HENSTOCK INTEGRABLE FUNCTIONS ARE LEBESGUE INTEGRABLE ON A PORTION

ZOLTÁN BUCZOLICH

(Communicated by R. Daniel Mauldin)

ABSTRACT. If a real function f defined on an interval $I \subset \mathbb{R}^m$ is Henstock integrable, then one can always find a nondegenerate subinterval $J \subset I$ on which f is Lebesgue integrable.

INTRODUCTION

In his memoir [O; 4.6.4, p. 87] Krzysztof M. Ostaszewski mentioned that “the question posed by Karták in [K]—whether, for a Perron-integrable function, one can find a nondegenerate interval on which it is Lebesgue-integrable—remains unanswered.” In this paper we give an affirmative answer to this question. The one-dimensional case of our theorem is an old and well-known result [S, Theorem 1.4, Chapter VIII].

Recall that the (classical) Henstock integral is equivalent to the Perron integral [O, Chapter 2, p. 20]. Since our proof is based on the Henstock integration process we shall use the name Henstock integral in the statement of our theorem.

PRELIMINARIES

By $\mathbb{R}^m$ we denote the m -dimensional Euclidean space. By an interval I we mean a set of the form $\times_{i=1}^m [a_i, b_i]$. The m -dimensional Lebesgue measure of the set $E \subset \mathbb{R}^m$ is denoted by $|E|$. The Lebesgue integral of a function f over a set E is denoted by $(L) \int_E f$. We say that the intervals I and J are non-overlapping if $\text{int}(I) \cap \text{int}(J) = \emptyset$. If the intervals $B_i \subset I$, $i = 1, \dots, k$, are non-overlapping and $I = \cup \{B_i : i = 1, \dots, k\}$, then we say that the set $D = \{B_i : i = 1, \dots, k\}$ is a division of I . A partition of an interval I is a collection $P = \{(B_i, x_i) : i = 1, \dots, k\}$, where $D(P) = \{B_i : i = 1, \dots, k\}$ is a division of I and $x_i \in B_i$, $i = 1, \dots, k$. If a function $\delta : I \rightarrow (0, +\infty)$ and a partition P of I is given we say that P is δ -fine if $B_i \subset \{y : \text{dist}(x_i, y) < \delta(x_i)\}$ for every $i = 1, \dots, k$. If f is a function defined on I and P is a

Received by the editors September 18, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A39, 26A42.

This paper was prepared while the author visited the University of California at Davis.

partition of I , then we put

$$\sigma(f, P) = \sum_{i=1}^k f(x_i) |B_i|.$$

We also recall the definition of the Henstock integral.

Definition. A function $f: I \rightarrow \mathbf{R}$ is Henstock integrable on I if there is a real number denoted by $(H) \int_I f$ with the following property: for every $\varepsilon > 0$, there is a $\delta: I \rightarrow (0, +\infty)$ such that

$$\left| \sigma(f, P) - (H) \int_I f \right| < \varepsilon$$

for each δ -fine partition of I .

We remark that Henstock integrable functions are measurable. This holds even for generalizations of the Henstock integral, see for example [P1, 4.3 Corollary] or [P2, 4.5 Corollary].

THE MAIN RESULT

Theorem. If $I \subset \mathbf{R}^m$ is an interval and f is Henstock integrable on I , then there exists a nondegenerate interval $J \subset I$ such that f is Lebesgue integrable on J .

Proof. Using the existence of $(H) \int_I f$, choose $\delta: I \rightarrow (0, +\infty)$ such that if P and P' are δ -fine partitions of I then

$$|\sigma(f, P) - \sigma(f, P')| < |I|.$$

Put $A_n = \{x \in I: |f(x)| < n, \delta(x) > 1/n\}$. Since $\delta(x)$ is positive and $f(x)$ is finite we have $\bigcup_{n \in \mathbf{N}} A_n = I$, and, by Baire's Category Theorem, there exists an $n \in \mathbf{N}$ and a portion $J \subset I$ such that A_n is dense in J . We can also assume that $\text{diam}(J) < 1/n$. We shall prove that $(L) \int_J f$ exists. For a contradiction suppose that this is not true. Then $(L) \int_J |f| = +\infty$, and we can choose a measurable set $E \subset J$ such that

$$\left| (L) \int_E f \right| > 3 \cdot n \cdot |I|.$$

Put

$$g(x) = \begin{cases} f(x), & \text{if } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Since $g(x)$ is Lebesgue and a fortiori Henstock integrable, we choose $\delta': J \rightarrow (0, +\infty)$ such that $\delta'(x) < \delta(x)$ for $x \in J$, and if P' is a δ' -fine partition of J then

$$\left| \sigma(g, P') - (L) \int_E f \right| < n|I|.$$

Suppose that P' is a fixed δ' -fine partition of J . Choose P^* a δ -fine partition of $I \setminus J$ and put $P_1 = P' \cup P^*$. Then P_1 is a δ -fine partition of I .

Suppose that $P' = \{(B_i, x_i) : i = 1, \dots, k'\}$. If $x_i \notin E$ then put $x_i'' = x_i$. If $x_i \in E$ then, using the density of A_n in J , choose an $x_i'' \in A_n \cap B_i$. Put $P'' = \{(B_i, x_i'') : i = 1, \dots, k'\}$. Since $\delta(x) > 1/n$ for $x \in A_n$, $\text{diam}(J) < 1/n$, and $\delta' < \delta$ the partition P'' is δ -fine on J . Put $P_2 = P'' \cup P^*$. Then P_2 is δ -fine on I . We obtain

$$\begin{aligned} |I| > |\sigma(f, P_1) - \sigma(f, P_2)| &= \left| \sum_{(B_i, x_i) \in P^*} f(x_i)|B_i| + \sum_{(B_i, x_i) \in P'} f(x_i)|B_i| \right. \\ &\quad \left. - \sum_{(B_i, x_i) \in P^*} f(x_i)|B_i| - \sum_{(B_i, x_i'') \in P''} f(x_i'')|B_i| \right| \\ &= \left| \sum_{(B_i, x_i) \in P'} f(x_i)|B_i| - \sum_{(B_i, x_i'') \in P''} f(x_i'')|B_i| \right| = \Psi_0. \end{aligned}$$

Using the fact that if $x_i \notin E$ then $x_i = x_i''$, we have

$$\begin{aligned} \Psi_0 &= \left| \sum_{x_i \in E, (B_i, x_i) \in P'} f(x_i)|B_i| - \sum_{x_i \in E, (B_i, x_i'') \in P''} f(x_i'')|B_i| \right| \\ &> |\sigma(g, P')| - \left| \sum_{x_i \in E, (B_i, x_i'') \in P''} f(x_i'')|B_i| \right| = \Psi_1 - \Psi_2. \end{aligned}$$

Recalling that $|(L) \int_J g| > 3 \cdot n \cdot |I|$ and $|\sigma(g, P') - (L) \int_J g| < n|I|$, we obtain $\Psi_1 > 2n|I|$. To estimate Ψ_2 recall that $x_i'' \in A_n$ and hence $|f(x_i'')| < n$. Thus $\Psi_2 < n|J| \leq n|I|$. Therefore we obtain $|I| > \Psi_1 - \Psi_2 > 2n|I| - n|I| = n|I| \geq |I|$, a contradiction proving our theorem. $\square$

REFERENCES

- [K] K. Karták, *K teorii vícerozměrného integrálu*, Časopis Pešt. Mat. **80** (1955), 400–414.
- [O] K. M. Ostaszewski, *Henstock integration in the plane*, Mem. Amer. Math. Soc., no. 63, Amer. Math. Soc., Providence, RI, 1986.
- [P1] W. F. Pfeffer, *The multidimensional fundamental theorem of calculus*, J. Austral. Math. Soc. Ser. A **43** (1987), 143–170.
- [P2] —, *The divergence theorem*, Trans. Amer. Math. Soc. **295** (1986), 665–685.
- [S] S. Saks, *Theory of the integral*, Hafner, New York, 1937.

EÖTVÖS LORÁND UNIVERSITY, DEPARTMENT OF ANALYSIS, BUDAPEST, MÚZEUM KRT 6-8, H-1088, HUNGARY