CONVEXITY PROPERTIES OF DISTINGUISHED EIGENVALUES OF CERTAIN CLASSES OF OPERATORS

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ABSTRACT. We prove two convexity results:

- (1) Let $A(\varepsilon) = A_0 + \varepsilon A_1$ be a family of selfadjoint operators in a Krein space with separated spectrum so that the maximum $\lambda_-(\varepsilon)$ of the spectrum of negative type of $A(\varepsilon)$ is an isolated simple eigenvalue. Then $\lambda_-(\varepsilon)$ is convex.
- (2) Let $\lambda_-(\varepsilon)$ be the left distinguished eigenvalue of the generalized eigenvalue problem $(A_0 + \varepsilon A_1)x = \lambda Bx$ where A_1 and B are real diagonal matrices and A_0 is an irreducible essentially nonnegative matrix. Then $\lambda_-(\varepsilon)$ is convex.

In both cases $\lambda_-(\varepsilon)$ is strictly convex unless it is linear.

INTRODUCTION

Let A_0 and A_1 be operators in a Banach space X such that A_1 is A_0 -bounded. Let λ_0 be an isolated simple eigenvalue of A_0 ; it is well known that for $|\varepsilon|$ small enough there exists a unique simple eigenvalue $\lambda(\varepsilon)$ of the operator $A(\varepsilon) = A_0 + \varepsilon A_1$ with the property $\lim_{\varepsilon \rightarrow 0} \lambda(\varepsilon) = \lambda_0$. Moreover $\lambda(\varepsilon)$ is an analytic function of ε in a neighborhood of $\varepsilon = 0$. We are interested in its convexity properties.

The following special cases have been investigated:

- (a) If X is a Hilbert space, $A(\varepsilon)$ self-adjoint for ε real and λ_0 is the maximum (the minimum, resp.) of the spectrum of A_0 then $\lambda(\varepsilon)$ is convex (concave, resp.). Moreover, it is strictly convex (strictly concave, resp.) unless $A_0 x_0 = \lambda_0 x_0$ implies $A_1 x_0 = \lambda_1 x_0$, hence $\lambda(\varepsilon)$ is linear.
- (b) If $X = \mathbb{C}^n$, A_0 is an irreducible essentially nonnegative matrix (i.e. all its off-diagonal elements are nonnegative), A_1 is a real diagonal matrix and λ_0 is the principal eigenvalue of A_0 (i.e. $\lambda_0 = \max \operatorname{Re} \sigma(A_0)$) then $\lambda(\varepsilon)$ is convex. Moreover, it is strictly convex unless A_1 is a scalar matrix and $\lambda(\varepsilon)$ is linear.

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See [2, 3] and also [4] where an extension to infinite dimensional spaces is given.

In this note we shall prove two related results:

- (c) If X is a Krein space, $A(\varepsilon)$ self-adjoint for ε real and has separated spectrum (for the definitions see §1) and λ_0 is the maximum of the negative type spectrum $\sigma_-(A_0)$ (minimum of the positive type spectrum $\sigma_+(A_0)$, resp.) then $\lambda(\varepsilon)$ is convex (concave, resp.). Moreover, it is strictly convex (strictly concave, resp.) unless $A_0x_0 = \lambda_0x_0$ implies $A_1x_0 = \lambda_1x_0$, hence $\lambda(\varepsilon)$ is linear.
- (d) If $X = \mathbb{C}^n$, A_0 is an irreducible essentially nonnegative matrix, A_1 and B are diagonal matrices and λ_0 is the left distinguished eigenvalue λ_- (the right distinguished eigenvalue λ_+) of the generalized eigenvalue problem

$$A_0x = \lambda Bx,$$

and if $\lambda(\varepsilon)$ is the eigenvalue of the generalized eigenvalue problem

$$(A_0 + \varepsilon A_1)x = \lambda Bx$$

such that $\lim_{\varepsilon \rightarrow 0} \lambda(\varepsilon) = \lambda_0$, then $\lambda(\varepsilon)$ is convex (concave, resp.). Moreover, it is strictly convex (strictly concave, resp.) unless A_1 and B are scalar matrices in which case $\lambda(\varepsilon)$ is linear.

For the definition of the distinguished eigenvalues of the generalized eigenvalue problem, see §2.

1. EXTREMAL EIGENVALUES OF A FAMILY OF SELF-ADJOINT OPERATORS WITH SEPARATED SPECTRUM IN A KREIN SPACE

Let X be a Krein space with inner product $[\cdot, \cdot]$ and intrinsic norm $\|\cdot\|$ (see [1] for these and subsequent notions). Let A_0 be a self-adjoint operator in X such that there exist a real number γ and a positive constant C such that

$$(1) \quad [(A_0 - \gamma I)x|x] \geq C\|x\|^2 \quad (x \in D(A_0))$$

holds. Then the spectrum $\sigma(A_0)$ is separated:

$$\sigma(A_0) = \sigma_-(A_0) \cup \sigma_+(A_0),$$

$\sigma_-(A_0) = (-\infty, \gamma) \cap \sigma(A_0)$, $\sigma_+(A_0) = (\gamma, \infty) \cap \sigma(A_0)$, and $\sigma_-(A_0)$ ($\sigma_+(A_0)$, resp.) is the set of spectral points of negative (positive, resp.) type. Denote

$$\lambda_- = \max \sigma_-(A_0), \quad \lambda_+ = \min \sigma_+(A_0).$$

Proposition 1. *Let A_0 be a self-adjoint operator in X such that (1) holds and let A_1 be a bounded self-adjoint operator. Assume that $\lambda_0 = \lambda_-$ ($\lambda_0 = \lambda_+$, resp.) is an isolated simple eigenvalue of A_0 . Then there exist an interval $J = (-\varepsilon_0, \varepsilon_0)$ and an analytic function $\lambda_-(\varepsilon)$ ($\lambda_+(\varepsilon)$, resp.) on J such that $\lambda_-(0) = \lambda_0$ ($\lambda_+(0) = \lambda_0$, resp.) and $\lambda_-(\varepsilon)$ ($\lambda_+(\varepsilon)$, resp.) is an isolated simple eigenvalue of $A(\varepsilon) := A_0 + \varepsilon A_1$ ($\varepsilon \in J$) which is the maximum of $\sigma_+(A(\varepsilon))$ (minimum of*

$\sigma_-(A(\varepsilon))$, resp.); $\lambda_-(\varepsilon)$ is convex ($\lambda_+(\varepsilon)$ is concave, resp.). It is strictly convex (strictly concave, resp.) unless $A_0x_0 = \lambda_0x_0$ implies $A_1x_0 = \lambda_1x_0$, hence $\lambda_-(\varepsilon)$ ($\lambda_+(\varepsilon)$, resp.) is linear.

Proof. First note that $A(\varepsilon)$ also satisfies (1) (with a different C) if $|\varepsilon|$ is small enough and that the analytic eigenvalue $\lambda(\varepsilon)$ such that $\lambda(0) = \lambda_0$ is a simple eigenvalue of negative type which is the maximum of $\sigma_-(A(\varepsilon))$. Hence $\lambda(\varepsilon) = \lambda_-(\varepsilon)$. Let $x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ be the analytic eigenvector function corresponding to $\lambda(\varepsilon)$ such that $[x_0|x_0] = -1$. Let $\lambda(\varepsilon) = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots$. The eigenvalue equation

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon)$$

yields the well-known equalities

$$\begin{aligned} A_0x_0 &= \lambda_0x_0, \\ A_1x_0 + A_0x_1 &= \lambda_1x_0 + \lambda_0x_1, \\ A_1x_1 + A_0x_2 &= \lambda_2x_0 + \lambda_1x_1 + \lambda_0x_2, \end{aligned}$$

implying

$$\begin{aligned} \lambda_1 &= -[A_1x_0|x_0], \\ (A_0 - \lambda_0)x_1 &= (\lambda_1 - A_1)x_0, \\ \lambda_2 &= -[(A_1 - \lambda_1)x_1|x_0]. \end{aligned}$$

Using the spectral function E of the definitizable operator A_0 (see [1]) it follows from the last two of these equalities that

$$\lambda_2 = \int_{-\infty}^{\infty} \frac{1}{\lambda_0 - t} d[E(t)y_0|y_0]$$

with $y_0 = (A_1 - \lambda_1)x_0$. Since $[E(\cdot)y_0|y_0]$ is monotone increasing on $\sigma_+(A_0)$, monotone decreasing on $\sigma_-(A_0)$, it follows that the integral is nonpositive; it is equal to zero if and only if $y_0 = 0$. This proves all the statements about $\lambda_+(\varepsilon)$; the other case is treated analogously.

The boundedness assumption on A_1 can be relaxed: we used only the analyticity of $\lambda(\varepsilon)$ and the fact that $A(\varepsilon)$ also satisfies (1).

Example. The Klein–Gordon operator with a small potential. Let $V \in L^\infty(\mathbf{R}^n)$ and $\lim_{|x| \rightarrow \infty} V(x) = 0$ and let $X = L^2(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. Endowed with the inner product

$$[x|y] = (x_1|y_2)_{L^2} + (x_2|y_1)_{L^2} \quad (x = (x_1, x_2), \quad y = (y_1, y_2)),$$

X is a Krein space. Let $m^2 > 0$. Define $H = (-\Delta + m^2)^{1/2}$ as the positive square root of the positive definite operator $-\Delta + m^2$ in $L^2(\mathbf{R}^n)$ (defined on $H^2(\mathbf{R}^n)$). Then $\mathcal{D}(H) = H^1(\mathbf{R}^n)$. Define the operator A_0 on $\mathcal{D}(A_0) = H^1(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$:

$$A_0(x_1, x_2) = (Vx_1 + Hx_2, Hx_1 + Vx_2) \quad (x = (x_1, x_2) \in \mathcal{D}(A_0)).$$

This operator is easily seen to be self-adjoint in X . Let A_1 be the bounded operator:

$$A_1(x_1, x_2) = (Vx_1, x_2) \quad ((x_1, x_2) \in X).$$

It is not hard to verify that A_0 satisfies the assumptions of Proposition 1 if $\|V\|_{L^\infty}$ is sufficiently small ($\|V\|_{L^\infty} < m$ is sufficient). The operator A_0 is connected with the Klein–Gordon equation and the form $[x|x]$ with the charge of the particle represented by the equation (cf. [6] for the equation and [5] for the eigenvalues of the Klein–Gordon operators). The family $A(\varepsilon) = A_0 + \varepsilon A_1$ describes the effect of changing potential. The eigenvalue $\lambda_+(\varepsilon)$ at the bottom of the positive spectrum is the analogue of the ground state energy of the Schrödinger operator. By Proposition 1 it follows that $\lambda_+(\varepsilon)$ is concave (the ground state energy is concave by the result in (a) in the introduction).

2. DISTINGUISHED EIGENVALUES OF A GENERALIZED EIGENVALUE PROBLEM

Let $X = \mathbb{C}^n$ and let A_0 be an irreducible essentially nonnegative matrix, A_1 and B real diagonal matrices. We consider the eigenvalue problem

$$(2) \quad (A_0 + \varepsilon A_1)x = \lambda Bx.$$

Let $\nu(\lambda)$ be the principal eigenvalue of the irreducible essentially nonnegative matrix $A_0 - \lambda B$ (see (b) in the introduction); it is a convex function. If additionally

$$\nu(0) < 0$$

and B is indefinite, then there are exactly two real values λ such that $\nu(\lambda) = 0$; denote them λ_- , λ_+ so that $\lambda_- < 0 < \lambda_+$. We call these two eigenvalues the *left* and *right distinguished eigenvalue* of the generalized eigenvalue problem $Ax = \lambda Bx$; this notion was used by Kato in [4]. These two eigenvalues have the following properties: λ_- (λ_+ , resp.) is geometrically simple, it is the largest negative (smallest positive, resp.) eigenvalue, its eigenvector is positive (up to a constant); moreover,

$$\nu'(\lambda_-) < 0 \quad (\nu'(\lambda_+) > 0, \text{ resp.}).$$

We are interested in the perturbation of the distinguished eigenvalues. Let $\mu(\lambda, \varepsilon)$ be the principal eigenvalue of $A_0 + \varepsilon A_1 - \lambda B$. Then $\nu(\lambda) = \mu(\lambda, 0)$ and for sufficiently small (real) ε the two distinguished eigenvalues $\lambda_\pm(\varepsilon)$ of the generalized eigenvalue problem (2) exist and are analytic in ε .

Proposition 2. *Let A_0 be an irreducible essentially nonnegative matrix with a negative principal eigenvalue. Let A_1 and B be real diagonal matrices and let $\lambda_\pm(\varepsilon)$ be the distinguished eigenvalues of the perturbed eigenvalue problem (2). Then $\lambda_-(\varepsilon)$ is convex and $\lambda_+(\varepsilon)$ is concave. Moreover $\lambda_-(\varepsilon)$ is strictly convex and $\lambda_+(\varepsilon)$ strictly concave unless A_1 and B are scalar matrices in which case both $\lambda_-(\varepsilon)$ and $\lambda_+(\varepsilon)$ are linear.*

Proof. We shall prove the proposition only for $\lambda_-(\varepsilon) =: \lambda(\varepsilon)$. Denote $\lambda(0) =: \lambda_0$. Since μ is analytic in λ , ε and $\lambda(\varepsilon)$ is defined implicitly by $\mu(\lambda, \varepsilon) = 0$

and since $(\partial\mu/\partial\lambda)(\lambda_0, 0) = \nu'(\lambda_0) < 0$ it is evident that $\lambda(\varepsilon)$ is analytic. Next we prove that $\mu(\cdot, \cdot)$ is a convex function; this follows from the fact that the function $\lambda \mapsto \mu(\lambda, 0)$ is convex as well as all the functions $\varepsilon \mapsto \mu(\lambda_0 + \alpha\varepsilon, \varepsilon)$ for $\alpha \in \mathbf{R}$. This follows from the result quoted in (b) in the introduction; all these functions are strictly convex unless there exist real constants α , β , and γ , such that $|\alpha| + |\beta| + |\gamma| > 0$ and $\alpha A_1 + \beta B = \gamma I$ and moreover, one of the above functions is linear. A simple calculation shows that in this case $\nu(\lambda)$ is also linear; hence A_1 and consequently B are scalar matrices and $\mu(\lambda, \varepsilon)$ is linear, also.

Next note the formula for $\lambda''(\varepsilon)$:

$$\lambda''(\varepsilon) = -\frac{1}{[(\partial\mu/\partial\lambda)(\lambda(\varepsilon), \varepsilon)]^3} (D^2\mu(\lambda(\varepsilon), \varepsilon) D\mu(\lambda(\varepsilon), \varepsilon) | D\mu(\lambda(\varepsilon), \varepsilon)).$$

We know that $(\partial\mu/\partial\lambda)(\lambda(\varepsilon), \varepsilon)$ is negative and $D^2\mu(\lambda(\varepsilon), \varepsilon)$ is positive definite near $(\lambda_0, 0)$ (unless $\mu(\lambda, \varepsilon)$ is linear). This proves that either $\lambda''(\varepsilon) = 0$ or $\lambda''(\varepsilon) > 0$ for all ε with $|\varepsilon|$ sufficiently small.

Remarks. 1. The conclusions of Proposition 2 also hold in the situation described in [4, Remark 6.4] where $X = L^p(\Omega)$ or $C_0(\Omega)$, A_0 is a second order elliptic partial differential operator in a domain Ω , defined using the Dirichlet boundary conditions, A_1 and B are multiplication operators.

2. The eigenvalues $\lambda_{\pm}(\varepsilon)$ have convexity properties analogous to those described in §1 and in case (a) in the introduction.

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