$$\tilde{S}(gh, k, x) = \tilde{S}(g, k, x) + \tilde{S}(h, kg, x), \quad g, h, k \in G, x \in \mathbf{X}.$$

Now put $\tilde{\alpha}(g, x) = \tilde{S}(g, e, x)$, $g \in G$, $x \in \mathbf{X}$. We have $\tilde{S}(g, k, x) = \tilde{S}(g, ek, x) = \tilde{\alpha}(kg, x) - \tilde{\alpha}(k, x)$. To complete the proof we have to show that, for each $x \in \mathbf{X}$, the functions $g, h \mapsto \tilde{S}(g, h, x)$ and $g \mapsto \tilde{\alpha}(g, x)$ are bounded measurable functions, and so we can define $\alpha \in (L^1(G) \hat{\otimes} X)^*$ by

$$\alpha(a \otimes x) = \int a(g) \tilde{\alpha}(g, x) dg.$$

This would give

$$\begin{aligned} (a\alpha - \alpha a, b, x) &= \iint (\tilde{\alpha}(hg, x) - \tilde{\alpha}(h, x)) a(g) b(h) dg dh \\ &= \iint \tilde{S}(g, h, x) a(g) b(h) dg dh, \end{aligned}$$

where the last integral has to be shown to be $S(a, b, x) = (D(a), b, x)$. For each $x \in \mathbf{X}$, there is $\sigma_x \in L^\infty(G \times G)$ with

$$S(a, b, x) = \iint \sigma_x(g, h) a(g) b(h) dg dh.$$

For each value of m and $g, h \mapsto \int \sigma_x(r, h) (\delta_g e_m)(r) dr$ is an $L^\infty(G)$ function, and so, by [1, Theorem 44.18], $\lim_n \iint \sigma_x(r, s) (\delta_g e_m)(r) (\delta_h e_n)(s) dr ds$ exists and equals $\int \sigma_x(r, h) (\delta_g e_m)(r) dr$ for almost all $h \in G$. Amalgamating the exceptional sets for the different values of m we see that, for each x , there is a subset E_1 of $G \times G$ such that

$$\lim_n S(\delta_g e_m, \delta_h e_n, x) = \int \sigma_x(r, h) (\delta_g e_m)(r) dr$$

for all m and all $(g, h) \in G \times G \setminus E_1$ and, for each g , $\{h; (g, h) \in E_1\}$ is of measure zero. However, $(g, h) \mapsto S(\delta_g e_m, \delta_h e_n, x)$ is a continuous function, so the set of points on which this limit exists is measurable. Thus the expression on the right is a measurable function and E_1 is a measurable set, hence a set of measure zero. A similar argument now shows that, for each x ,

$$\lim_m \int \sigma_x(r, h) (\delta_g e_m)(r) dr = \sigma_x(g, h)$$

except for a set of values of (g, h) of measure zero. Thus

$$\sigma_x(g, h) = \lim_m \lim_n S(\delta_g e_m, \delta_h e_n, x) = \tilde{S}(g, h, x)$$

except for a set of values of (g, h) of measure zero. This shows that $\tilde{S}$ is a measurable function of g and h and that

$$S(a, b, x) = \iint \tilde{S}(g, h, x) a(g) b(h) dg dh.$$

The function $g \mapsto \tilde{S}(g, h, x)$ is thus measurable for one value h_0 (in fact for almost all values) of h . The identity

$$\tilde{S}(g, h, x) = \tilde{S}(g, h_0 h_0^{-1} h, x) = \tilde{S}(h_0^{-1} h g, h_0, x) - \tilde{S}(h_0^{-1} h, h_0, x)$$

then shows that it is measurable for all values of h and hence that $\tilde{\alpha}(g, x)$ is a measurable function of g for each $x \in \mathbf{X}$. $\square$

Corollary 2.2. *Under the same conditions as in Theorem 2.1, let H be a closed subgroup of G . The space $L^1(G/H)$ of bounded measures on the right coset space G/H , which are absolutely continuous with respect to the quotient measure space structure on G/H coming from the Haar measure space structure on G , is a quotient bimodule of the bimodule $L^1(G)$, and every continuous derivation from $L^1(G)$ into $(L^1(G/H) \hat{\otimes} \mathbf{X})^*$ is inner.*

Proof. Let q be the quotient map of $L^1(G)$ onto $L^1(G/H)$ and D_0 be a continuous derivation from $L^1(G)$ into $(L^1(G/H) \hat{\otimes} \mathbf{X})^*$. Then $q_0 = (q \otimes \text{Id}_{\mathbf{X}})^*$ is a bimodule map from $(L^1(G/H) \hat{\otimes} \mathbf{X})^*$ into $(L^1(G) \hat{\otimes} \mathbf{X})^*$, so $D = q_0 D_0$ is a derivation and hence inner, by the theorem. Applying the proof of the theorem to the present case, we have $S(a, b, x) = (D_0(a), qb, x)$. For any h in H , $q(b) = q(\delta_h b)$ and so $\tilde{S}(g, hk, x) = \tilde{S}(g, k, x)$ for all $g, k \in G, h \in H$ and $x \in \mathbf{X}$. We thus have

$$\tilde{S}(hg, e, x) = \tilde{S}(h, e, x) + \tilde{S}(g, h, x) = \tilde{S}(h, e, x) + \tilde{S}(g, e, x)$$

for $g \in G, h \in H$ and, putting $g = h^n$, we see by induction that $\tilde{S}(h^n, e, x) = n\tilde{S}(h, e, x)$, implying $\tilde{S}(h, e, x) = 0$ because $\tilde{S}$ is bounded. Hence $\tilde{\alpha}(hg, x) = \tilde{S}(hg, e, x) = \tilde{S}(g, e, x)$, that is, $g \mapsto \tilde{\alpha}(g, x)$ is constant on the right cosets of H and can be considered as an element of $L^\infty(G/H)$. Thus defining $\alpha_0(a_0, x) = \int \tilde{\alpha}(Hg, x) da_0(Hg)$, we get $\alpha_0 \in (L^1(G/H) \hat{\otimes} \mathbf{X})^*$ and $\alpha_0(q(a), x) = \alpha(a, x)$. Thus $D(a) = a\alpha - \lambda(a)\alpha$ implies $D_0(a) = a\alpha_0 - \lambda(a)\alpha_0$. $\square$

Lemma 2.3. *Let G be a real Lie group. Then G contains a D' sequence.*

Proof. Suppose G is d dimensional. We take the Euclidean metric on $\mathbf{R}^d$ and a cononical chart α mapping a neighbourhood U of e in G onto an open ball B_ϵ of radius ϵ centre at 0 in $\mathbf{R}^d$. There is a neighbourhood V of e such that $VV^{-1} \subseteq U$ and α maps V onto B_η for some $\eta < \epsilon$. The map $\pi: B_\eta \times B_\eta \rightarrow B_\epsilon$ given by $\pi(x, y) = \alpha((\alpha^{-1}x)(\alpha^{-1}y)^{-1})$ is a smooth map with $\|\pi(x, y) - (x - y)\| \leq K(\|x\|^2 + \|y\|^2)$ for all $x, y \in B_\eta$ for some K . Thus, for $r < \min\{\eta, (2K)^{-1}\}$, if $x, y \in B_r$ then $\|\pi(x, y)\| \leq \|x\| + \|y\| + 2Kr^2 \leq 3r$, that is $\pi(B_r, B_r) \subseteq B_{3r}$.

Under the mapping α , left Haar measure λ on U is carried into a measure on B_ϵ which is absolutely continuous with respect to Lebesgue measure L , and we have, for a suitably chosen Haar measure,

$$\int \phi(\alpha(g)) d\lambda(g) = \int \phi(x) J(x)^{-1} dx,$$

for all $\phi \in C_0(B_\varepsilon)$ where $J(x) = \det[\partial \pi'_i(x, y)/\partial y_j]_{y=0}$ and $\pi'(x, y) = \alpha((\alpha^{-1}x)(\alpha^{-1}y))$. Thus J is continuous and $J(0) = 1$; so, for sufficiently small values of r , we have $L(B_r)/2 \leq \lambda(\alpha^{-1}B_r) \leq 2L(B_r)$. This shows that $\lambda((\alpha^{-1}B_r)(\alpha^{-1}B_r)^{-1}) \leq 2L(\pi(B_r, B_r)) \leq 2L(B_{3r}) \leq 2 \cdot 3^d L(B_r) \leq 4 \cdot 3^d \lambda(\alpha^{-1}B_r)$ and we can construct a D' sequence by taking $U_n = \alpha^{-1}(B_{r_n})$ for a suitable null sequence of values of r . $\square$

Corollary 2.4. $L^1(\text{Gl}(n, \mathbb{C}))$ is weakly amenable.

Proof. Starting with $\mathbf{X} = L^1(G)$ considered as an $L^1(G)$ -bimodule, where the left and right actions are algebra multiplication, and carrying out the transformations in [3, p.31], we get a new $L^1(G)$ -bimodule structure on $L^1(G)$ given by

$$(a \circ f)(s) = \lambda(a)f(s)$$

$$(f \circ a)(s) = \int f(gsg^{-1})a(g)\Delta(g^{-1})dg, \quad a \in L^1(G), f \in L^1(G), s \in G$$

We will have proved the theorem if we show that every continuous derivation from $L^1(G)$ into $(L^1(G)^\circ)^*$ is inner.

The new module action on $L^\infty(G)$ is

$$F \circ a = \lambda(a)F$$

$$(a \circ F)(s) = \int a(g)F(g^{-1}sg)dg, \quad a \in L^1(G), F \in L^\infty(G), s \in G.$$

Let H be the subgroup of $G = \text{Gl}(n, \mathbb{C})$ consisting of diagonal matrices, and consider the map γ of $G/H \times H$ onto G given by $(Hg, h) \mapsto g^{-1}hg$ (this map is well defined because H is abelian). If we restrict this map to a map $G/H \times H_0 \rightarrow G_0$, where G_0, H_0 are the subsets of G, H respectively of elements with n distinct eigenvalues, we have an $n!$ -fold covering map; replacing H_0 by X , the set of matrices $\text{diag}(a_1, a_2, \dots, a_n)$, where $\text{Re } a_i < \text{Re } a_j$ or $\text{Re } a_i = \text{Re } a_j$ and $\text{Im } a_i < \text{Im } a_j$ whenever $i < j$, we get a bijective map γ_0 of $G/H \times X$ onto G_0 . If we take the induced Haar measure space structure on G/H and G_0 and Lebesgue measure on X , γ_0 is a homomorphism and $G \setminus G_0$ is of measure zero. Thus γ_0 induces an isometric isomorphism γ of $L^1(G/H) \hat{\otimes} (L^1(X))$ onto $L^1(G)$ by

$$[\gamma(a \otimes x)](k) = a(Hg)x(\xi),$$

where $k \in G_0, \xi \in X$ and $g \in G$ with $g^{-1}\xi g = k$. Because

$$\gamma[(a \otimes x)b](k) = \int a(Hg)b(g^{-1}r)x(\xi)dg,$$

where $r^{-1}\xi r = k$, and

$$\begin{aligned} [\gamma(a \otimes x) \circ b](k) &= \int (\gamma a \otimes x)(tk t^{-1})b(t)\Delta(t^{-1})dt \\ &= \int a(Hs)x(\eta)b(t)\Delta(t^{-1})dt, \end{aligned}$$

where $s^{-1}\eta s = tkt^{-1}$ so that $\eta = \xi$ and $t = s^{-1}r$, we see that $\gamma[(a \otimes x) \circ b] = \gamma(a \otimes x) \circ b$, $a \in L^1(G/H)$, $b \in L^1(G)$, $x \in L^1(X)$. Trivially the left module action is also preserved by γ , and the result follows from 2.2 and 2.3. $\square$

3. THE MAIN THEOREM

Theorem 3.1. *Let G be a connected complex semisimple Lie group. Then $L^1(G)$ is weakly amenable.*

Proof. We consider open sets R in G for which there is a homeomorphism and measure space isomorphism γ from $G/H \times X$ onto R , where H is some closed subgroup of G and X is an open set in a Euclidean space with Haar/Lebesgue measure structure such that $\gamma(Hgk, \xi) = k^{-1}\gamma(Hg, \xi)k$, $g, k \in G$, $\xi \in X$. As in the proof of Corollary 2.4, we need to prove that every continuous derivation D from $L^1(G)$ into $(L^1(G)^\circ)^* = L^\infty(G)$ is inner. By restriction of $L^\infty(G)$ functions to R , D gives a derivation from $L^1(G)$ into $L^\infty(R)$ which is isomorphic as an $L^1(G)$ bimodule with $L^\infty(G/H \times X) = (L^1(G/H) \hat{\otimes} L^1(X))^*$. Using 2.2 and 2.3 we see that there is an $F \in L^\infty(R)$, with $D(a) = a \circ F - F \circ a$ on R for all $a \in L^1(G)$ and $\|F\| \leq \|D\|$. The main part of the proof will be to establish that the union of all such sets R has complement in G of measure zero. Because the topology of G has a countable base, this would imply that there is a sequence $\{R_i\}$ of such sets whose union has complement of measure zero. If we take the elements F_i of $L^\infty(R_i)$ given by the above and define $F(g) = F_i(g)$, for $g \in R_i \setminus \cup_{j < i} R_j$, then $F \in L^\infty(G)$ and $D(a) = a \circ F - F \circ a$, $a \in L^1(G)$, completing the proof.

Let L be the Lie algebra of G . First of all we consider the case in which the adjoint representation is faithful and G is a closed subgroup of $\text{Gl}(L)$. Taking a basis in L , G is a closed subgroup of $\text{Gl}(n, \mathbb{C})$ where $n = \dim L$. We shall need the following well-known result.

Lemma 3.2. *Let f be an analytic function on the open polydisc $D_1 \times D_2 \times \cdots \times D_d$ in $\mathbb{C}^d$. Then, either f is identically zero or its zero set is of $2d$ Lebesgue measure zero.*

Proof. We prove this by induction on d . The case $d = 1$ is an elementary result. If it holds for $d-1$ and f is analytic on $D_1 \times \cdots \times D_d$ and not identically zero, then consider the functions $(z_2, \dots, z_d) \mapsto f(z_1, z_2, \dots, z_d)$. Suppose the set of $z_1 \in D_1$, for which this is identically zero, has a point of accumulation in D_1 . Then, for each $(z_2, \dots, z_d) \in D_2 \times \cdots \times D_d$, the function $z_1 \mapsto f(z_1, z_2, \dots, z_d)$ is identically zero, that is, f is identically zero. Thus, for all but a countable set of values of z_1 , the zero set of f intersects $\{z_1\} \times D_2 \times D_3 \times \cdots \times D_d$ in a set of $2(d-1)$ measure zero, and the proof is completed by Fubini's Theorem. $\square$

An element g of G is called regular if the dimension of the null space of $g - I$ is the rank of L . We denote the set of regular elements of G by G_1 . Let

$p \in \{1, \dots, n\}$, and let $m(g)$ be the determinant of a fixed $p \times p$ minor of $g - I$. Then, either $m(g) \equiv 0$ or the zero set of m is a closed nowhere dense set of zero measure. If $p > n - r$, then $\dim \text{range ad } \ell \leq n - r < p$ for all $\ell \in L$, so, for all ℓ in a neighbourhood of 0 in L , $\dim \text{range}(\exp \text{ad } \ell - I) < p$ and $m(\exp \text{ad } \ell) = 0$, hence $m \equiv 0$. Thus $\dim \text{range}(g - I) \leq n - r$ for all $g \in G$. If $g \in G_1$ then there is an $(n - r) \times (n - r)$ minor m , with $m(g) \neq 0$ and $m(g') \neq 0$ for all g' near g , that is $\dim \text{range}(g' - I) \geq n - r$. Then $\dim \text{range}(g' - I) = n - r$, and hence G_1 is open. Its complement is the set of common zeros of all the $(n - r) \times (n - r)$ minors and is of measure zero, because the only alternative is that all these minors are identically zero and, if ℓ is a regular element of L then, for small positive t , $\dim \text{range}(\exp \text{ad } t\ell - I) = \dim \text{range } \ell = n - r$ so that one of the $(n - r) \times (n - r)$ minors of $(\exp \text{ad } t\ell) - I$ is not zero.

By the elementary theory of equations, if $P \in \mathbb{C}[x]$ then there is a $Q \in \mathbb{C}[x]$ such that the zeros of Q are $(z_i - z_j)^2$, $1 \leq i < j \leq \deg P$ where the z_i are the zeros of P . The coefficients of Q are polynomials in the coefficients of P . If P is the characteristic function of $g \in G$ then Q is of the form $a_0(g) + a_1(g)x + a_2(g)x^2 + \dots$, where the $a_i(g)$ are analytic functions on G . Thus

$$F_k = \{g : 0 \text{ is a zero of } Q \text{ of multiplicity } \geq k\}$$

is either G or a closed nowhere dense subset of G of measure zero. As $\dim \ker(g - I) \geq r$ for all $g \in G$, $F_{k_0} = G$ for $k_0 = r(r - 1)/2$. We shall show that $F_{k_0+1} \neq G$. Suppose C is a Cartan subalgebra of L and $\mathcal{R}$ is the set of roots of C (including the zero root). Put

$$E = \{c; c \in C, r(c) \neq s(c) \text{ whenever } r, s \in \mathcal{R} \text{ and } r \neq s\}.$$

Since E is the complement of a finite set of hyperplanes, it is a dense open subset of C . If $c \in E$ then 0 is an eigenvalue of $\text{ad } c$ with multiplicity r and the eigenvalues are otherwise distinct. Thus, for sufficiently small $c \in E$, 1 is an eigenvalue of $\exp \text{ad } c$ and the eigenvalues are otherwise distinct, so $\exp \text{ad } c \notin F_{k_0+1}$. We put $G_0 = G_1 \cap (G \setminus F_{k_0+1})$ so that G_0 is an open dense subset of G with $G \setminus G_0$ of measure zero. We have $g \in G_0$ if and only if there is a basis for L with respect to which g has a diagonal matrix with 1 in the first r places and $n - r$ distinct elements of $\mathbb{C} \setminus \{0, 1\}$ on the rest of the diagonal. We denote the set of diagonal matrices of this form by Δ_r .

Let $g \in G_0$ and choose a basis of L such that $g \in \Delta_r$. Then the set $Z_g = \{h, h \in G, gh = hg\}$ is a closed subgroup of G with Lie algebra

$$\begin{aligned} K &= \{\ell : \ell \in L, \exp(t \text{ad } \ell)g = g \exp(t \text{ad } \ell), t \in \mathcal{R}\} \\ &= \{\ell : \ell \in L, \text{ad } \ell g = g \text{ad } \ell\} \\ &= \{\ell : \ell \in L, g\ell = \ell g\}, \end{aligned}$$

that is, K is the linear span of the first r basis vectors. Let V be a neighbourhood of 0 in K such that, for all $k \in V$, all matrix entries in $\text{ad } k$ have modulus less than $\delta = (\inf \log |\alpha \beta^{-1}|)/2$, where the infimum is taken over all pairs of distinct eigenvalues α, β of g . Because $g \in G_0$, the matrix of $\text{ad } k$ for any $k \in K$ has nonzero off-diagonal entries k_{ij} only for $i, j \leq r$. However, if $k \in K$, then, choosing $t \in \mathbf{R}$ with $tk \in V$, $g \exp \text{ad } tk$ also has this form and has distinct entries different from 1 on the diagonal outside the first r rows. Then, since $\dim \ker(g \exp \text{ad } tk - I) \geq r$, we have $\exp \text{ad } tk = I$ on K , for all such t , showing that $\text{ad } k$ has a diagonal matrix with zeros in the first r rows.

Replacing V by a smaller neighbourhood of 0 in K if necessary, the exponential map maps V homeomorphically onto a neighbourhood of I in Z_g . We now consider the map $\gamma: G/Z_g \times V \rightarrow G$ given by $\gamma(Z_g g', v) = (g')^{-1} \exp v g'$. Because $g \exp v \in Z_g$, this map depends only on the coset $Z_g g'$ and is a well defined continuous map of $G/Z_g \times V$ into G with $\gamma(Z_g g' k, v) = k^{-1} \gamma(Z_g g', v) k$. We now show that it is injective. If $\gamma(Z_g k, v) = \gamma(Z_g k', v')$ then $k(k')^{-1} g \exp v' = g \exp v k(k')^{-1}$, where $g, \text{ad } v$ and $\text{ad } v'$ are given by diagonal matrices $g = \text{diag}(g_1, \dots, g_n)$, $\text{ad } v = \text{diag}(v_1, \dots, v_n)$ and $\text{ad } v' = \text{diag}(v'_1, \dots, v'_n)$. If $k(k')^{-1}$ is given by $[\kappa_{ij}]$, then this equation is

$$\kappa_{ij} g_j \exp v'_j = g_i \exp v_i \kappa_{ij} \quad i, j = 1, \dots, n.$$

By the definition of V , $|v_i| < \delta$ and $|v'_i| < \delta$, so $g_j \exp v'_j = g_i \exp v_i$ if and only if $i \leq r$ and $j \leq r$ or $i = j$ and $v_i = v'_i$. Thus $[\kappa_{ij}]$ has nonzero off-diagonal entries only in the principal $r \times r$ minor, which implies that $k(k')^{-1}$ commutes with g , $\exp v$ and $\exp v'$. Thus $Z_g k = Z_g k'$, $\exp v = \exp v'$ and hence $v = v'$. Finally we show that the range of γ is open and γ is a measure space isomorphism by showing its derivative is nonsingular everywhere. Since the map $k' \mapsto k^{-1} k' k$ is an analytic homeomorphism on G , it is enough to show that γ' is nonsingular at the points (Z_g, v) , $v \in V$. Let $v_0 \in V$ and M be the subspace of L spanned by the eigenvectors $e_{r+1}, \dots, e_n$ of g not in K . The maps $\ell \mapsto g \exp v_0 \exp \ell$, $k \mapsto v_0 + k$, $m \mapsto Z_g \exp m$ are diffeomorphisms of neighbourhoods L_0, K_0 and M_0 of 0 in L, K and M onto neighbourhoods of $g \exp v_0$ in G , of v_0 in K and of Z_g in G/Z_g , and so γ gives rise to a diffeomorphism Γ of $M_0 \times K_0$ into L_0 . We have

$$\exp(-m) g \exp(v_0 + k) \exp m = g \exp v_0 \exp \Gamma(m, k).$$

Thus the differential Γ'_0 of Γ at $m = 0, k = 0$ is given by $\Gamma'_0(m', k') = (g \exp v_0 - I)m' + k'$ so that Γ'_0 is given by a diagonal matrix with 1 in the first r places and nonzero entries everywhere else on the diagonal. We have thus shown that there is a set R in G with the properties described at the beginning of this proof with $g \in R$, and the result follows for G .

We now turn to the case in which G has a possibly nontrivial centre Z , a finite subgroup. Let q denote the quotient map $G \rightarrow G/Z$. We apply the above analysis to G/Z and put $G_0 = q^{-1}(G/Z)_0$. For any $g \in G$ we have $Z_g \subseteq q^{-1}Z_{qg}$, and if $h \in q^{-1}Z_{qg}$ then $q(ghg^{-1}h^{-1}) = e$, that is, $ghg^{-1}h^{-1} \in Z$ so that if h is close to e then $h \in Z_g$. Thus Z_g is an open subgroup of $q^{-1}Z_{qg}$, and Z_g and Z_{qg} have the same Lie algebra. Also G and G/Z have the same Lie algebra. Take $g \in G_0$. The construction of V and γ is as before (α, β in the definition of V refer to the eigenvalues of $\text{ad } g$). In showing that γ is injective, we start with $\gamma(Z_g k, v) = \gamma(Z_g k', v')$ and work as above in G/Z to see that $v = v'$ and $q(k(k')^{-1}) \in Z_{qg}$. We thus have $k(k')^{-1}g \exp v = g \exp vk(k')^{-1}$. Consider the map $\alpha: V \rightarrow G$ given by $\alpha(w) = (g \exp w, k(k')^{-1})$ where $(\ , \)$ denotes the commutator. We have $q\alpha(w) = e$ for all w so that α is a continuous map of the connected space V into the discrete space Z and is constant. Since $\alpha(v) = e$, we have $\alpha(0) = e$, that is, $k(k')^{-1} \in Z_g$. The remainder of the proof being of a local nature, it applies to the more general situation. $\square$

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