THE ENVELOPE OF HOLOMORPHY OF A TRUNCATED TUBE

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ABSTRACT. A counterexample is constructed to show an essential difference between local and global CR extensions in tube domains.

In $I^4 = \{(z, w) \in \mathbb{C}^2 \mid |\operatorname{Re} z| < 1, |\operatorname{Im} z| < 1, |\operatorname{Re} w| < 1, |\operatorname{Im} w| < 1\}$, let us consider the following type of truncated tube domain $\Omega_h \subset \mathbb{C}^2$:

$$\Omega_h = \{(z, w) \in I^4 \mid \operatorname{Re} w > h(\operatorname{Re} z)\},$$

with boundary

$$\Sigma_h = b\Omega_h = \{(z, w) \in I^4 \mid \operatorname{Re} w = h(\operatorname{Re} z)\},$$

where h satisfies $h \in C^\infty(I)$, $I = (-1, 1)$, $h(0) = 0$, $h'(0) = 0$, and h attains both positive and negative values in any neighborhood of 0. It is clear from Trepreau's theorem [T] that any CR function on Σ_h extends holomorphically to at least one side, but Trepreau's theorem does not indicate to which side. In the situation described above, due to the changes of sign of h , one might have expected that every CR function extends to a full neighborhood of 0 as would happen in the case of an unlimited tube. Baouendi and Rothschild suggested that one look for an example where the extendability to both sides fails. We present such an example in this note.

Proposition. *There exists a function h as described above such that the envelope of holomorphy of Ω_h does not contain a neighborhood of 0.*

In fact, we obtain a pseudoconvex domain (putting $z = x + iy$)

$$\tilde{\Omega} = \{(z, w) \in I^4 \mid \operatorname{Re} w > \phi(x, y)\}$$

with $\phi \in C^\infty$, $\phi(0) = 0$, which contains some Ω_h as above. Hence, there are holomorphic functions in Ω_h which do not extend holomorphically to a neighborhood of 0. It implies the failure of the extendability of CR functions on Σ_h to both sides. It is routine to check that $\tilde{\Omega}$ is pseudoconvex if and only

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if ϕ is subharmonic on $I^2 =: \{z = x + iy \mid |x| < 1, |y| < 1\}$. The proposition, therefore, follows from:

Lemma. *There is a C^∞ subharmonic function ϕ defined in $\overline{I^2}$ satisfying*

- (i) $\phi(0) = 0$.
- (ii) *There are two sequences of positive numbers $\{a_n\}, \{b_n\}$ so that $a_n \rightarrow 0, b_n \rightarrow 0$ as $n \rightarrow \infty$, and for all $|y| < 1, n \in \mathbb{N}$,*

$$\phi(\pm a_n + iy) < 0, \quad \phi(\pm b_n + iy) > 0.$$

- (iii) *There exists an $h \in C^\infty(I)$ so that $h(0) = h'(0) = 0$,*

$$h(x) \geq \max_{y \in I} \phi(x + iy), \quad \forall x \in I \text{ and } h(\pm a_n) < 0.$$

Proof. The basic building blocks for constructing ϕ are the following subharmonic functions $\psi_n \in C^\infty(I \times [-2^{n+1}, 2^{n+1}])$ satisfying

- (1) $-1 \leq \psi_n(x, y) \leq -\frac{1}{2}$, for $|x| < \frac{1}{8}$,
- (2) $\frac{1}{2} \leq \psi_n(x, y) \leq 1$, for $|x| = \frac{1}{4}$,
- (3) $\psi_n(x, y) = 0$, for $|x| > \frac{3}{8}$.

The existence of such ψ_n is a simple consequence of the Runge approximation theorem. As we know, there is an entire function g such that

$$\begin{aligned} |g(z) + \frac{3}{4}| &< \frac{1}{8} \quad \text{for } |\operatorname{Re} z| < \frac{1}{8}, |\operatorname{Im} z| \leq 2^{n+2}, \\ |g(z) - \frac{3}{4}| &< \frac{1}{8} \quad \text{for } |\operatorname{Re} z| = \frac{1}{4}, |\operatorname{Im} z| \leq 2^{n+2}, \\ |g(z) + \frac{3}{4}| &< \frac{1}{8} \quad \text{for } \frac{3}{8} \leq |\operatorname{Re} z| \leq \frac{3}{2}, |\operatorname{Im} z| \leq 2^{n+2}. \end{aligned}$$

Then, we put

$$\psi_{n,0}(x, y) =: \begin{cases} \operatorname{Re} g(z), & \text{if } |x| < \frac{1}{4}, |y| < 2^{n+2}, \\ \max\{0, \operatorname{Re} g(z)\}, & \text{if } \frac{1}{4} \leq |x| < \frac{3}{2}, |y| < 2^{n+2}. \end{cases}$$

Finally, ψ_n is obtained by smoothing $\psi_{n,0}$ (by convolution). Now, we set

$$\phi(x, y) =: \begin{cases} \varepsilon_n \psi_n(2^{n+1}(|x| - \frac{3}{2^{n+1}}), 2^{n+1}y), & \frac{1}{2^n} \leq |x| < \frac{1}{2^{n-1}}, n = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon_n > 0$. At the same time, we define

$$h(x) =: \begin{cases} \varepsilon_n h_n(2^{1+n}(|x| - \frac{3}{2^{n+1}})), & \frac{1}{2^n} \leq |x| < \frac{1}{2^{n-1}}, n = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

where $h_n \in C_0^\infty(I)$, $\max_{y \in [-2^{n+1}, 2^{n+1}]} \psi_n(x, y) \leq h_n(x) < 2\|\psi_n\|_{L^\infty(I \times [-2^{n+1}, 2^{n+1}])}$, $\forall x \in I, h_n(0) < 0$. Then, one can see that ϕ satisfies all the required properties in the lemma provided that ε_n is chosen to be very small, say

$$\varepsilon_n = e^{-2^n} / \max\{1, \|\psi_n\|_{C^n}, \|h_n\|_{C^n}\},$$

$$\text{where } \|f\|_{C^n} = \max_{\substack{p+q \leq n, p, q \geq 0 \\ |x| \leq 1, |y| \leq 2^{n+1}}} \left| \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(x, y) \right|,$$

for any smooth function f defined on $I \times [-2^{n+1}, 2^{n+1}]$. $\square$

Remark. The oscillation of h is an important factor in this example. One can check that if $h(x) > 0$ for $x > 0$ and $h(x) < 0$ for $x < 0$ then every CR function on $\sum_h$ in a neighborhood of 0 extends holomorphically to both sides, regardless how flat h may be (see Theorem 1.1 in [BT]). Notice also that if h is odd and nonlinear, it follows from Trepreau's theorem and by symmetry that every CR function on $\sum_h$ can be holomorphically extended to both sides.

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