ULTRASEPARATING FUNCTION SPACES AND OPERATING FUNCTIONS FOR THE REAL PART OF A FUNCTION ALGEBRA

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ABSTRACT. We show how a local version of a result due to A. Bernard on ultraseparating Banach function spaces can be used to give a short proof of the theorem stating that only affine functions operate on the real part of a function algebra.

The paper [5] by K. de Leeuw and Y. Katznelson contains the following generalization of the Stone-Weierstrass Theorem:

Let X be a compact Hausdorff space and B a subspace of $C_{\mathbf{R}}(X)$, which separates the points of X and contains the constant functions. If there is a continuous non-affine function h defined on a subinterval I of $\mathbf{R}$ which operates on B , i.e., $h \circ b \in B$ if $b \in B$ and $h \circ b$ is defined, then B is dense in $C_{\mathbf{R}}(X)$.

At the same time J. Wermer [8] proved the following theorem:

Let A be a function algebra on a compact Hausdorff space X . Then if $\operatorname{Re} A$ is an algebra, $A = C(X)$.

Here, $\operatorname{Re} A$ denotes the space of the real parts of functions in A . Another formulation of Wermer's result is that if the function $t \mapsto t^2$ operates on $\operatorname{Re} A$, then $A = C(X)$. These two results lead to the following conjecture:

Let A be a function algebra on the compact Hausdorff space X . If there is a non-affine function defined on a subinterval of $\mathbf{R}$ which operates on $\operatorname{Re} A$, then $A = C(X)$.

(Here, it is not necessary to assume that the operating function is continuous because a function which operates on the real part of a function algebra is automatically continuous [6].)

If one tries to apply directly the theorem of de Leeuw and Katznelson one only finds that $\operatorname{Re} A$ is a dense subspace of $C_{\mathbf{R}}(X)$, i.e., that A is a Dirichlet algebra. Thus, more is needed to prove the conjecture.

In [1] A. Bernard introduced a powerful method to attack the conjecture and related problems: Let $\beta(\mathbf{N} \times X)$ denote the Stone-Čech compactification of $\mathbf{N} \times X$, and let $l^\infty(\mathbf{N}, C(X))$ denote the space of all bounded sequences

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of continuous functions on X . This space can be identified in a natural way with $C(\beta(\mathbf{N} \times X))$, the space of all continuous complex-valued functions on $\beta(\mathbf{N} \times X)$. Now, if A is a function algebra on X , one can introduce a norm on $\operatorname{Re} A$ in the following manner:

$$N(u) = \inf\{\|a\| : a \in A \text{ and } \operatorname{Re} a = u\} \quad \text{for all } u \in \operatorname{Re} A,$$

where $\|\cdot\|$ denotes the supremum norm. This norm dominates the supremum norm and turns $\operatorname{Re} A$ into a Banach space. The space $l^\infty(\mathbf{N}, \operatorname{Re} A)$ of all N -bounded sequences of elements from $\operatorname{Re} A$ satisfies the inclusion $l^\infty(\mathbf{N}, \operatorname{Re} A) \subseteq l^\infty(\mathbf{N}, C_{\mathbf{R}}(X)) = C_{\mathbf{R}}(\beta(\mathbf{N} \times X))$. It turns out that if $\operatorname{Re} A$ is a Dirichlet algebra then $l^\infty(\mathbf{N}, \operatorname{Re} A)$ separates the points of $\beta(\mathbf{N} \times X)$, (cf. [1]). From the theorem of de Leeuw and Katznelson it thus follows that if there is a non-affine function which operates on $l^\infty(\mathbf{N}, \operatorname{Re} A)$ then $l^\infty(\mathbf{N}, \operatorname{Re} A)$ is dense in $C_{\mathbf{R}}(\beta(\mathbf{N} \times X))$. This has some interesting consequences, as the next theorem, which is due to Bernard [1], shows.

Theorem. *Let E be a subspace of $C(X)$ which is a Banach space in a norm that dominates the supremum norm. Then, if $l^\infty(\mathbf{N}, E)$ is dense in $C(\beta(\mathbf{N} \times X))$, it follows that $E = C(X)$.*

This theorem holds for the real as well as the complex case. Thus one concludes that if $l^\infty(\mathbf{N}, \operatorname{Re} A)$ is dense in $C_{\mathbf{R}}(\beta(\mathbf{N} \times X))$, then $\operatorname{Re} A = C_{\mathbf{R}}(X)$, and hence $A = C(X)$ by a well known theorem of Bishop, Hoffman and Wermer.

In general it is not clear that a function which operates on $\operatorname{Re} A$ also operates on $l^\infty(\mathbf{N}, \operatorname{Re} A)$. This led Bernard to introduce the following concept:

A real-valued function h , defined on an open subinterval of $\mathbf{R}$ containing 0, operates boundedly on $\operatorname{Re} A$ if there exist numbers $\alpha > 0$ and $M > 0$ such that $N(h \circ u) < M$ whenever $u \in \operatorname{Re} A$ and $N(u) < \alpha$.

As an example, take the function $t \mapsto t^2$. If this function operates on $\operatorname{Re} A$ then it operates boundedly on $\operatorname{Re} A$ (cf. [1]). If h operates boundedly on $\operatorname{Re} A$ then $h \circ \{u_n\} \in l^\infty(\mathbf{N}, \operatorname{Re} A)$ whenever $\{u_n\} \in l^\infty(\mathbf{N}, \operatorname{Re} A)$ and $N(u_n) < \alpha$ for all n . The proof of the theorem of de Leeuw and Katznelson shows that if h is also non-affine, then it follows that $l^\infty(\mathbf{N}, \operatorname{Re} A)$ is dense in $C_{\mathbf{R}}(\beta(\mathbf{N} \times X))$.

Using the Baire Category Theorem in a clever way S. Sidney [6] succeeded in proving the conjecture without the condition that the operating function should operate boundedly. However, he had to add another condition; the operating function must not be affine on any subinterval of the one on which it is defined. The finishing touch is due to O. Hatori [3], who proved the conjecture for the remaining case.

Recently O. Hatori [4] has given a local version of the theorem of Bernard stated above. In this note we show how a variant of Hatori's local version can be used to give a simple proof of the above conjecture.

RESULTS AND APPLICATIONS

Let X be a compact Hausdorff space. A subspace B of $C(X)$ which separates the points of X , contains the constant functions and is complete in a norm which dominates the supremum norm is called a *Banach function space* on X . Bernard calls such a space *ultraseparating* on X if $l^\infty(\mathbf{N}, B)$ separates the points of the Stone-Čech compactification $\beta(\mathbf{N} \times X)$ of $\mathbf{N} \times X$. As mentioned earlier, if A is a Dirichlet algebra on X , then $\text{Re } A$ is ultraseparating (see [1]).

If $f \in C(X)$ we let $\{f\}$ denote the element $\{f_n\} \in l^\infty(\mathbf{N}, C(X))$ where $f_n = f$ for all n . For $x \in X$ and $f \in C(X)$ we let $\mathcal{F}_x = \mathcal{F}_x(f)$ denote the set of constancy for $\{f\}$ given by

$$\mathcal{F}_x = \{\xi \in \beta(\mathbf{N} \times X) : \{f\}(\xi) = f(x)\}.$$

The following theorem is a variant of Hatori's local version of Bernard's Theorem:

Theorem 1. *Let B be a Banach function space on a compact Hausdorff space X , and let $x \in X$. If $l^\infty(\mathbf{N}, B)|_{\mathcal{F}_x}$ is dense in $C(\mathcal{F}_x)$ then there exists a compact neighbourhood K_x of x such that $B|_{K_x} = C(K_x)$.*

Proof. Let $g \in C(X)$ be a function such that $\mathcal{F}_x = \mathcal{F}_x(g)$ and let $K_n = \{x' \in X : |g(x') - g(x)| \leq 1/n\}$ for each $n \in \mathbf{N}$. We assert that there exists a natural number n_0 having the following property:

For all $f \in C(X)$ for which $\|f\|_\infty \leq 1$ there is a function $u \in B$ such that $N(u) \leq n_0$ and $|f - u| < 1/2$ on K_{n_0} .

If not, then there is, for each $n \in \mathbf{N}$, a function $f_n \in C(X)$ such that $\|f_n\|_\infty \leq 1$ and such that $N(u) > n$ if $u \in B$ and $|f_n - u| < 1/2$ on K_n . Since $\{f_n\} \in l^\infty(\mathbf{N}, C(X))$ there is a sequence $\{u_n\} \in l^\infty(\mathbf{N}, B)$ such that $|\{f_n\} - \{u_n\}| < 1/2$ on $\mathcal{F}_x$ and hence on some neighbourhood $\mathcal{U}$ of $\mathcal{F}_x$. Since $\mathcal{U}$ is open the definition of $\mathcal{F}_x$ shows that there is a number $n_0 \in \mathbf{N}$ such that

$$\{\xi \in \beta(\mathbf{N} \times X) : |\{g\}(\xi) - g(x)| \leq 1/n_0\} \subseteq \mathcal{U}.$$

Now, $\mathbf{N} \times K_{n_0}$ is contained in the set on the left hand side and hence in $\mathcal{U}$. We conclude that $|\{f_n\} - \{u_n\}| < 1/2$ on $\mathbf{N} \times K_{n_0}$, which means that $|f_n - u_n| < 1/2$ on K_{n_0} for all n . Since $\{u_n\} \in l^\infty(\mathbf{N}, B)$ there is a number $n_1 > n_0$ such that $N(u_n) < n_1$ for all n . Since $K_{n_1} \subseteq K_{n_0}$ we have reached a contradiction, and the assertion has been proved.

Let $f \in C(X)_1$, the unit ball of $C(X)$. By induction we can construct elements $v_0, v_1, \dots, v_n, \dots$ in B such that $N(v_n) < n_0$ for all n and such that

$$\left| f - \left(v_0 + \frac{1}{2}v_1 + \dots + \frac{1}{2^n}v_n \right) \right| < \frac{1}{2^{n+1}} \quad \text{on } K_{n_0}.$$

Let $v = \sum v_n/2^n$. Then $v \in B$ and $v = f$ on K_{n_0} . $\square$

Using this theorem, the Baire Category Theorem in the same way as Sidney in [6] and a few results from the theory of function algebras, we obtain a simple proof of the conjecture.

Theorem 2. ([1, 3, 6]). *Let A be a function algebra on a compact Hausdorff space X and suppose there is a non-affine function defined on a subinterval of $\mathbf{R}$ which operates on $\operatorname{Re} A$. Then $A = C(X)$.*

Proof. We have already noted that $\operatorname{Re} A$ is ultraseparating on X . Suppose $A \neq C(X)$, let $\mu \in \operatorname{ext}((A^\perp)_1)$, and let E be the smallest p -set for A which contains the support of μ . Then $u(E)$ is an interval for any $u \in \operatorname{Re} A$. To see this let $u \in \operatorname{Re} A$ and pick $a \in A$ such that $u = \operatorname{Re} a$. If $u(E)$ is not an interval, then we can find a real number t such that $a(E) = C_1 \cup C_2$, where C_1 and C_2 are nonempty disjoint compact subsets of the complex plane such that $\operatorname{Re} z < t$ for all $z \in C_1$ and $\operatorname{Re} z > t$ for all $z \in C_2$. By Runge's Theorem there is a sequence of polynomials converging uniformly to 0 on C_1 and to 1 on C_2 . It follows that $E = a^{-1}(C_1) \cup a^{-1}(C_2)$, a union of two disjoint p -sets. Since μ is an extreme point of $(A^\perp)_1$, the support of μ is contained in $a^{-1}(C_1)$ or in $a^{-1}(C_2)$, contradicting the fact that E is the smallest p -set for A with this property.

We may assume that the operating function h is defined on an interval containing 0 as an interior point and is not affine on any open interval which contains 0. We choose a positive number δ such that $h \circ u$ is defined if $N(u) \leq \delta$, put $H = \{u \in \operatorname{Re} A : 0 \in u(E) \text{ and } N(u) \leq \delta\}$ and write $H = \bigcup H_n$, where $H_n = \{u \in H : N(h \circ u) \leq n\}$. By the Baire Category Theorem there is a number $n_0 \in \mathbf{N}$ such that $\overline{H}_{n_0}$ contains a set which is open in H . (The bar denotes the closure w.r.t. the norm N .) Thus there is a function $u_0 \in H$ and a positive number ε such that $\{u \in H : N(u - u_0) < \varepsilon\} \subseteq \overline{H}_{n_0}$. Adding to or subtracting from u_0 a small multiple of the constant function 1, one obtains a function u'_0 such that 0 is an interior point of $u'_0(E)$. This assumes that u_0 is not constant on E . In that case we replace u_0 by another function u in H for which $N(u - u_0) < \varepsilon$ and ε by a smaller number. Unless E is a singleton, in which case we already have a contradiction, we can thus assume that 0 is an interior point of $u_0(E)$. Since $u(E)$ is an interval for all $u \in \operatorname{Re} A$ we conclude that $u \in H$ if u is sufficiently close to u_0 , and hence we can assume that

$$\{u \in \operatorname{Re} A : N(u - u_0) < \varepsilon\} \subseteq \overline{H}_{n_0}.$$

It follows that $N(h \circ (u_0 + u)) < n_0$ if u belongs to a dense subset of $\operatorname{Re} A_\varepsilon := \{u \in \operatorname{Re} A : N(u) < \varepsilon\}$ and thus $h \circ \{u_0 + u_n\} \in l^\infty(\mathbf{N}, \operatorname{Re} A)$ if the functions u_n belong to a dense subset of $\operatorname{Re} A_\varepsilon$. We conclude that $h \circ \{u_0 + u_n\} \in \overline{l^\infty(\mathbf{N}, \operatorname{Re} A)}$ if the functions u_n belong to $\operatorname{Re} A_\varepsilon$. (The bar denotes closure w.r.t. the sup-norm on $\beta(\mathbf{N} \times X)$.)

We now pick a point $x \in E$ such that $u_0(x) = 0$. Let $\mathcal{F}_x = \{\xi \in \beta(\mathbf{N} \times X) : \{u_0\}(\xi) = u_0(x)\}$ and look at the space $l^\infty(\mathbf{N}, \operatorname{Re} A)|_{\mathcal{F}_x}$. Since $\{u_0 + u_n\}|_{\mathcal{F}_x} = \{u_n\}|_{\mathcal{F}_x}$, the proof of the theorem of de Leeuw and Katznel-

son shows that $\{u_n\}^2|_{\mathcal{F}_x} \in \overline{l^\infty(\mathbf{N}, \operatorname{Re} A)|_{\mathcal{F}_x}}$ (where the closure is taken w.r.t. the supremum norm on $\mathcal{F}_x$) if the functions u_n belong to $\operatorname{Re} A_\varepsilon$, and hence $\overline{l^\infty(\mathbf{N}, \operatorname{Re} A)|_{\mathcal{F}_x}} = C(\mathcal{F}_x)$ by the Stone-Weierstrass Theorem.

By Theorem 1 there is a compact neighbourhood K_x of x such that $\operatorname{Re} A|_{K_x} = C(K_x)$. Using a theorem of S. Sidney and E. Stout [7] we infer that $A|_{K_x} = C(K_x)$. Since A is a Dirichlet algebra, a theorem of I. Glicksberg [2] shows that $\nu|_{K_x} = 0$ for all $\nu \in A^\perp$, and thus $\nu|(X \setminus U_x) \in A^\perp$ for all $\nu \in A^\perp$, where U_x is the interior of K_x . Referring again to a theorem of I. Glicksberg [2], the set $X \setminus U_x$ is a p -set for A supporting μ , hence $E \setminus U_x$ is a p -set which supports μ , contradicting the fact that E is the smallest p -set with this property. This concludes the proof.

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