CHARACTERIZATION OF FUNCTIONS IN TERMS OF RATE OF CONVERGENCE OF A QUADRATURE PROCESS

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ABSTRACT. We discuss the relationship between the regularity properties of a continuous 2π -periodic function f and the behavior of the remainder of the trapezoidal rule on $[0, 2\pi]$ applied to f .

1. INTRODUCTION

It is well known that certain classes of periodic functions can be characterized in terms of the degree of best approximation by trigonometric polynomials. To illustrate this, we quote the following results.

Theorem A [10, p. 61]. *A continuous 2π -periodic function g is infinitely differentiable if and only if*

$$\lim_{n \rightarrow \infty} n^k E_n^T(g) = 0$$

for every exponent k . Here

$$E_n^T(g) = \inf_{x \in \mathbf{R}} \sup |g(x) - t_n(x)|,$$

where the infimum is taken over all trigonometric polynomials t_n of degree at most n .

Theorem B [10, p. 61]. *The property*

$$E_n^T(g) = O(n^{-k-1}), \quad k \text{ an integer } \geq 0,$$

is a necessary and sufficient condition that the continuous 2π -periodic function g is k times differentiable and the k th derivative belongs to the Zygmund class $\mathcal{Z}$.

In this paper we wish to investigate whether classes of periodic functions can be similarly characterized in terms of the speed of convergence of a quadrature

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procedure. We treat this question for the trapezoidal rule

$$(1) \quad \int_0^{2\pi} f(x) dx = \frac{2\pi}{n} \sum_{\nu=1}^n f\left(\frac{2\pi\nu}{n}\right) + R_n[f],$$

where f belongs to $C_{2\pi}(\mathbf{R})$, the space of continuous 2π -periodic functions $f: \mathbf{R} \rightarrow \mathbf{C}$. Our choice is motivated by the fact that for periodic functions this rule is optimal in certain ways. For example it is the “best” quadrature formula of the form

$$\int_0^{2\pi} f(x) dx = \sum_{\nu=1}^n w_\nu f(x_\nu) + \rho_n[f]$$

for the Sobolev space $W_{L^2}^k$ of 2π -periodic functions (see Definition 2 below) where the weights w_ν and the nodes $0 < x_1 < \cdots < x_n \leq 2\pi$ are arbitrary (see [12, Theorem 3]).

We start with certain observations that will shed some light on the problem under consideration and will also motivate the formulation of the desired characterizations.

Let $\mathcal{T}_k$ be the space of all trigonometric polynomials of degree at most k . Using the well-known fact that $\sum_{\nu=0}^{n-1} \omega^\nu = 0$ for an n th root of unity $\omega \neq 1$, we obtain

$$(2) \quad R_n[f] = 0 \quad \text{for all } n > k$$

and all $f \in \mathcal{T}_k$. However, (2) does not characterize $\mathcal{T}_k$ since (2) also clearly holds for any function $f \in C_{2\pi}(\mathbf{R})$ which is odd on $[0, 2\pi]$; i.e., $f(x) \equiv -f(2\pi - x)$. It may even hold if f is not of the form $t + \varphi$ where $t \in \mathcal{T}_k$ and φ is odd on $[0, 2\pi]$. In fact, the function

$$g(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{inx}$$

considered by C. H. Ching (see [1, 9]) wherein μ is the Möbius function, has the remarkable property that $R_n[g] = 0$ for all $n \in \mathbf{N}$ although $g + c$ is not odd on $[0, 2\pi]$ for any constant c . This shows that the property (2) does not contain “much” information about the function f . Broadly speaking, the remainder is small if the function either has strong regularity properties or else possesses certain symmetries. We note that the regularity properties of a function f are inherited by its translates

$$f_h: x \mapsto f(x + h), \quad h \in \mathbf{R},$$

but the same is not true for the symmetries. Since we are interested in regularity properties, it seems natural to us to involve translates of f into our consideration. We shall in fact prove the following:

Theorem 1. *Let $f \in C_{2\pi}(\mathbf{R})$. Then $f \in \mathcal{T}_k$ if and only if $R_n[f_h] = 0$ for all $n > k$ and $|h| \leq \pi/n$.*

Certain other spaces of functions can also be characterized via $R_n[f_h]$.

Definition 1. Let $B_{2\pi, d}^2$ denote the set of all 2π -periodic functions f holomorphic in

$$S_d := \{z \in \mathbb{C} : |\Im z| < d\}$$

and satisfying

$$(3) \quad \|f\| := \sup_{|y| < d} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x + iy)|^2 dx \right)^{1/2} < +\infty.$$

As usual we shall denote by ℓ^2 the Hilbert space of sequences $(a_n)_{n \in \mathbb{N}}$ for which $\sum_{n \in \mathbb{N}} |a_n|^2 < \infty$.

Theorem 2. For $f \in C_{2\pi}(\mathbb{R})$, the following statements are equivalent:

- (i) f is the restriction to $\mathbb{R}$ of a function belonging to $B_{2\pi, d}^2$,
- (ii) $|R_n[f_h]| \leq b_n e^{-nd}$ for all $n \in \mathbb{N}$ and $|h| \leq \pi/n$, where d is a positive number and $(b_n)_{n \in \mathbb{N}} \in \ell^2$.

Definition 2. Let $W_{L^2}^k$ be the space of functions (Sobolev space) $f: \mathbb{R} \rightarrow \mathbb{C}$ of period 2π for which $f^{(k-1)}$ exists and is absolutely continuous with $f^{(k)} \in L^2(0, 2\pi)$.

Theorem 3. For $f \in C_{2\pi}(\mathbb{R})$, the following statements are equivalent:

- (i) $f \in W_{L^2}^k$ for some integer $k \geq 2$,
- (ii) $R_n[f_h] \leq b_n n^{-k}$ for all $n \in \mathbb{N}$ and $|h| \leq \pi/n$ where $(b_n)_{n \in \mathbb{N}} \in \ell^2$.

From the proof of Theorem 3, it will be clear that (ii) implies (i) for $k = 1$ as well. However, (i) does not imply (ii) in the case $k = 1$, as the next theorem shows.

Theorem 4. There exists a function $f \in W_{L^2}^1$ such that

$$\sum_{n \in \mathbb{N}} n^2 |R_n[f_h]|^2 = \infty.$$

It has been pointed out by Professor C. K. Chui through the editor that our results have some relationship with those in [2] and [4–7] but it seems to us that there is no overlap. In order to illustrate the connection to those papers we quote the following:

Theorem C [5, Theorem 1]. Let f be a function holomorphic in the open unit disk with $f' \in H^p$, $1 < p \leq \infty$, and

$$\frac{1}{n} \sum_{k=1}^n f(\exp(i2\pi k/n)) = 0$$

for each $n = 1, 2, \dots$. Then f is the zero function.

Theorem D [4, Theorem 1]. Denote by B^s , where $s > 0$, the class of periodic functions f with period one whose Fourier coefficients satisfy $\hat{f}(n) = O(n^{-s})$

and define

$$S_{\alpha,n}(f) := \frac{1}{n} \sum_{k=1}^n f\left(\alpha + \frac{k}{n}\right), \quad R_{\alpha,n}(f) := S_{\alpha,n}(f) - \int_0^1 f(t) dt.$$

Let α and β be any two real numbers in $[0, 1]$.

- (i) Suppose $\alpha - \beta$ is irrational and f is in B^s with $s > 1$ such that $S_{\alpha,n}(f) = S_{\beta,n}(f) = 0$ for all $n > 0$. Then f is the zero function.
- (ii) Suppose $\alpha - \beta$ is rational. Then there is a trigonometric function $g \neq 0$ such that $S_{\alpha,n}(g) = S_{\beta,n}(g) = 0$.
- (iii) Suppose $\alpha - \beta$ is irrational. Then there exist trigonometric polynomials $P_{\alpha,m}$ such that

$$\begin{aligned} R_{\beta,n}(P_{\alpha,m}) &= S_{\beta,n}(P_{\alpha,m}) = 0, \\ R_{\alpha,n}(P_{\alpha,m}) &= S_{\alpha,n}(P_{\alpha,m}) = \delta_{m,n}, \end{aligned}$$

for all positive integers m and n .

2. AUXILIARY RESULTS

We start by recalling the following classical result of L. Fejér.

Lemma 1. Let $f \in C_{2\pi}(\mathbf{R})$. If

$$f^\wedge(\mu) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-i\mu x} dx, \quad \mu \in \mathbf{Z},$$

then uniformly for all $x \in \mathbf{R}$ we have

$$\lim_{m \rightarrow \infty} \sum_{\mu=-m}^m \left(1 - \frac{|\mu|}{m+1}\right) f^\wedge(\mu) e^{i\mu x} = f(x).$$

We use Lemma 1 to obtain a representation for $R_n[f_h]$.

Lemma 2. If $f \in C_{2\pi}(\mathbf{R})$, then uniformly for all $h \in \mathbf{R}$ we have

$$(4) \quad \lim_{m \rightarrow \infty} -2\pi \sum'_{|j| \leq [m/n]} \left(1 - \frac{|jn|}{m+1}\right) f^\wedge(jn) e^{ijnh} = R_n[f_h].$$

Here and in the rest of the paper, the prime indicates that the term corresponding to the value 0 of the index is omitted in the summation.

Proof. By the result of Fejér,

$$f_h(x) = f(x+h) = \lim_{m \rightarrow \infty} \sum_{\mu=-m}^m \left(1 - \frac{|\mu|}{m+1}\right) f^\wedge(\mu) e^{i\mu(x+h)}$$

uniformly for $x, h \in \mathbf{R}$. Hence

$$\begin{aligned} R_n[f_h] &= \lim_{m \rightarrow \infty} \sum_{\mu=-m}^m \left(1 - \frac{|\mu|}{m+1}\right) f^\wedge(\mu) \\ &\quad \times \left\{ \int_0^{2\pi} e^{i\mu(x+h)} dx - \frac{2\pi}{n} \sum_{\nu=1}^n e^{i\mu(h+2\pi\nu/n)} \right\}. \end{aligned}$$

Since $\int_0^{2\pi} e^{i\mu(x+h)} dx$ is equal to 2π or 0 depending on whether μ is equal to zero or not, and $\frac{1}{n} \sum_{\nu=1}^n e^{i2\pi\mu\nu/n}$ is equal to 1 or 0 depending on whether μ is a multiple of n or not, we obtain (4). $\square$

The next lemma contains a characterization of the class $B_{2\pi, d}^2$ which represents an important step toward the proof of Theorem 2.

Lemma 3. *A function $f \in C_{2\pi}(\mathbf{R})$ is the restriction to $\mathbf{R}$ of a function belonging to $B_{2\pi, d}^2$ if and only if $f^\wedge(n) = a_n e^{-|n|d}$ where $(a_{\pm n})_{n \in \mathbf{N}} \in \ell^2$.*

Proof. Let $f \in B_{2\pi, d}^2$, and set

$$\alpha_n(y) := \frac{1}{2\pi} \int_0^{2\pi} f(x + iy) e^{-inx} dx \quad \text{for } n \in \mathbf{Z}.$$

Contour integration of $f(z) e^{-inz}$ along the rectangle with vertices $0, 2\pi, iy$, and $2\pi + iy$, where $|y| < d$ and $ny < 0$, yields

$$\begin{aligned} (5) \quad f^\wedge(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x + iy) e^{-in(x+iy)} dx = e^{-|ny|} \alpha_n(y), \end{aligned}$$

since the contributions coming from the two vertical segments cancel each other out because of the assumed periodicity. Using (3), Bessel's inequality shows that

$$\sum_{n \in \mathbf{Z}} |f^\wedge(n)|^2 e^{-2|ny|} = \sum_{n \in \mathbf{Z}} |\alpha_n(y)|^2 \leq \|f\|^2 \quad \text{for } y \in (-d, d).$$

Hence, setting $a_n := f^\wedge(n) e^{|n|d}$, we obtain

$$\sum_{n \in \mathbf{Z}} |a_n|^2 e^{-2|n|(d-|y|)} \leq \|f\|^2 \quad \text{for } y \in (-d, d),$$

which implies that $\sum_{n \in \mathbf{Z}} |a_n|^2 \leq \|f\|^2$.

Conversely, if $f^\wedge(n) = a_n e^{-|n|d}$, where $(a_{\pm n})_{n \in \mathbf{N}} \in \ell^2$, then the function f has a uniformly and absolutely convergent Fourier series, and f may be extended to S_d by the holomorphic function

$$f(x + iy) := \sum_{n=-\infty}^{\infty} f^\wedge(n) e^{in(x+iy)}$$

which, by Parseval's relation, satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x + iy)|^2 dx = \sum_{n=-\infty}^{\infty} |f^\wedge(n) e^{-ny}|^2 \leq \sum_{n=-\infty}^{\infty} |a_n|^2 e^{-2|n|(d-|y|)}$$

for all $y \in (-d, d)$. Consequently, $\|f\| < \infty$ and $f \in B_{2\pi, d}^2$. $\square$

For the proof of Theorem 4, we need to recall a notation and a result about infinite matrices.

Definition 3 [8, p. 303]. An infinite symmetric matrix $(\mu_{jk})_{j,k \in \mathbb{N}}$ with real entries is said to be *bounded* if there exists a constant M such that

$$\left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu_{jk} x_j x_k \right| \leq M$$

for every sequence $(x_k)_{k \in \mathbb{N}} \in \ell^2$. Equivalently (see [8, p. 301; 11, p. 571]), the matrix is bounded if for every $(x_k)_{k \in \mathbb{N}} \in \ell^2$ the sequence $(y_k)_{k \in \mathbb{N}}$, where

$$y_j = \sum_{k=1}^{\infty} \mu_{jk} x_k,$$

also belongs to ℓ^2 .

Now denote by (j, k) the greatest common divisor of j and k .

Lemma 4 [11, pp. 577–578]. *The infinite matrix*

$$\left(\frac{(j, k)^{2\sigma}}{(jk)^\sigma} \right)_{j, k \in \mathbb{N}}$$

is bounded if and only if $\sigma > 1$.

With this we are ready to prove

Lemma 5. *The transformation*

$$b_n = \sum_{j=1}^{\infty} \frac{a_{nj}}{j^\sigma} \quad \left(\sigma > \frac{1}{2} \right)$$

maps every sequence $(a_n)_{n \in \mathbb{N}} \in \ell^2$ into a sequence $(b_n)_{n \in \mathbb{N}} \in \ell^2$ if and only if $\sigma > 1$.

Proof. Using the Cauchy–Schwarz inequality, it is readily seen that b_n exists for all $\sigma > 1/2$. For $N \in \mathbb{N}$, we may write

$$\begin{aligned} B_N &:= \sum_{n=1}^N |b_n|^2 = \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{nj}}{j^\sigma} \cdot \frac{\bar{a}_{nk}}{k^\sigma} \\ &= \sum_{n=1}^N \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{nj}}{(nj)^\sigma} \cdot \frac{\bar{a}_{nk}}{(nk)^\sigma} \cdot n^{2\sigma} \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \bar{a}_k \frac{1}{j^\sigma} \cdot \frac{1}{k^\sigma} \sum_{d \leq N, d|(j, k)} d^{2\sigma}, \end{aligned}$$

where the last summation extends to all divisors d of (j, k) that do not exceed N . Hence B_N is the value of the quadratic form

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk}(N, \sigma) a_j \bar{a}_k$$

with the nonnegative coefficients

$$c_{jk}(N, \sigma) = \frac{1}{(jk)^\sigma} \sum_{d \leq N, d|(j,k)} d^{2\sigma}.$$

For $\sigma > 1$, we find

$$(6) \quad c_{jk}(N, \sigma) \leq \frac{1}{(jk)^\sigma} \sum_{d|(j,k)} d^{2\sigma} = \frac{(j, k)^{2\sigma}}{(jk)^\sigma} \sum_{\nu=1}^{(j,k)} \frac{1}{\nu^{2\sigma}} \\ \leq \frac{\pi^2}{6} \cdot \frac{(j, k)^{2\sigma}}{(jk)^\sigma}.$$

On the other hand, for $1/2 < \sigma \leq 1$ we get

$$(7) \quad c_{jk}(N, \sigma) \geq \frac{(j, k)^{2\sigma}}{(jk)^\sigma} \quad \text{if } N \geq \max\{j, k\}.$$

The estimates (6) and (7) in conjunction with Lemma 4 show that for *all* $(a_n)_{n \in \mathbb{N}} \in \ell^2$ the corresponding B_N remains bounded as $N \rightarrow \infty$, if and only if $\sigma > 1$. $\square$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. If $f \in \mathcal{T}_k$, then $f_h \in \mathcal{T}_k$ for all $h \in \mathbf{R}$ and hence $R_n[f_h] = 0$ for all $n > k$ as indicated in the introduction.

Now suppose that $R_n[f_h] = 0$ for all $n > k$ and $|h| \leq \pi/n$. Then, given $\varepsilon > 0$ and $n > k$, there exists by Lemma 2 an $m_0 \in \mathbb{N}$ such that

$$\left| 2\pi \sum'_{|j| < [m/n]} \left(1 - \frac{|jn|}{m+1} \right) f^\wedge(jn) e^{ijnh} \right| < \varepsilon$$

for all $m \geq m_0$ and $|h| \leq \pi/n$. Hence the trigonometric polynomial

$$(8) \quad t(\theta) = 2\pi \sum'_{|j| \leq [m/n]} \left(1 - \frac{|jn|}{m+1} \right) f^\wedge(jn) e^{ij\theta}$$

is bounded in absolute value by ε for all $\theta \in \mathbf{R}$ if $m \geq m_0$. In particular,

$$\left| 2\pi \left(1 - \frac{n}{m+1} \right) f^\wedge(\pm n) \right| < \varepsilon \quad \text{for } m \geq m_0, \quad n > k.$$

From this we readily conclude that $f \in \mathcal{T}_k$. $\square$

Proof of Theorem 2. Let $f \in B_{2\pi, d}^2$. According to Lemma 3, $f^\wedge(n) = a_n e^{-|n|d}$, where $(a_n)_{n \in \mathbb{N}} \in \ell^2$. Hence, in view of Lemma 2, we obtain for all $n \in \mathbb{N}$ and $h \in \mathbf{R}$

$$|R_n[f_h]| \leq 2\pi \sum'_{j \in \mathbb{Z}} |a_{jn}| e^{-n|j|d} := b_n e^{-nd}.$$

Using the Cauchy–Schwarz inequality we find

$$|b_n|^2 \leq \frac{8\pi^2}{1 - e^{-d}} \sum'_{j \in \mathbf{Z}} |a_{jn}|^2 e^{-n(|j|-1)d},$$

and hence, for all $N \in \mathbf{N}$,

$$\begin{aligned} \sum_{n=1}^N |b_n|^2 &\leq \frac{8\pi^2}{1 - e^{-d}} \sum'_{\ell \in \mathbf{Z}} |a_\ell|^2 \sum_{\nu=0}^{|\ell|-1} e^{-\nu d} \\ &\leq \frac{8\pi^2}{(1 - e^{-d})^2} \sum'_{\ell \in \mathbf{Z}} |a_\ell|^2, \end{aligned}$$

which means that (i) implies (ii).

Now let us assume that (ii) holds, and let n be an arbitrary but fixed positive integer. Then by Lemma 2 there exists an $m_0 \in \mathbf{N}$ such that, for all $m \geq m_0$,

$$\left| 2\pi \sum'_{|j| \leq [m/n]} \left(1 - \frac{|jn|}{m+1}\right) f^\wedge(jn) e^{ijnh} \right| \leq \left(b_n + \frac{1}{n}\right) e^{-nd}.$$

Hence the right-hand side is a bound for the coefficients of the trigonometric polynomial (8); in particular,

$$\left| 2\pi \left(1 - \frac{n}{m+1}\right) f^\wedge(\pm n) \right| \leq \left(b_n + \frac{1}{n}\right) e^{-nd}$$

for all $m \geq m_0$; i.e.,

$$|f^\wedge(\pm n)| \leq \frac{1}{2\pi} \left(b_n + \frac{1}{n}\right) e^{-nd}.$$

Finally, with the help of Lemma 3, we see that f is the restriction to $\mathbf{R}$ of a function belonging to $B_{2\pi, d}^2$. $\square$

Proof of Theorem 3. According to a well-known result [3, Theorem 4.1.10], statement (i) is equivalent to

$$(i^*) \quad f^\wedge(n) = (in)^{-k} c_n \quad \text{for all } n \in \mathbf{Z}, \text{ where } (c_{\pm n})_{n \in \mathbf{N}} \in \ell^2.$$

Hence, by Lemma 2,

$$|R_n[f_h]| \leq 2\pi \sum'_{j \in \mathbf{Z}} |c_{jn}|/|jn|^k =: b_n n^{-k}$$

for all $n \in \mathbf{N}$ and $h \in \mathbf{R}$. If $k \geq 2$ and $M_k := \sum_{j \in \mathbf{N}} j^{-k}$, we obtain, using the Cauchy–Schwarz inequality,

$$|b_n|^2 \leq 8\pi^2 \sum'_{j \in \mathbf{Z}} |j|^{-k} |c_{jn}|^2 \quad \text{for } n \in \mathbf{N}.$$

Consequently,

$$\sum_{n=1}^N |b_n|^2 \leq 8\pi^2 M_k \sum'_{\ell \in \mathbf{Z}} |c_\ell|^2 \sum_{j=1}^{\ell} j^{-k} \leq 8\pi^2 M_k^2 \sum_{\ell \in \mathbf{Z}} |c_\ell|^2,$$

and (ii) follows.

Now suppose that (ii) holds. For every $n \in \mathbb{N}$ there corresponds by Lemma 2 an $m_0 \in \mathbb{N}$ such that

$$\left| 2\pi \sum'_{|j| \leq [m/n]} \left(1 - \frac{|jn|}{m+1} \right) \widehat{f}(jn) e^{ijnh} \right| \leq \left(b_n + \frac{1}{n} \right) n^{-k}$$

for all $m > m_0$ and $|h| \leq \pi/n$. Considering again the trigonometric polynomial (8), we find that (i*) holds which is, of course, equivalent to (i). $\square$

Proof of Theorem 4. By Lemma 5, there exists a sequence $(a_n)_{n \in \mathbb{N}} \in \ell^2$ such that, for

$$b_n := \sum_{j=1}^{\infty} \frac{a_{nj}}{j},$$

we have $\sum_{n=1}^{\infty} |b_n|^2 = \infty$. Defining

$$f(x) := \sum_{n=1}^{\infty} \frac{a_n}{n} e^{inx},$$

we obtain a function $f \in W_{L^2}^1$ for which, by Lemma 2,

$$nR_n[f] = -2\pi \sum_{j=1}^{\infty} \frac{a_{nj}}{j} = -2\pi b_n.$$

This leads immediately to the desired result. $\square$

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