SHARPLY TRANSITIVE LINEAR GROUPS OVER ALGEBRAICALLY CLOSED FIELDS

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(Communicated by Warren J. Wong)

ABSTRACT. We classify the groups from the title. The result is interpreted in terms of nearfields, and applied to a problem in the model theory of permutation groups.

1. INTRODUCTION

We prove the following theorem on sharply transitive linear groups.

Theorem. *Let K be an algebraically closed field and G a subgroup of $GL_n(K)$ which acts sharply transitively on the set of nonzero vectors in K^n . Then either $n = 1$ and $G = K^*$, or $n = 2$ and the following holds: $G = ZG \cdot G_0$ is the product of its center ZG (a group of scalars) with $G_0 = G \cap SL_2(K)$, where $ZG \cap G_0 = \{\pm 1\}$; furthermore, there is a real closed subfield L of K such that $K = L(\sqrt{-1})$, and an L -subalgebra D of the matrix algebra $M_2(K)$, isomorphic to the quaternion division algebra over L , such that $G_0 \cong \text{SU}_2(K, L)$ is the group of elements of D of norm 1.*

We remark that for $n = 2$ it would be enough to assume that K is quadratically closed; then the only change is that L is no longer real closed, but Euclidean (i.e., an ordered field whose positive elements are all squares).

The second and fourth author's interest in this theorem came from the theory of nearfields and planes coordinatized by them. In these terms, the theorem amounts to the classification of the nearfields over algebraically closed fields. This can be seen as a counterpart to Zassenhaus' well-known classification of the finite nearfields [Z].

The first and third author's interest in this issue arose from a problem in the model theory of permutation groups. A permutation group G is said to be sharply n -transitive on the set X if it acts regularly on the set of ordered n -tuples of distinct elements of X . The third author has proved that an infinite sharply 3-transitive superstable group is of the form $PSL_2(k)$ with k

Received by the editors December 22, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 03C60, 20H20, 51J20.

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0002-9939/91 \$1.00 + \$.25 per page

algebraically closed, in its natural action on the projective line [Ne1], and has conjectured that infinite sharply 2-transitive groups of finite U-rank are similarly of *standard form*, namely: $k_+ \rtimes k^\times$ with k an algebraically closed field. This conjecture naturally divides in two parts: that any such group *splits* as $N \rtimes H$ with N some normal subgroup and with H the stabilizer of a point; and that the split groups are standard. All finite sharply 2-transitive groups are split, and it is possible that this also holds for all infinite ones (with no stability hypothesis). Our theorem yields the following information concerning the split case (Corollary 4, §2):

Corollary. *Let $G = N \rtimes H$ be a split infinite sharply 2-transitive superstable group of finite U-rank. If the centralizer of H in $\text{End } N$ is infinite, then G is of standard form. In particular superstable nearfields of finite U-rank of characteristic 0 are algebraically closed fields.*

2. NEARFIELDS OVER ALGEBRAICALLY CLOSED FIELDS

The main theorem can be rephrased in the language of nearfields, and applied to the model theoretic problem considered in the introduction. We make those consequences of the theorem explicit here.

A nearfield $(F, +, *)$ is called a *Dickson nearfield* if there is a division ring $D = (F, +, \cdot)$ and a mapping $\alpha : D^* \rightarrow \text{Aut}(D)$ such that $x * y = x \cdot \alpha(x)y$ for all $x, y \in D^*$; then the map α satisfies $\alpha(x \cdot \alpha(x)y) = \alpha(x)\alpha(y)$. We say that such a Dickson nearfield is of *Kalscheuer type* if D is a quaternion division algebra over a real closed field L and α is constant on the norm classes of D . Then α is the composition of the norm map $D^* \rightarrow L^+$ (= the group of positive elements of L) and an arbitrary homomorphism $\beta : L^+ \rightarrow D^*/L^* (= \text{Aut}(D))$. (Since the image of β is abelian, it is actually contained in K^*/L^* for some 2-dimensional subfield K of D). As an Abelian group, L^+ is just a vector space over the rationals; hence there is a vast number of such maps β .

Nearfields of this type were first studied by Kalscheuer [K] in his work on continuous nearfields, where D is the division ring of real quaternions and β is continuous, hence of the form $\beta(t) = \exp(i\lambda \log(t)) \bmod L^*$ for some real parameter λ .

Recall that the *kernel* of a nearfield is the subnearfield (actually a division ring) consisting of the elements x satisfying both $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ for all y, z . Now we can restate our theorem as

Corollary 1. *Let F be a nearfield that is of finite dimension n over an algebraically closed field K contained in the kernel of F . Then either $n = 1$ and $F = K$, or $n = 2$ and F is a Dickson nearfield of Kalscheuer type.*

An immediate consequence of this is

Corollary 2. *Let F be a nearfield that is finite-dimensional over a division ring H contained in the kernel of F . If H is a quaternion algebra over some real closed field, then $F = H$.*

We now turn to the model theory of permutation groups. Let $N \rtimes H$ be an infinite sharply 2-transitive group. Then N is abelian and H acts regularly on $N^\times$ (see e.g. [Ke], 5.11-12). Let E be the additive endomorphism ring of N and view H as a subgroup of the invertible elements of E , acting on the right.

Lemma. $C_E(H)$ is interpretable in G .

Proof. Fix $x \in N^\times$. If $f \in C_E(H)$ and $f(x) = x'$, then for $y \in N^\times$ we have $y = x^h$ for some unique $h \in H$, and $y^f = x'^h$. It is easy to derive an interpretation for $C_E(H)$ from this. $\square$

Corollary 3. Let $\mathcal{G} = N \rtimes H$ be a split sharply 2-transitive group which is superstable of finite U-rank. If $K = C_E(H)$ is infinite then K is an algebraically closed field and $\mathcal{G} \cong K_+ \rtimes K^\times$.

Proof. K is a superstable ring. As N is H -irreducible, K is a division ring, hence an algebraically closed field [Ch]. Let V be N with its K -structure, and $G = H$. Then $n = \dim V \leq \text{U-rk } G$. This restores the notation of the theorem. If $n = 1$ we are done. If $n = 2$ then L is interpretable in $\mathcal{G}$ and hence is superstable, hence algebraically closed, a contradiction. $\square$

Corollary 4. A superstable nearfield of finite U-rank with infinite center is an algebraically closed field.

This applies in particular if F is of characteristic zero.

It is reasonable to conjecture that infinite superstable sharply 2-transitive groups are all of standard form, and in particular that superstable nearfields are commutative. (It is known that superstable fields are either finite or algebraically closed.)

One extreme possibility we have not been able to eliminate is the following: $G = A \rtimes H$, A is abelian with no proper infinite definable subgroups, $|ZH| = 2$, H/ZH a bad group of Morley rank 3. In particular H/ZH would be a simple linear group with no involution; it is not known whether such groups exist at all.

The model theory of nearfields has been investigated by Felgner [Fe], who looked at the pseudofinite case, and also by Schulz [S].

3. THE PROOF OF THE MAIN THEOREM

The hypotheses and notation of the main theorem are in force throughout this section. Let K^* denote the multiplicative group of K . As K is algebraically closed, all $g \in G$ have nontrivial eigenvectors. By the regularity hypothesis, elements of G commute if and only if they have a common eigenvector, and are conjugate if and only if they have a common eigenvalue. We refer to these criteria as the *commutation* and *conjugacy* criteria below. Clearly G acts irreducibly on $V := K^n$; hence by Schur's Lemma, the center ZG acts as a group of scalars.

We will now proceed to prove the theorem in a series of lemmas, using in-

duction on n . The case $n = 1$ is trivial, so assume $n > 1$. For $g \in G$ let $C(g)$ denote the centralizer of g in G .

Lemma 1. *If $g \in G - ZG$ and $V_\alpha \neq (0)$ is the α -eigenspace of g in V (for some $\alpha \in K$), then V_α has dimension 1, and the map that sends each $h \in C(g)$ to the eigenvalue of h on V_α is an isomorphism from $C(g)$ to K^* .*

Proof. $C(g)$ acts on V_α , and we claim first that it acts regularly on the nonzero vectors of V_α . If $v, w \neq 0$ are in V_α and h is the (unique) element of G with $h \cdot v = w$, then $[g, h]v = v$ and thus $h \in C(g)$. The regularity of $C(g)$ on $V_\alpha \setminus \{0\}$ follows. Now if the dimension d of V_α is 1, the claim follows. So assume $d > 1$. Then it follows by induction that $d = 2$ and $C(g)$ has the structure specified in the conclusion of the theorem. In particular G contains a noncentral element g' of order 4, which has $\pm\sqrt{-1}$ as its eigenvalues. Since the characteristic of K is not 2 and g' is of order 4, g' is diagonalizable with two distinct eigenvalues, and it must have an eigenspace V' of dimension ≥ 2 since $n > 2$. Again it follows that $\dim V' = 2$ and $C(g')$ has the structure specified in the conclusion of the theorem. Thus $C(g')$ has both a central element (namely, g') and a noncentral element of order 4. By the conjugacy criterion they are conjugate in $C(g')$ (since $C(g')$ acts regularly on V'), which is a contradiction. $\square$

Lemma 2. *Let $g, g' \in G - ZG$.*

(a) *The elements g, g' commute if and only if they have the same set of eigenspaces.*

(b) *We have either $C(g) = C(g')$ or $C(g) \cap C(g') = ZG$.*

(c) *If $N(g)$ denotes the normalizer of $C(g)$ in G , then $N(g)/C(g)$ acts regularly on the set of eigenspaces of g .*

(d) *$C(g)$ and $C(g')$ are conjugate.*

Proof. (a) follows from the commutation criterion and the fact that the eigenspaces are 1-dimensional. Claim (b) follows from (a).

For the transitivity in (c), let v, w be eigenvectors for g and let h be the element of G with $hv = w$. Then g and g^h have v as an eigenvector; hence $g \in C(g) \cap C(g^h)$ (by the commutation criterion); thus $C(g) = C(g^h)$ (by (b)), and so $h \in N(g)$. This shows the transitivity. On the other hand if $hV_\alpha = V_\alpha$ with V_α the α -eigenspace of g , then $h \in C(g)$ (by the commutation criterion), which shows the regularity.

For (d), after a suitable conjugation we can assume g, g' share a common eigenvector, and apply (a,b). $\square$

Lemma 3. (a) $n = 2$, and $\text{char}(K) \neq 2$.

(b) *Every $g \in G - ZG$ has 2 distinct eigenvalues, and $[N(g) : C(g)] = 2$.*

Proof. Let $g \in G - ZG$. By Lemma 2(c) the number m of distinct eigenvalues of g is $[N(g) : C(g)]$, which is independent of g by Lemma 2(d). Suppose $m = 1$, and write $n = q^r e$ where $q = \text{char}(K)$ (or $q = 1$, if $\text{char}(K) = 0$), and $e \not\equiv 0 \pmod{q}$. Then any element f of the commutator subgroup G' of

G has as its only eigenvalue an e th root of unity (since $\det(f) = 1$); hence f^e has eigenvalue 1 and so $f^e = 1$. Since $X^e - 1$ is a separable polynomial, f is diagonalizable, hence is a scalar $\lambda \cdot \text{id}_V$, with λ an e th root of unity. So G' is finite and central in G . Hence the commutator yields a bi-multiplicative map $G \times G \rightarrow G'$, and this map must be trivial since G is divisible and G' is finite. Thus G is abelian, and so $G = ZG$ a group of scalars, which is absurd.

Thus $m > 1$, and so there is $h \in N(g)$ whose image in $N(g)/C(g)$ has prime order p . Then Lemma 2(b) shows

$$(1) \quad C(g) \cap C(h) = ZG.$$

Since $h^p \in C(g)$ it follows that

$$(2) \quad h^p \in ZG.$$

In particular since h has more than one eigenvalue, the characteristic of K is not p . The element h induces an automorphism σ of order p of $C(g)$. Let U_p be the group of elements of $C(g)$ of p -power order. Since $C(g) \cong K^*$, U_p is a Prüfer group, and its automorphism group is isomorphic to $\mathbb{Z}_p^\times$, the invertible p -adic integers.

Now we show $p = 2$. If $p > 2$ there is no element of order p in $\mathbb{Z}_p^\times$ (cf. [Am]), so $U_p \subseteq C(g) \cap C(h) = ZG$. Now ZG is the group of fixed points of the automorphism σ of prime order p acting on a divisible Abelian group, whose p -torsion lies in ZG ; hence ZG is p -divisible (in additive notation, if $px \in ZG$ and $py = x$ then $\sigma y = y + z$ with $p^2 z = 0$; hence $y = \sigma^p y = y + pz$ and $pz = 0$, $\sigma x = x$). As $h^p \in ZG$ and ZG is p -divisible, we can take $h^p = 1$. Now if v is an eigenvector for g , then h fixes the vector $v + hv + \cdots + h^{p-1}v$, which is nontrivial by Lemma 2(c). This is a contradiction, which shows that $p = 2$.

Now h^2 is a scalar (by (2)); hence h has at most two distinct eigenvalues. Thus $m \leq 2$. Since $m > 1$, we have $m = 2$. Furthermore since $\text{char}(K) \neq 2$, h is diagonalizable, with 1-dimensional eigenspaces (Lemma 1), so $n = 2$. This proves (a).

As $m = 2$, (b) follows from Lemma 2(c). $\square$

Lemma 4. Let $G_0 = G \cap \text{SL}(2, K)$. Then $G = ZG \cdot G_0$ with $ZG \cap G_0 = \{\pm 1\}$.

Proof. Clearly $-1 \in G$ (since an element of G of order 2 can only have one eigenvalue); hence $ZG \cap G_0 = \{\pm 1\}$. Let $g \in G - ZG$. As $[N(g) : C(g)] = 2$ and $C(g)$ is divisible abelian with a unique involution, if $h \in N(g) - C(g)$ then h induces an automorphism of order 2 on $C(g)$ and $g = g_+ g_-$ where h centralizes g_+ and inverts g_- . Then $g_+ \in ZG$ and g_- is conjugate to its inverse, and has two distinct eigenvalues. Since these eigenvalues cannot be the pair $\{1, -1\}$ they must be a pair $\{\alpha, \alpha^{-1}\}$ and thus $g_- \in G_0$. $\square$

To identify the group $\overline{G}_0 := G_0 / \langle -1 \rangle$, we will use the geometry of involutions of $\overline{G}_0$, inspired by Bachmann [Ba].

Notation. For $g \in G_0$ let $C_g = C_{G_0}(g)$ and $N_g = N_{G_0}(C_g)$. Let $\overline{G}_0 = G_0/\langle -1 \rangle$.

Lemma 5. Let $g \in G_0 - \{\pm 1\}$. Then:

- (1) C_g is abelian and 2-divisible.
- (2) $[N_g : C_g] = 2$.
- (3) $N_g - C_g$ consists of elements of order 4, each of which acts on C_g by inversion.
- (4) All elements of order 4 in G lie in G_0 , and are conjugate in G_0 .
- (5) The involutions of $\overline{G}_0$ are exactly the images in $\overline{G}_0$ of elements of order 4 in G_0 .

Proof. (5) is clear, and we know C_g is abelian.

If $h \in N(g) - C(g)$ then $h \equiv h' \pmod{ZG}$ for some $h' \in N_g - C_g$ (by Lemma 4). Thus $[N_g : C_g] \geq 2$. Also $N_g - C_g \subseteq N(g) - C(g)$; hence $[N_g : C_g] = 2$. By the proof of Lemma 4, each $h \in N_g - C_g$ acts on C_g by inversion; in particular, $h^2 \in C_g$ must be an involution, and hence h is of order 4. This proves (2) and (3).

By the conjugacy criterion one sees that all elements of G of order 4 are conjugate, and (4) follows.

It remains to show that C_g is 2-divisible. By Lemma 4, $C_g/(\pm 1) \cong K^\times/ZG$ is 2-divisible, and by (4) it follows that C_g is also 2-divisible. $\square$

Lemma 6. For any two distinct involutions of $\overline{G}_0$ there is another involution in $\overline{G}_0$ distinct from the two and centralizing both.

Proof. Let $i, j \in G_0$ be elements of order 4 representing distinct involutions $\bar{i}, \bar{j}$ of $\overline{G}_0$. Let $g = ij \notin \{\pm 1\}$. Then $C(g)$ contains a unique cyclic subgroup $\langle k \rangle$ of order 4, and this k is in G_0 . As i, j invert g , they normalize $C(g)$, hence either invert or centralize k . In either case $\bar{i}, \bar{j}$ centralize $\bar{k}$. If $\bar{i}, \bar{j}$ do not commute, then clearly they are distinct from $\bar{k}$; if they do commute, then $\bar{k} = \bar{g} = \bar{ij}$, which again implies $\bar{k} \neq \bar{i}, \bar{k} \neq \bar{j}$. $\square$

Lemma 7. G_0 equals its own commutator subgroup G'_0 .

Proof. By Lemma 5.3, we have for each $g \in G_0$ that $g^{-1} = g^j$ for some $j \in G_0$. Thus $g^{-2} = [g, j] \in G'_0$. So G'_0 contains all squares of G_0 ; but by Lemma 5.1, each element of G_0 is a square. $\square$

Now let A be the 3-dimensional K -vector space of all trace 0 matrices in $M_2(K)$. The determinant yields a nondegenerate quadratic form Q on A . The group G_0 acts on A by conjugation, leaving Q invariant. This gives an embedding of $\overline{G}_0 = G_0/\langle -1 \rangle$ into the special orthogonal group $SO_3(Q)$, and via this embedding we will regard $\overline{G}_0$ as a subgroup of $SO_3(Q)$. Let I (resp., I_0) be the set of involutions in $SO_3(Q)$ (resp., in $\overline{G}_0$).

Lemma 8. (a) Each involution $\nu \in I$ has a 1-dimensional eigenspace $A^+(\nu)$ and a 2-dimensional eigenspace $A^-(\nu)$, corresponding to the eigenvalue $+1$ and -1 , respectively. These eigenspaces are nondegenerate (relative to Q), and for each nondegenerate 1-space (resp., 2-space) X in A there is a unique $\nu \in I$ with $X = A^+(\nu)$ (resp., $X = A^-(\nu)$).

(b) For distinct $\nu, \nu_1, \nu_2 \in I$, the following are equivalent:

- (i) ν centralizes ν_1 and ν_2 .
- (ii) $A^+(\nu) = A^-(\nu_1) \cap A^-(\nu_2)$.
- (iii) $A^-(\nu) = A^+(\nu_1) + A^+(\nu_2)$.

Proof. An elementary exercise in Linear Algebra (all contained in [Ba]). $\square$

Now let $\mathcal{P}$ (resp., $\mathcal{L}$) denote the set of all $A^+(\nu)$ (resp., $A^-(\nu)$) for $\nu \in I_0$. Then $\mathcal{P}$ (resp., $\mathcal{L}$) is a set of points (resp., lines) of the projective plane π_A associated with A , and we are going to show that $(\mathcal{P}, \mathcal{L})$ is a subplane of π_A . Take $\nu_1 \neq \nu_2$ in I_0 . By Lemma 6 there is $\nu \in I_0$ distinct from ν_1, ν_2 , and centralizing ν_1, ν_2 . Hence by Lemma 8(b), the lines of $\mathcal{L}$ (resp., the points of $\mathcal{P}$) associated to ν_1 and ν_2 intersect in the point of $\mathcal{P}$ (resp., are joined by the line of $\mathcal{L}$) associated to ν . Clearly, $(\mathcal{P}, \mathcal{L})$ is not contained in a triangle; hence it is a subplane of π_A .

Choose points $Kb_1, \dots, Kb_4$ in $\mathcal{P}$ forming a (nondegenerate) quadrangle. Then b_1, b_2, b_3 form a basis of A , and we may assume $b_4 = b_1 + b_2 + b_3$. This gives homogeneous coordinates on π_A such that the above points have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, respectively. Then it follows from the usual geometric interpretation of addition and multiplication that the elements $t \in K$ for which the point $(t, 0, 1)$ lies in $\mathcal{P}$ form a subfield L of K . Furthermore, $\mathcal{P}$ consists exactly of those points in π_A with some coordinate triple in L^3 . In other words, if π_B denotes the projective plane associated to the L -vector space $B = Lb_1 + Lb_2 + Lb_3$, then the canonical image of π_B in π_A equals the subplane $(\mathcal{P}, \mathcal{L})$.

Lemma 9. $\overline{G}_0$ leaves B invariant.

Proof. Viewing $\text{PGL}(A)$ as a collineation group of π_A , we first note that the identity is the only element of $\text{PGL}(A)$ acting trivially on the subplane π_B (namely, such an element would in particular fix four points forming a quadrangle). Hence the subgroup Φ of $\text{PGL}(A)$ fixing π_B is isomorphic to a collineation group of π_B containing $\text{PGL}(B)$. But the stabilizer in Φ of any quadrangle in π_B is trivial; hence Φ coincides with $\text{PGL}(B)$ (embedded naturally into $\text{PGL}(A)$). Thus the inverse image of Φ in $\text{GL}(A)$ is $\Psi := K^* \cdot \text{GL}(B)$.

Now $\overline{G}_0$ is contained in Ψ , hence also in the commutator subgroup of Ψ (by Lemma 7). Thus $\overline{G}_0 \subseteq \text{SL}(B)$, which means that $\overline{G}_0$ leaves B invariant. $\square$

Every 1-dimensional subspace of B is of the form $A^+(\nu) \cap B$ for some $\nu \in I_0$. Since all $\nu \in I_0$ are conjugate under $\overline{G}_0$ (Lemma 5.4, 5.5) it follows

that $\overline{G}_0$ acts transitively on the 1-spaces in B . Thus we may assume $\det(b) \in L$ for all $b \in B$ (replacing B by αB for some $\alpha \in K$, if necessary); further, if $b \neq 0$ then $\det(b) \neq 0$ (since b lies in some space $A^+(\nu)$, which is Q -anisotropic). Thus $\det$ gives an anisotropic quadratic form on B , invariant under $\overline{G}_0$. Viewing B as an anisotropic quadratic space via $\det$, we get $\overline{G}_0 \subseteq \text{SO}_3(B)$.

Lemma 10. (a) $[K : L] = 2$; hence L is a real closed field and $\text{char}(K) = 0$.

(b) The eigenvalues x of each $g \in G_0$ have norm $N_{K/L}(x) = 1$.

Proof. Let x, x^{-1} be the eigenvalues of some $g \in G_0$. By Lemma 5.1 we have $x = y^2$ where y, y^{-1} are the eigenvalues of some $h \in G_0$. Then $y^2, y^{-2}, 1$ are the eigenvalues of its image $\overline{h} \in \overline{G}_0$ acting on A .

Assuming (a) momentarily, we derive (b). For x, y, g, h as above, since $\overline{h}|_B \in \text{SO}_3(B)$, it follows from (a) that $1 = N_{K/L}(y^2) = N_{K/L}(x)$, proving (b).

Thus it only remains to prove (a). With x, y, g, h as above, since $\overline{h}$ fixes B , the trace $\text{tr}(\overline{h}) = y^2 + y^{-2} + 1$ lies in L . Now $\text{tr}(g) = x + x^{-1} = y^2 + y^{-2} \in L$, so we get

(1) For all $g \in G_0$ we have $\text{tr}(g) \in L$.

Since x is an eigenvalue of $\overline{h}$ on A , if $x \in L$ then $\overline{h}$ has an eigenvector v in B with eigenvalue x , and $\det(v^h) = x^2 \det(v)$; since $\det$ is anisotropic on B , we find $x = \pm 1$. This shows:

(2) If $x \neq \pm 1$ is an eigenvalue of some $g \in G_0$, then $x \notin L$.

By Burnside's theorem [Ja, p. 213], G_0 contains a basis $g_1, \dots, g_4$ of the matrix algebra $M_2(K)$. Since the trace form on $M_2(K)$ is nondegenerate, the map $\phi : M_2(K) \rightarrow K^4$ sending any $m \in M_2(K)$ to the 4-tuple $(\text{tr}(mg_1), \dots, \text{tr}(mg_4))$ is a linear isomorphism. Additionally, $\phi(G_0) \subseteq L^4$ by (1). Hence $G_0 \subseteq Lm_1 + \dots + Lm_4$ for certain $m_1, \dots, m_4 \in M_2(K)$. Thus for $w \neq 0$ in K^2 we have $G_0 \cdot w \subseteq Lm_1(w) + \dots + Lm_4(w)$. So the L -vector space W spanned by $G_0 \cdot w$ has dimension ≤ 4 over L .

Clearly, the set k of all $t \in K$ with $tW \subseteq W$ is a subfield of K , with $[k : L] \leq 4$. Since kw ($\subseteq Kw$) cannot exhaust all of W , we even get $[k : L] \leq 2$. If x is an eigenvalue of some $g \in G_0$ then $xw \in G_0 \cdot w$; hence $xW \subseteq W$ and so $x \in k$. Now (2) shows that $k \neq L$. Hence $[k : L] = 2$. Thus W has dimension 2 over k .

Choose a basis w_1, w_2 of W over k . Then this is also a basis of K^2 (over K). Now any line in K^2 through 0 whose slope in w_1, w_2 -coordinates does not lie in k intersects W trivially. Hence such a line cannot exist (since G_0 acts transitively on the lines in K^2 through 0, and leaves W invariant). This proves $k = K$, hence $[K : L] = [k : L] = 2$. The rest of the claim follows. $\square$

Let $i \in K$ with $i^2 = -1$. Since $\det|_B$ is an anisotropic quadratic form over the real closed field L , its nonzero values are all positive or all negative (in the unique order on L). Replacing B by iB if necessary, we may thus assume $\det(b) > 0$ for all $b \neq 0$ in B . Any $g \in M_2(K)$ can be written uniquely as $g = \mu \cdot 1 + b_1 + ib_2$ with $\mu \in K$, $b_1, b_2 \in B$. For $g \in G_0$, since g centralizes itself and normalizes B , we get $b_1^g = b_1$ and $b_2^g = b_2$. If $g \neq \pm 1$, it follows that b_1 and b_2 are linearly dependent over L (since each nonidentity element of $\text{SO}_3(B)$ has a 1-dimensional fixed space in B). Thus every $g \in G_0$ is of the form $g = \mu \cdot 1 + \lambda b$ for some $b \in B$, $\lambda, \mu \in K$. Now $\mu = \text{tr}(g)/2$, and $\text{tr}(g) \in L$ (by (1) from the proof of Lemma 10). Furthermore, $1 = \det(g) = \mu^2 + \lambda^2 \det(b)$; hence $\lambda^2 \in L$. Now we will show that $\lambda^2 \geq 0$, so that $\lambda \in L$, and we may conclude that:

Lemma 11. G_0 is contained in $D := L \cdot 1 + B$.

Proof. To show that $\lambda^2 \geq 0$ it suffices to show that $\mu^2 \leq 1$ (since $\det(b) \geq 0$ by the choice of B above). But the eigenvalues x, x^{-1} of g have norm 1 (by Lemma 10) and satisfy $\mu = (x + x^{-1})/2$. Thus $|\mu| = |x + x^{-1}|/2 \leq (|x| + |x^{-1}|)/2 = 1$ (where $|\cdot|$ is the usual absolute value, i.e., the square root of $N_{K/L}$). Thus also $\mu^2 \leq 1$. $\square$

Since G_0 contains 4 linearly independent elements, it follows that the L -span of G_0 equals D . Thus D is a subalgebra of $M_2(K)$. Any $d \neq 0$ in D is of the form $d = \mu \cdot 1 + b$ with $\mu \in L$, $b \in B$ not both zero; hence $\det(d) = \mu^2 + \det(b) > 0$. Thus D is a (4-dimensional) division algebra over L , hence is the quaternion (division) algebra over L . Furthermore G_0 is contained in the group $\text{SU}_2(K, L)$ of quaternions of norm 1, and from the transitive action of G_0 on the 1-spaces in K^2 it follows that G_0 contains all elements of $\text{SU}_2(K, L)$ of order 4. These elements generate the group, and hence $G_0 = \text{SU}_2(K, L)$.

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