MATRIX MAPS AND THE ISOMORPHIC STRUCTURE OF BK SPACES

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ABSTRACT. This note gives a characterization of BK spaces that contain isomorphic copies of c_0 in terms of matrix maps and a sufficient condition for a matrix map from l_∞ into a BK space to be a compact operator. The primary tool used in this note is the Bessaga-Pelczynski characterization of Banach spaces which contain isomorphic copies of c_0 . It is shown that weakly compact matrix maps on l_∞ are compact and that, if E is a BK space such that there is a matrix A such that $c_0 \subseteq E_A$ and E_A is not strongly conull, then E must contain an isomorphic copy of c_0 .

1. INTRODUCTION

In this note, we give a characterization of Banach-Koordinat (BK) spaces that contain isomorphic copies of c_0 and a sufficient condition for a matrix map from the bounded sequences into a BK space to be a compact operator. As a corollary, we show that any matrix map from the bounded sequences into a BK space that does not contain an isomorphic copy of l_∞ must be compact. These results are obtained by studying the "column space" of a matrix map. As an application, we give an indication of how the isomorphic structure of a BK space affects its "summability" structure.

Before proceeding with the discussion, we pause to collect some definitions and notation. We set

$$\omega = \{\text{all scalar sequences}\}$$

$$\varphi = \{x \in \omega: x \text{ is finitely nonzero}\}$$

$$bv = \{x \in \omega: \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty\}$$

$$c_0 = \{x \in \omega: \lim_n x_n = 0\}$$

$$l_\infty = \{x \in \omega: \sup_n |x_n| < \infty\}$$

$$e^k = \text{the } k\text{th unit vector}$$

$$e = (1, 1, 1, \dots),$$

and, if E is a Banach space, we let E' denote its continuous dual and B_E denote E 's unit ball. We call the linear functional $p_n: \omega \rightarrow \mathbb{C}$, defined by $p_n(x) = x_n$ for $x = (x_n)_{n=1}^{\infty} \in \omega$, the n th coordinate functional.

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Definition. Let E be a Banach space and $(x^n)_{n=1}^\infty \subseteq E$. The series $\sum_{n=1}^\infty x^n$ is unconditionally convergent if any of the following equivalent conditions hold:

- (1) $\sum_{n=1}^\infty x^{\pi(n)}$ exists for every permutation π of $\mathbf{N}$.
- (2) $\sum_{k=1}^\infty x^{n_k}$ exists for every increasing sequence $n_1 < n_2 < n_3 < \dots$ of positive integers.
- (3) $\sum_{k=1}^\infty x^{n_k}$ exists weakly for every increasing sequence $n_1 < n_2 < \dots$ of positive integers.
- (4) $\sum_{n=1}^\infty \beta_n x^n$ for every $(\beta_n) \in l_\infty$.

Condition (1) is the usual definition of unconditional convergence. The equivalence of (2) and (3) is the Orlicz–Pettis Theorem. Condition (4) is known as the bounded multiplier test.

Definition. Let E be a Banach space and $(x^n)_{n=1}^\infty \subseteq E$. The series $\sum_{n=1}^\infty x^n$ is weakly unconditionally Cauchy if any of the following equivalent conditions holds:

- (1) $\sum_{n=1}^\infty |f(x^n)| < \infty$ for all $f \in E'$.
- (2) $\sum_{n=1}^\infty \alpha_n x^n$ exists for all $(\alpha_n) \in c_0$.

In general, a weakly unconditional Cauchy series is not unconditionally convergent. In fact,

Theorem [Bessaga-Pelczynski]. *Let E be a Banach space. E contains a weakly unconditionally Cauchy series which is not unconditionally convergent if and only if E contains an isomorphic copy of c_0 .*

(See [D] for a thorough discussion of the convergence of series in Banach spaces.)

We now center our attention on BK spaces. Recall that a Banach space E is a BK space if it is a vector subspace of ω and $p_n \in E'$ for all $n \in \mathbf{N}$. c_0 and l_∞ , with their usual norms, are BK spaces. Let $x, y \in \omega$ and $A = (a_{n,k})_{n,k=1}^\infty$ be an infinite array of scalars. We write $y = A(x)$ if $y_n = \sum_{k=1}^\infty a_{n,k} x_k$ for every $n \in \mathbf{N}$. If E and F are BK spaces and $A(x) \in E$ for every $x \in F$, then A defines a continuous linear map from F into E and we call A a matrix map from F into E . If $A(B_F)$ is relatively (weakly) compact in E , we say that A is a (weakly) compact map.

If $A = (a_{n,k})_{n,k=1}^\infty$ defines a matrix map from F into E , we let $a^k = A(e^k)$ denote the k th column of A . Observe that if $\varphi \subseteq F$, then the columns of A are elements of E ; hence, it makes sense to discuss the convergence of $\sum_{k=1}^\infty a^k$ in E .

For the remainder of the note, we let E and F denote BK spaces and A denote the array $(a_{n,k})_{n,k=1}^\infty$. Now we are ready to begin.

2. WEAKLY COMPACT MATRICES ON l_∞ ARE COMPACT

We start with the key lemma.

Lemma 1. *Suppose A defines a matrix map from l_∞ into E . If A is weakly compact, then $\sum_{k=1}^\infty a^k$ is unconditionally convergent in E .*

Proof. We show that $\sum_{k=1}^\infty a^k$ is weakly subseries convergent and hence, via the Orlicz–Pettis Theorem, unconditionally convergent.

Let $\theta = (\theta_k)$ be an arbitrary sequence of 0's and 1's, and consider $\sum_{k=1}^\infty \theta_k a^k$. First observe that

$$\begin{aligned} C &= \left\{ A \left(\sum_{k=1}^n \theta_k e^k \right) : n \in \mathbf{N} \right\} \\ &= \left\{ \sum_{k=1}^n \theta_k a^k : n \in \mathbf{N} \right\} \end{aligned}$$

is a relatively weakly compact set in E since $\{\sum_{k=1}^n \theta_k e^k : n \in \mathbf{N}\}$ is bounded in l_∞ . Since E is a BK space, we have that E 's weak topology and the topology of coordinatewise convergence inherited from ω are equivalent on the weak closure of C : this follows from the fact that comparable Hausdorff topologies on a compact set are equal. We claim that the coordinatewise limit point, hence the limit point, of C is $A(\theta)$. Observe that $x = \sum_{k=1}^\infty \theta_k a^k$ coordinatewise if and only if $x_n = \lim_\eta \sum_{k=1}^\eta \theta_k a_{n,k}$ for all $n \in \mathbf{N}$ since $p_n(a^k) = a_{n,k}$. Now, since $\theta \in l_\infty$, $A(\theta) = (\sum_{k=1}^\infty \theta_k a_{n,k})_{n=1}^\infty$ is an element of E and $A(\theta)_n = \lim_\eta \sum_{k=1}^\eta \theta_k a_{n,k} = \sum_{k=1}^\infty \theta_k a_{n,k}$, it follows that $A(\theta)$ is the coordinatewise, hence weak, limit point of C . Note that the use of the word “the” is proper in that C can only have one coordinatewise limit point. Since θ was arbitrary, we now have that $\sum_{k=1}^\infty a^k$ is weakly subseries convergent and hence unconditionally convergent. $\square$

Lemma 2. *If $\sum_{k=1}^\infty a^k$ is unconditionally convergent in E , then A defines a matrix map from l_∞ into E and $A(\beta) = \sum_{k=1}^\infty \beta_k a^k$ for every $\beta = (\beta_k)_{k=1}^\infty \in l_\infty$.*

Proof. Let $\beta = (\beta_k) \in l_\infty$. First observe that $\sum \beta_k a^k$ converges since $\sum a^k$ is unconditionally convergent, and note that

$$p_n \left(\sum_{k=1}^\infty \beta_k a^k \right) = \sum_{k=1}^\infty a_{n,k} \beta_k = p_n(A(\beta))$$

for every $n \in \mathbf{N}$. Now, since $\{p_n\}_{n=1}^\infty$ is total over E , it follows that $A(\beta) = \sum \beta_k a^k$.

Since β was arbitrary, we also have that $A(\beta) \in E$ for every $\beta \in l_\infty$, hence A defines a matrix map from l_∞ into E . $\square$

Theorem 3. *If $A: l_\infty \rightarrow E$ is a weakly compact matrix map, then A is compact.*

Proof. First note that it suffices to show that, for every $\varepsilon > 0$ there is a compact set K_ε in E such that $A(B_{l_\infty}) \subseteq K_\varepsilon + \varepsilon B_E$. Let $\beta = (\beta_k) \in l_\infty$ and recall that, for every $\delta > 0$, there is an N_δ such that

$$(*) \quad \left\| \sum_{k=n}^{\infty} \beta_k a^k \right\|_E < \delta \sup_k |\beta_k|$$

for all $n \geq N_\delta$, since $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent.

Now suppose that $A(B_{l_\infty}) \subseteq MB_E$, $M > 0$, and set $\delta = \varepsilon M^{-1}$. Select N such that $(*)$ is satisfied and let F_N be the linear span of $\{a^k\}_{k=1}^N$. Note that $K_\varepsilon = F_N \cap MB_E$ is compact since F_N is finite dimensional and that

$$\begin{aligned} A(B_{l_\infty}) &= \left\{ \sum_{k=1}^{\infty} \beta_k a^k : (\beta_k) \in B_{l_\infty} \right\} \\ &\subseteq K_\varepsilon + \delta M^{-1} B_E \\ &= K_\varepsilon + \varepsilon B_E \end{aligned}$$

which completes the proof since ε was arbitrary. $\square$

Rosenthal [R] has shown that any continuous linear operator from l_∞ into a Banach space that does not contain an isomorphic copy of l_∞ must be weakly compact. This, in tandem with Theorem 3, yields the following corollary.

Corollary 4. *Let E be a BK space that does not contain an isomorphic copy of l_∞ . If A is a matrix map from l_∞ into E , then A is compact.*

3. THE ISOMORPHIC STRUCTURE OF BK SPACES

We add one more piece of notation before proceeding. We set

$$E_A = \{x \in \omega : Ax \in E\}.$$

Observe that A is a matrix map from F into E if and only if $F \subseteq E_A$ and that Lemma 2 shows that if $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E , then $l_\infty \subseteq E_A$. In our next lemma, we characterize the matrix maps that take c_0 into E .

Lemma 5. *A is a matrix map from c_0 into E if and only if $\sum_{k=1}^{\infty} a^k$ is weakly unconditionally Cauchy in E .*

Proof. First, suppose that A is a matrix map from c_0 into E . Since A is continuous and $(e^k)_{k=1}^{\infty}$ is a Schauder basis for c_0 , we have that

$$\begin{aligned} A(\alpha) &= \lim_n A \left(\sum_{k=1}^n \alpha_k e^k \right) \\ &= \lim_n \sum_{k=1}^n \alpha_k A e^k = \sum_{k=1}^{\infty} \alpha_k a^k \end{aligned}$$

for any $\alpha = (\alpha_k) \in c_0$, hence $\sum_{k=1}^{\infty} a^k$ is weakly unconditionally Cauchy in E .

Conversely, suppose $\sum a^k$ is weakly unconditionally Cauchy in E . An argument similar to the one given in Lemma 2 shows that $A(\alpha) = \sum_{k=1}^{\infty} \alpha_k a^k$ for every $\alpha = (\alpha_k) \in c_0$, hence $c_0 \subseteq E_A$ and A defines a matrix map from c_0 into E . $\square$

Theorem 6. *E does not contain an isomorphic copy of c_0 if and only if whenever A is a matrix such that $c_0 \subseteq E_A$ then $l_{\infty} \subseteq E_A$ and A defines a compact map from l_{∞} into E .*

Proof. Suppose that E does not contain an isomorphic copy of c_0 and A is a matrix such that $c_0 \subseteq E_A$. Lemma 5 yields that $\sum_{k=1}^{\infty} a^k$ is weakly unconditionally Cauchy in E and therefore, thanks to the Bessaga–Pelczynski theorem, unconditionally convergent in E . Lemma 2 and the proof of Theorem 3 yield that $l_{\infty} \subseteq E_A$ and that A is a compact map from l_{∞} into E .

Conversely, if $\sum_{k=1}^{\infty} a_k$ is weakly unconditionally Cauchy in E , then the matrix A defined by $a_{n,k} = p_n(a^k)$ for all $n, k \in \mathbb{N}$ has the property that $c_0 \subseteq E_A$. Consequently, $l_{\infty} \subseteq E_A$ and $A: l_{\infty} \rightarrow E$ is a compact map, which forces $\sum_{k=1}^{\infty} a^k$ to be unconditionally convergent in E (Lemma 1). It follows from the Bessaga–Pelczynski Theorem that E cannot contain an isomorphic copy of c_0 . $\square$

In the event that E does not contain an isomorphic copy of l_{∞} (i.e., E is separable), Theorem 6 can be given slightly more succinctly as

Corollary 7. *Let E be a BK space that does not contain an isomorphic copy of l_{∞} . E does not contain an isomorphic copy of c_0 if and only if, whenever A is a matrix such that $c_0 \subseteq E_A$, then $l_{\infty} \subseteq E_A$.*

This follows immediately from Theorem 6 and Corollary 4.

4. SOME APPLICATIONS TO THE SUMMABILITY STRUCTURE OF FK SPACES

First recall that an FK space is a locally convex Frechet space which is a vector subspace of ω with continuous coordinate functionals, and that if E is a BK space, then E_A can be metrized in such a fashion that it becomes an FK space. An FK space X is said to be strongly conull if $e = \sum_{k=1}^{\infty} e^k$ in X 's metric topology. In this section, we restrict our attention to FK spaces of the form E_A where E is a BK space.

We also use the following result in the sequel:

Theorem (Bennett–Sembr) [B, S]. *Let E be a BK space and A be a matrix. E_A is strongly conull if and only if $A: bv \rightarrow E$ is compact.*

We can now use Theorem 3 to establish a weak version of a result of Bennett.

Proposition 8. *Let E be a BK space that does not contain an isomorphic copy of l_{∞} . If A is a matrix such that $l_{\infty} \subseteq E_A$, then E_A is strongly conull.*

Proof. Corollary 4 asserts that $A: l_\infty \rightarrow E$ is compact. Now, since the inclusion map between FK spaces is continuous and the diagram

$$\begin{array}{ccc} bv & \longrightarrow & l_\infty \\ & \searrow A & \downarrow A \\ & & E \end{array}$$

commutes, it follows from the Bennett–Sember result that E_A is strongly conull. $\square$

Bennett and Kalton [BK] have shown that any separable FK space that contains l_∞ setwise is strongly conull. We use Proposition 8 to establish a similar result.

Theorem 9. *Let E be a BK space that does not contain an isomorphic copy of c_0 . If A is a matrix such that $c_0 \subseteq E_A$, then E_A is strongly conull.*

Proof. Since E contains no isomorphic copy of c_0 , we have that $l_\infty \subseteq E_A$ and that E does not contain an isomorphic copy of l_∞ . Proposition 8 yields that E_A is strongly conull. $\square$

Theorem 9 yields some of the “bigness” theorems for conservative weakly sequentially complete spaces (see [W]) since a weakly sequentially complete space cannot contain an isomorphic copy of c_0 .

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