SIMPLY CONNECTED MANIFOLDS WITH NO CONJUGATE POINTS WHICH ARE FLAT OUTSIDE A COMPACT SET

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ABSTRACT. In this note we improve a previous result to show that euclidean n -space ($n \geq 3$) is the only simply connected manifold without conjugate points which is flat outside a compact set.

Our goal is to prove the following:

Theorem. *If M is a complete, simply connected Riemannian manifold of dimension $n \geq 3$ which has no conjugate points and is flat outside a compact set, then M is isometric to $\mathbb{R}^n$.*

This is an improvement of Theorem A of [C], where we prove the same theorem under the assumption that the complement of a compact set is isometric to the complement of a compact set in $\mathbb{R}^n$. The proof here consists of reducing the above theorem to Theorem A of [C].

The theorem is clearly false for $n = 2$. However, as had been shown earlier in [G-G], Theorem A of [C] does hold in this case as well.

Proof. Pick $x \in M$, and choose R so large that $M - B(x, R/2)$ is flat, where $B(x, R)$ represents the ball of radius R about x . Let $S(x, R)$ be the boundary of $B(x, R)$. Since M has no conjugate points and is simply connected, M is diffeomorphic to $\mathbb{R}^n$, $B(x, R)$ is diffeomorphic to an n -ball, and $S(x, R)$ is diffeomorphic to an $n - 1$ sphere.

We claim that the second fundamental form of $S(x, R)$ is definite. Let $y \in S(x, R)$ and γ be the geodesic with $\gamma(0) = x$, and $\gamma(R) = y$. Let e be an eigenvalue of the second fundamental form of $S(x, R)$ at y , and let $X \in T_y S(x, R)$ be a corresponding eigenvector. If $\tau(s) \in S(x, R)$ is a curve with $\tau'(0) = X$, then the variation of geodesics $H(s, t) = \text{Exp}_{\tau(s)}(t - R)N(\tau(s))$, where $N(\tau(s))$ is the outward normal to $S(x, R)$ at $\tau(s)$, has as variation field the Jacobi field $J(t)$ with $J(R) = X$ and $J'(R) = eX$. Further, since $H(s, 0) = x$, $J(0) = 0$. Since there are no conjugate points, J cannot vanish for $t \neq 0$ and, by [E-O], must get arbitrarily large as $t \rightarrow \infty$. For $t \geq R$, M

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is flat at $\gamma(t)$, and hence $J(t) = (1 + e \cdot t)\bar{X}$, where $\bar{X}$ is the parallel field along γ with $\bar{X}(R) = X$. Thus $e > 0$.

The rest of the argument proceeds as in Lemma 1 of [S-S]. Since $n \geq 3$, $M - B(x, R/2)$ is simply connected and hence admits an isometric immersion (the development map) F into $\mathbb{R}^n$. Since the second fundamental form of $S(x, R)$ is definite, $F(S(x, R))$ must be an embedded convex surface [S]. Since F takes geodesics normal to $S(x, R)$ to geodesics normal to $F(S(x, R))$, it is easy to see that F is an isometry from $M - B(x, R)$ to the unbounded component of $\mathbb{R}^n - F(S(x, R))$.

The theorem now follows from Theorem A of [C].

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