HOW TO COMPUTE THE SQUARE ROOT OF THE NON-EUCLIDEAN WAVE OPERATOR

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ABSTRACT. In this paper we derive a first-order differential operator that can serve as an alternative to the non-Euclidean wave operator to study $SL(2, R)$.

Extensive research has been done on the harmonic analysis of $SL(2, R)$. It has been shown [1], [2] that most of the results can be obtained using the theory of the non-Euclidean wave equation and an operator associated with it which behaves roughly like its square root. The question of how to compute a first-order differential operator that can be thought of as the square root of the non-Euclidean wave operator naturally arises. (Obviously, the Dirac operator does not do the job [3].) In this note, we show how it may be done. The operator constructed can be used to provide an analysis of $SL(2, R)$ different from the approach developed in [1].

But first, let us introduce some notation. Let $\mathbb{H}$ be the set $\{y \geq 0, -\infty < t, x < +\infty\}$ with the metric form

$$(1) \quad d\sigma^2 = dt^2 - \frac{dx^2 + dy^2}{y^2}.$$

The wave operator on $\mathbb{H}$ is defined to be [1]

$$\square_H = \frac{\partial^2}{\partial t^2} - y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{4}.$$

Our objective is to construct a differential operator M , such that $M^2 = \square_H$.

In order to do this, we introduce an auxiliary Minkowski space with coordinates τ, χ, η and the metric form

$$ds^2 = d\tau^2 - d\chi^2 - d\eta^2.$$

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The transformation

$$(2) \quad \begin{aligned} \tau &= \frac{e^t}{2} \left(\frac{1}{y} + y + \frac{x^2}{y} \right), \\ \chi &= \frac{e^t}{2} \left(\frac{1}{y} - y - \frac{x^2}{y} \right), \\ \eta &= e^t \frac{x}{y} \end{aligned}$$

maps Π onto the interior of the future light cone $\tau^2 - \chi^2 - \eta^2 > 0$ of the Minkowski space in such a way that the metric forms satisfy the identity

$$(3) \quad d\tau^2 - d\chi^2 - d\eta^2 = e^{2t} \left(dt^2 - \frac{dx^2 + dy^2}{y^2} \right).$$

Let $v(t, x, y)$ be a solution of the hyperbolic wave equation

$$(4) \quad \square_H v = \frac{\partial^2 v}{\partial t^2} - y^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{1}{4} v = 0.$$

Then the function

$$(5) \quad u(\tau, \chi, \eta) = e^{-t/2} v(t, x, y)$$

is a solution of the Euclidean wave equation

$$(6) \quad \square_E u(\tau, \chi, \eta) = \frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial \chi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0,$$

provided t, x, y are expressed in terms of τ, χ, η via the transformation (2).

One can construct the Dirac operator for $\square_E$ as

$$(7) \quad \mathcal{D} = \sigma^0 \frac{\partial}{\partial \tau} + \sigma^1 \frac{\partial}{\partial \chi} + \sigma^2 \frac{\partial}{\partial \eta},$$

where

$$(8) \quad \sigma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Expressing $\mathcal{D}$ in terms of t, x, y we obtain

$$(9) \quad \begin{aligned} \mathcal{D} &= \left(\sigma^0 \frac{\partial t}{\partial \tau} + \sigma^1 \frac{\partial t}{\partial \chi} + \sigma^2 \frac{\partial t}{\partial \eta} \right) \frac{\partial}{\partial t} \\ &\quad + \left(\sigma^0 \frac{\partial x}{\partial \tau} + \sigma^1 \frac{\partial x}{\partial \chi} + \sigma^2 \frac{\partial x}{\partial \eta} \right) \frac{\partial}{\partial x} \\ &\quad + \left(\sigma^0 \frac{\partial y}{\partial \tau} + \sigma^1 \frac{\partial y}{\partial \chi} + \sigma^2 \frac{\partial y}{\partial \eta} \right) \frac{\partial}{\partial y} \\ &= e^{-t} \left\{ T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \right\}, \end{aligned}$$

where

$$(10) \quad \begin{aligned} T &= \begin{pmatrix} -\frac{x}{y} & \frac{y^2+x^2}{y} \\ -\frac{1}{y} & \frac{x}{y} \end{pmatrix}, \\ X &= \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix}, \\ Y &= \begin{pmatrix} -x & x^2-y^2 \\ -1 & x \end{pmatrix}. \end{aligned}$$

In terms of variables $\tau, \chi, \eta, u = (u_1, u_2)$, the Dirac operator $\mathcal{D}$ satisfies the equation

$$\mathcal{D}^2 u = \square_E u = 0.$$

Thus, in terms of variables $t, x, y, v = (v_1, v_2)$, it must satisfy

$$\mathcal{D}^2 e^{t/2} v = \square_H v = 0,$$

or

$$\widetilde{M}^2 v = \square_H v = 0,$$

where

$$(11) \quad \widetilde{M} = e^{t/2} \mathcal{D} e^{-t/2} = e^{-t/2} \left\{ T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \right\} e^{-t/2}.$$

Using the fact that $X = -TY$ and $Y = TX$, we rewrite (11) as

$$\begin{aligned} \widetilde{M} &= e^{-t/2} T \left\{ \frac{\partial}{\partial t} - Y \frac{\partial}{\partial x} + X \frac{\partial}{\partial y} \right\} e^{-t/2} \\ &= e^{-t} T \left\{ \frac{\partial}{\partial t} - Y \frac{\partial}{\partial x} + X \frac{\partial}{\partial y} - \frac{1}{2} \right\}. \end{aligned}$$

Define

$$(12) \quad \begin{aligned} M &= \frac{\partial}{\partial t} - Y \frac{\partial}{\partial x} + X \frac{\partial}{\partial y} - \frac{1}{2} \\ &= \frac{\partial}{\partial t} + \begin{pmatrix} x & -x^2+y^2 \\ 1 & -x \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix} \frac{\partial}{\partial y} - \frac{1}{2}. \end{aligned}$$

One can also compute all eigenfunctions of the time-independent part of (12). Let

$$(13) \quad L = \begin{pmatrix} x & -x^2+y^2 \\ 1 & -x \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix} \frac{\partial}{\partial y} - \frac{1}{2}.$$

Suppose $u(x, y)$ is an eigenfunction of L corresponding to an eigenvalue λ . Then $e^{-\lambda t} u(x, y)$ is in the kernel of (12), and therefore it is a solution of (4), which implies that $u(x, y)$ is an eigenfunction of the non-Euclidean Laplacian

$$(14) \quad y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{4}$$

corresponding to the eigenvalue λ^2 . But according to [4, p. 60] any eigenfunction of (14) corresponding to an eigenvalue λ can be represented as

$$f_\lambda(x, y) \int_0^{2\pi} \left[\frac{y}{y^2 + (x - \tan \varphi)^2} \right]^{\lambda+1/2} dT(\varphi),$$

where $T(\varphi)$ is a measure on S^1 . The question is elaborated in [4].

Thus we can conclude that any eigenfunction of (13) corresponding to an eigenvalue λ can be represented as

$$f_\lambda(x, y) = \int_0^{2\pi} \left[\frac{y}{y^2 + (x - \tan \varphi)^2} \right]^{\lambda+1/2} \begin{pmatrix} \sin & \varphi \\ \cos & \varphi \end{pmatrix} dT(\varphi).$$

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