ABSOLUTE CONTINUITY OF HAMILTONIANS WITH VON NEUMANN WIGNER POTENTIALS

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ABSTRACT. For separated Dirac and Schrödinger operators whose potentials have oscillatory decaying terms the asymptotics of the eigenfunctions is determined. From this the absolute continuity of the spectrum off a finite resonance set and a limiting absorption principle is derived.

The behaviour of eigenfunctions of a differential operator is closely related to its spectral properties. For Sturm-Liouville differential operators this connection has been made more precise by the notion of subordinacy, introduced recently by Gilbert and Pearson [3, 4]. Here we apply these ideas to separated Schrödinger- and Dirac operators with oscillating potentials decaying at infinity. Such operators have absolutely continuous spectra off the set of resonances. The set of resonances is countable and can be computed explicitly from the asymptotic behaviour of the potential. The absolute continuity of the spectrum can also be deduced from the continuity of the Weyl-Titchmarsh m -function, which is shown to be continuous in the closed upper half plane off the set of resonances.

This extends results of Heinz [7], Weidmann [10], Devinatz, Rejto [3], and Behncke, Rejto [1]. The paper closes with a few concrete examples, which show that the main result is sharp and which exhibit typical resonance phenomena.

This paper is divided into three sections. These are

- I. Subordinacy
- II. Absolute Continuity
- III. Examples

The main technique to analyze the asymptotics of eigenfunctions is asymptotic integration in the form developed by Harris, Lutz [6] and Ben Artzi, Devinatz [2].

The notation is mostly standard. For a conditionally integrable function f , $\tilde{f}$ is defined by

$$\tilde{f}(t) = \int_{\infty}^t f(s) ds.$$

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This notation will also be used for matrix valued functions. Generic $\mathcal{L}^1$ -(matrix) functions will be denoted by r , respectively R . If all matrix elements of a matrix A belong to $\mathcal{L}^p$, we write $A \in \mathcal{L}^p$.

In the theory of asymptotic integration, equations and expressions are considered only for sufficiently large argument values. We abridge this by the notion "far out".

I. SUBORDINACY

For spherically symmetric potentials the separated Schrödinger (respectively Dirac) equation can be written as

$$-y'' + V_1 y = \lambda y \quad \text{on } \mathcal{L}^2(0, \infty),$$

respectively

$$\begin{pmatrix} V_1 & V_2 \\ V_2 & V_3 \end{pmatrix} u + DJu = \lambda u \quad \text{on } \mathcal{L}^2(0, \infty) \otimes C^2,$$

where $D = d/dt$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Under suitable conditions on the potentials these differential operators can be extended and closed and lead to selfadjoint operators on the corresponding Hilbert spaces [10]. These selfadjoint operators we shall call Hamiltonians.

In the systems notation the Dirac equation becomes

$$(1) \quad u' = \begin{pmatrix} -V_2 & -V_3 + \lambda \\ V_1 - \lambda & V_2 \end{pmatrix} u = (V - \lambda J)u.$$

(1) also covers the Schrödinger case, if one lets $V_2 = 0$ and replaces $-V_3 + \lambda$ by 1. For this reason we shall consider mostly the Dirac equation and indicate only how the results extend to the simpler Schrödinger equation.

We assume that the potentials V_1 , V_2 , and V_3 are real valued and locally integrable. Moreover, far out they are supposed to admit a decomposition.

$$(2) \quad V_i = P_i + S_i + W_i \quad \text{or} \quad V = \mathcal{P} + \mathcal{S} + W, \quad i = 1, 2, 3,$$

into a short range part $\mathcal{S}$,

$$(3.1) \quad \mathcal{S} \in \mathcal{L}^1,$$

and a smooth long range part $\mathcal{P}$,

$$(3.2) \quad \mathcal{P}' \in \mathcal{L}^1.$$

The Wigner von Neumann part W is assumed to be conditionally integrable and satisfy

$$(3.3) \quad W(t) \rightarrow 0, \quad \text{for } t \rightarrow \infty.$$

Since $\mathcal{P}' \in \mathcal{L}^1$, $\mathcal{P}(t)$ converges for t tending to infinity. For $\mathcal{P}$ we require, in addition,

$$(3.4) \quad -P_1(t), P_3(t) \rightarrow m \geq 0, \quad P_2(t) \rightarrow 0, \quad \text{for } t \rightarrow \infty.$$

Definition. Let τ be a Sturm-Liouville differential operator on (a, b) , $-\infty \leq a < b \leq \infty$, which is singular at b [10]. A solution y of $\tau y = \lambda y$ is subordinate at b , if, for any linear independent solution of $\tau z = \lambda z$,

$$\lim_{d \rightarrow b-} \|y\|_d / \|z\|_d = 0,$$

where

$$\|f\|_d^2 = \int_c^d |f(s)|^2 ds.$$

This definition is clearly independent of c , $a \leq c < b$.

If τ is singular at a , subordinacy at a can be defined similarly. In the same manner subordinacy can be defined for Dirac systems.

By abusing the notation slightly we shall also speak of subordinacy for Schrödinger, respectively Dirac Hamiltonians.

Subordinate solutions are thus functions which decay faster near the singularity than the other solutions. Eigenfunctions are in general subordinate. Subordinacy generalizes therefore the notion of a principal solution and eigenfunction. By physical intuition, eigenfunctions of the absolutely continuous spectrum are supposed to look like generalized plane waves. They are thus not subordinate.

In order to make this more precise and state the main result of Gilbert and Pearson [4, 5], we need two more definitions.

Definition. Let S be a measurable subset of $\mathbf{R}$, and let μ be a Borel measure on $\mathbf{R}$. Then S is a set of minimal support of μ if

- (i) $\mu(\mathbf{R} \setminus S) = 0$;
- (ii) for any measurable subset $T \subset S$ with $\mu(T) = 0$ one has $\lambda(T) = 0$, where λ is the Lebesgue measure.

Minimal supports are clearly not unique. But this lack of nonuniqueness will not concern us here.

Definition. For a Borel measure μ on $\mathbf{R}$ the absolutely continuous (singular continuous, discrete) part of μ will be denoted by μ_{ac} (μ_{sc} , μ_d).

In the same fashion these suffixes will be used to denote corresponding minimal supports. If μ is a spectral measure, minimal supports M_{ac} , M_{sc} , and M_d of μ_{ac} , μ_{sc} , and μ_d , respectively, can be chosen such that their closure contains the corresponding spectra σ_{ac} , σ_{sc} , and σ_d [4, Lemma 5].

Lemma 1. (a) Let $\mathcal{H}$ be a Hamiltonian which is derived from a Schrödinger, or Dirac, differential expression τ on $(0, \infty)$ with two singular limit point endpoints. Then minimal supports M_{ac} and M_{sc} of the absolutely continuous, respectively singular continuous, part of the spectral measure μ of $\mathcal{H}$ are given by

- (i) $M_{ac} = \{\lambda \in \mathbf{R} \mid \text{There is no solution of } \tau u = \lambda u \text{ which is subordinate at } 0 \text{ or } \infty \text{ or both}\}$
- (ii) $M_{sc} = \{\lambda \in \mathbf{R} \mid \text{There is a solution of } \tau u = \lambda u \text{ which is subordinate at } 0 \text{ and } \infty, \text{ but which is not square integrable}\}$

- (b) If 0 is a regular endpoint, (a) remains valid if any solution of $\tau u = \lambda u$, which satisfies the boundary condition at 0, is considered to be subordinate at 0.
- (c) (b) remains valid also if 0 is a singular limit circle endpoint.

Remark. It is now obvious how to extend Lemma 1 to the case of a differential operator τ on (a, b) .

In this situation one obtains for the two regular endpoints case $M_{ac} = M_{sc} = \emptyset$, which is well known.

Proof. (b) Gilbert and Pearson prove this result only for Schrödinger differential operators. Their proof in [4] however can easily be extended to more general Sturm-Liouville or Dirac operators. In fact only minor modifications for the inequalities are needed.

(a) For the two singular endpoint case the Titchmarsh-Weyl matrix M is given by

$$M(z) = (m_0(z) - m_\infty(z))^{-1} \begin{pmatrix} m_0(z)m_\infty(z) & (1/2)(m_0(z) + m_\infty(z)) \\ (1/2)(m_0(z) + m_\infty(z)) & 1 \end{pmatrix},$$

where m_0 , respectively m_∞ , are the Titchmarsh-Weyl coefficients near 0, respectively infinity. This expression can easily be derived from the definition of the resolvent, see, e.g., [10, Chapter 9]. In particular this expression holds for Sturm-Liouville and Dirac expressions alike. The discussion can thus be completed as in [5].

(c) If 0 is a limit circle singular endpoint, the domain of $\mathcal{H}$ is characterized by a boundary condition at 0, which is of the form $[u, w](0) = 0$ [10, p. 156], where w is a solution of $\tau w = 0$. From the representation of the resolvent [10, pp. 119, 141], one deduces that M is given as above, only m_0 has to be chosen such that the corresponding function $u = m_0 u_1 + u_2$ satisfies the boundary condition at 0. This implies that m_0 is meromorphic with simple poles located on the real axis. This of course also follows from the fact that the Hamiltonian has discrete spectrum on $[0, c]$ for any $c > 0$. Now the problem can be discussed along the lines of [5]. It is even simpler, because the normal limits of $\text{Im } m_0$ are either 0 or ∞ .

Remark. The advantage of Lemma 1 for the characterization of the spectrum rests in the fact that properties of the eigenfunctions are only needed for real λ . This is particularly valuable, because most operators τ are real. It should also be possible to extend Lemma 1 to higher order systems, because the m function admits an appropriate generalization to this situation. The notion of subordinacy however has to be modified in this case.

II. ABSOLUTE CONTINUITY

Lemma 1 will now be used as follows. For a given λ conditions will be derived such that all solutions of (1) behave like generalized plane waves. This will be achieved by successively transforming (1) into simpler systems. Since

each of these transformations is uniformly bounded with a uniformly bounded inverse, subordinacy or nonsubordinacy is preserved. Thus it suffices to look for subordinate solutions of the transformed equations only.

Since we want to investigate the continuum eigenfunctions Schrödinger and Dirac operators only, $\lambda \in \mathbb{R}$ will be chosen such that $\mathcal{P}(\infty) - \lambda J$ has two distinct imaginary eigenvalues. This means $|\lambda| > m$ in the Dirac case and $\lambda > 0$ in the Schrödinger case. By continuity of the eigenvalues we may then choose $c > 0$ such that $\mathcal{P}(t) - \lambda J$ has two distinct imaginary eigenvalues $\pm i\mu(t)$ for $t \geq c$, where

$$\mu = [(P_1 - \lambda)(P_3 - \lambda) - P_2^2]^{1/2}.$$

Now let

$$a_{\pm} = (\pm i\mu + P_2)/(\lambda - P_3)$$

and

$$a = |a_{\pm}|;$$

then the transformation

$$A = (a_- - a_+)^{-1/2} \begin{pmatrix} 1 & 1 \\ a_+ & a_- \end{pmatrix}$$

diagonalizes $\mathcal{P} - \lambda J$ for $t \geq c$. Thus the transformation

$$w = A^{-1}u \quad \text{or} \quad u = Aw$$

leads to

$$w' = \{A^{-1}(\mathcal{P} - \lambda J)A + A^{-1}WA + A^{-1}SA - A^{-1}A'\}w.$$

By (3.1) and (3.2) the last two matrix summands are $\mathcal{L}^1$, and thus (1) simplifies to

$$(4) \quad w' = \{\text{diag}(i\mu, -i\mu) + A^{-1}WA + R\}w.$$

Now let

$$\sigma(t) = \int_c^t \mu(s) ds$$

and $\Sigma = \text{diag}(\exp i\sigma, \exp -i\sigma)$; then the transformation

$$x = \Sigma^{-1}w \quad \text{or} \quad w = \Sigma x$$

gives

$$(5) \quad x' = \{\Sigma^{-1}A^{-1}WA\Sigma + R\}x.$$

Here

$$\begin{aligned} (6) \quad & \Sigma^{-1}A^{-1}WA\Sigma \\ &= (a_- - a_+)^{-1} \begin{pmatrix} -W_1 - a^2W_3 - (a_+ + a_-)W_2 & -e^{-2i\sigma}(2a_-W_2 + W_1 + a_-^2W_3) \\ (2a_+W_2 + W_1 + a_+^2W_3)e^{2i\sigma} & W_1 + a^2W_3 + (a_+ + a_-)W_2 \end{pmatrix} \\ &= \text{diag}(l, -l) + \begin{pmatrix} 0 & q_1e^{-2i\sigma} \\ q_2e^{2i\sigma} & 0 \end{pmatrix} = \Lambda + Q. \end{aligned}$$

For later use, let us write

$$(7) \quad q_1 = \sum a_i W_i \quad \text{and} \quad q_2 = \sum b_i W_i.$$

For Schrödinger operators these formulas remain valid if one sets $W_2 = W_3 = 0$ and $\pm i\mu = a_{\pm}$. In order to simplify (5) and (6) further, we shall use a transformation introduced and studied by Harris, Lutz [6]. Its general form is as follows. Let

$$x' = (A + B)x$$

be a system of differential equations on $[a, \infty)$ with A, B uniformly bounded and B conditionally integrable. Then $(1 + \tilde{B})^{-1}$ exists far out and $y = (1 + \tilde{B})^{-1}x$ satisfies

$$y' = (1 + \tilde{B})^{-1}[A + A\tilde{B} + B\tilde{B}]y.$$

Here, essential use is made of $\tilde{B}' = B$. Actually Harris and Lutz never used the full strength of this relation. If, more generally, $\tilde{B}' = B + C$ and $C \in \mathcal{L}^1$, we get

$$(8) \quad y' = (1 + \tilde{B})^{-1}[A + A\tilde{B} + B\tilde{B} + R]y,$$

which suffices in most cases. In this situation we call C a counterterm. In this paper we shall also assume $\tilde{B}^2 A, \tilde{B}A\tilde{B}, \tilde{B}B\tilde{B} \in \mathcal{L}^1$. Then (8) can be written as

$$y' = [A + A\tilde{B} - \tilde{B}A + B\tilde{B} + R]y.$$

To the system, the above transformation can be applied again and again until it is transformed into Levinson's form [6, 8]. If $\tilde{B}$ decays more slowly, a more general expression can easily be derived.

Applying these transformations to (5) and (6) with $A = \Lambda$ and $B = Q$ twice yields the analog of Theorem 3.2 of [6].

Lemma 2. Assume for $i = 1, 2, 3$, $h_i = W_i \sin 2\sigma$ and $h_{i+3} = W_i \cos 2\sigma$ are conditionally integrable with

$$(i) \quad \tilde{h}_i \tilde{h}_j W_k \in \mathcal{L}^1, \quad i, j = 1, \dots, 6, \quad l, k = 1, 2, 3.$$

Moreover assume that $W_i \tilde{h}_j$, $i = 1, 2, 3$, $j = 1, \dots, 6$, are conditionally integrable with

$$(ii) \quad W_k (W_l \tilde{h}_j)^\sim \in \mathcal{L}^1.$$

Then a fundamental matrix X of (5) is given by

$$(9) \quad X = (1 + o(1)) \exp \left(\text{diag} \left(\int_c^t \rho_1 ds, \int_c^t \rho_2 ds \right) \right).$$

Here ρ_1 and ρ_2 are given by (7) and

$$\rho_1 = \sum a_l b_k W_l e^{-2i\sigma} (W_k e^{2i\sigma})^\sim$$

and

$$\rho_2 = \sum a_i b_k W_k e^{2i\sigma} (W_i e^{-2i\sigma})^\sim.$$

In particular, (1) has no subordinate solution if $\text{Re } \rho_1$ and $\text{Re } \rho_2$ are integrable.

In this form Lemma 2 is not very useful. In order to obtain a result for which the conditions of Lemma 2 can be shown easily, we shall define a set of Wigner von Neumann potentials, which is still large enough to cover the classes of Devinatz, Rejto [2] or Behncke, Rejto [1]. We shall assume that the potentials in this set can be written as

$$(10) \quad W_i = \sum f_j^{(i)} \sin g_j$$

where $f_j^{(i)}$, g_i and g_j' are absolutely continuous with

$$(10.1) \quad g_j(t) \nearrow \infty \quad \text{or} \quad g_j'(t) \rightarrow g_j'(\infty) > 0 \quad \text{for } t \rightarrow \infty,$$

$$(10.2) \quad f_j^{(i)} \in \mathcal{L}^3, \quad \frac{f_j^{(i)'}}{g_j'}, \quad f_j^{(i)} \cdot \frac{g_j''}{g_j'^2} \in \mathcal{L}^1 \text{ far out,}$$

$$(10.3) \quad \liminf |g_j'(t) - g_k'(t)| > 0, \quad \text{for } j \neq k.$$

Terms for which $g_j'(t) \nearrow \infty$ we shall call rapidly oscillating, while the others will be called evenly oscillating. The reason for the definition of this class lies in the fact that, modulo absolutely integrable counterterms,

$$(f \sin g)^\sim = -\frac{f}{g'} \cos g + \text{counterterms}.$$

Since μ also depends implicitly on λ , we shall write $\mu(t, \lambda)$ in order to indicate this dependence. This allows us to define the resonance sets for the Wigner von Neumann potential W , defined by (10) above, by

$$(11) \quad \begin{aligned} \mathcal{R}_1 &= \{\lambda \in [-m, m]^c \mid \lim g_j'(t) - 2\mu(t, \lambda) = 0, \text{ for } t \rightarrow \infty \text{ for some } j\} \\ \mathcal{R}_2 &= \{\lambda \in [-m, m]^c \mid \lim(g_j' \pm g_i')(t) - 2\mu(t, \lambda) = 0, \\ &\quad \text{for } t \rightarrow \infty \text{ for some } i, j\}. \end{aligned}$$

For the Schrödinger case the definitions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are similar. While $\mathcal{R}_1$ describes first order resonances between the potential and the wave function, second order effects like interference and resonance in beats correspond to $\mathcal{R}_2$.

Our main result can now be stated as follows.

Theorem. *Let $\mathcal{P}$, $\mathcal{S}$, and W be as above satisfying (2), (3), and (10). Then the spectrum of any Dirac (Schrödinger) Hamiltonian derived from the corresponding differential expression is absolutely continuous in $\mathbf{R} \setminus ([-m, m] \cup \mathcal{R}_1 \cup \mathcal{R}_2)$ respectively $(0, \infty) \setminus (\mathcal{R}_1 \cup \mathcal{R}_2)$.*

Proof. For simplicity we shall only treat the Dirac case. Because of Lemma 1 it suffices to show that no solution of (1) for $\lambda \in [-m, m]^c \setminus \mathcal{R}_1 \setminus \mathcal{R}_2$ is subordinate at infinity. Since A and $\sum$ are uniformly bounded with uniformly bounded inverse, it suffices to show this for the solutions of (5) only. Following the proof of Lemma 2 one sees easily that (i) holds because $\lambda \notin \mathcal{R}_1$ and because $f_j^{(i)} \in \mathcal{L}^3$. This implies also that the first Harris and Lutz transformation with

counterterms can be applied. Condition (ii) holds because $\lambda \notin \mathcal{R}_2$, and the second transformation can be applied. Because of (6) and (7) we obtain, with the aid of (10),

$$\rho_1 = \sum a_i b_j f_l^{(i)} f_l^{(j)} \frac{i\mu}{g_l'^2 - 4\mu^2} = -\rho_2$$

modulo integrable terms. Since this sum is easily seen to be completely imaginary, Lemma 2 shows that $X = (1 + o(1))$. $\square$

Thus both solutions of (1) look like generalized plane waves, and there is no subordinate solution in this case.

Remark 1. The proof of the theorem could have been completed also as in [1] without recourse to subordinacy.

Remark 2. The conditions of the theorem tell us that the essential spectrum of the Hamiltonian contains $[-m, m]^c$, respectively $(0, \infty)$. In fact, under very mild conditions on the behaviour of the potentials at 0, one has equality. In this case the theorem states that the Hamiltonian has no singular continuous spectrum and that eigenvalues embedded in the continuum can at most be found in $\mathcal{R}_1 \cup \mathcal{R}_2$.

It is quite obvious that this theorem can be generalized in several ways. Condition (10) for example can be altered, while still retaining the result. For rapidly oscillating summands for example, in most cases, the condition $f/g' \in \mathcal{L}^3$ will suffice.

Secondly it is possible to allow infinite sums in (10) as long as all convergence conditions needed for the proof are satisfied. These are

- (i) $\mathcal{R}_1 \cup \mathcal{R}_2$ has no point of accumulation in $[-m, m]^c$;
- (ii) $\sum |f_j^{(i)}|$ converges;
- (iii) $\sum \|f_j^{(i)}\|_3$ and $\sum \|f_j^{(i)'}\|_1$ converge.

Finally the result can be extended to $\mathcal{L}^p$ -potentials with $p \geq 4$, leading to higher order resonances.

Gilbert and Pearson have shown their main result in [4] by an analysis of Titchmarsh Weyl m -functions. It is therefore natural to investigate the m -function also in this case. This means that one has to solve (1) for complex λ . In this case one can show

Lemma 3. Let $\mathcal{P}$, $\mathcal{S}$ and W be as above satisfying (2), (3), and (10). Moreover let λ_1 and $\varepsilon > 0$ be such that $[\lambda_1 - \varepsilon, \lambda_1 + \varepsilon] \subset \mathbb{R} \setminus ([-m, m] \cup \mathcal{R}_1 \cup \mathcal{R}_2)$. Then, for any $\lambda \in K_\varepsilon = \{\lambda | 0 \leq \operatorname{Im} \lambda \leq \varepsilon, \lambda_1 - \varepsilon \leq \operatorname{Re} \lambda \leq \lambda_1 + \varepsilon\}$, (1) has a fundamental matrix

$$U(t, \lambda) = A(t, \lambda)(1 + o(1)) \operatorname{diag}(e^{i\sigma(t, \lambda)}, e^{-i\sigma(t, \lambda)}),$$

which is continuous (analytic) in K_ε , respectively $\overset{\circ}{K}_\varepsilon$.

Proof. The proof follows the lines of Lemma 2. $\mu = \mu(t, \lambda)$ however will be defined such that $\operatorname{Re} i\mu(\lambda, t) \leq 0$ for $\lambda \in K_\varepsilon$. Since Σ is not bounded any

more, the Harris and Lutz transformation [6] in the form used by Ben Artzi, Devinatz [2] will be applied to (4) directly. In this case the transformation matrix corresponding to $\tilde{B}$ is given by

$$\tilde{B} = \begin{pmatrix} 0 & m_1 \\ m_2 & 0 \end{pmatrix},$$

where, with the notation of (6),

$$m_1(t) = e^{2i\sigma(t)} \left[\int_c^t e^{-2i\sigma} (q_1 + l_1) ds - \int_c^\infty e^{-2i \operatorname{Re} \sigma} (q_1 + l_1) ds \right]$$

and

$$m_2(t) = \int_\infty^t e^{-2i\sigma(t)+2i\sigma(s)} [q_2 + l_2] ds.$$

For the necessary estimates appropriate counterterms l_1 and l_2 play an important role. Using this transformation twice reduces (4) to Levinson's form. $\square$

A simple consequence of this lemma is

Corollary. *The m -function for (1) with a regular left hand endpoint 0 is continuous in K_ε . In particular,*

$$\lim_{\lambda_2 \rightarrow 0+} m(\lambda_1 + i\lambda_2) = m_+(\lambda_1)$$

exists for all $\lambda_1 \in \mathbf{R} \setminus ([-m, m] \cup \mathcal{R}_1 \cup \mathcal{R}_2)$.

This corollary and Lemma 3 also lead to a limiting absorption principle.

Proposition. *Let $\mathcal{P}$, $\mathcal{S}$, and W be as above with (2), (3), and (10). Moreover assume 0 to be a regular endpoint of differential operator (1). Then*

$$\sup_{\lambda \in K_\varepsilon^0} \|M_s(\lambda - \mathcal{H})^{-1} M_s\| < \infty, \quad \text{for } s > 1/2,$$

where M_s is the operator of multiplication by $(1 + t^2)^{-s}$. Moreover

$$\lim_{\lambda_2 \rightarrow 0+} M_s(\lambda_1 + i\lambda_2 - \mathcal{H})^{-1} M_s = M_s R^+ M_s$$

exists in the norm topology.

Proof. For $\operatorname{Im} \lambda = \lambda_2 > 0$ the resolvent $(\lambda - \mathcal{H})^{-1} = R_\lambda$ is an integral operator. Thus, also, $M_s R_\lambda M_s$ is an integral operator. Its boundedness can be shown by the Schur test [9, Chapter 4.1, Theorem 4.12] with $\phi \equiv 1$ as a test function. Because of Lemma 3 it suffices to prove this only for the functions $e^{i\sigma(t, \lambda)}$. The same method also shows the convergence result. $\square$

Remark. Lemma 3, the corollary and the proposition are valid also for Schrödinger operators. The corollary and the proposition can also be shown for potentials which are singular at 0, e.g., Coulomb-like potentials. In this case ϕ should be chosen as $\phi(t) = t^{-\beta}$ for $t \in (0, 1]$ and $\phi(t) = 1$ for $t \geq 1$.

The corollary and the proposition also imply that the spectrum of the Hamiltonian is absolutely continuous off $\mathcal{R}_1 \cup \mathcal{R}_2$. In principle this would be a third proof of this result.

III. EXAMPLES

In these examples we shall investigate various resonance phenomena connected with $\mathcal{R}_1 \cup \mathcal{R}_2$. Mathematically these systems are treated as above, i.e., the conditionally integrable terms are transformed away with a Harris and Lutz transformation, while nonintegrable terms, which appear intermittently, are treated by a diagonalization procedure.

It should be noted however that an analysis like Lemma 3 cannot be carried over to this situation, because, in general, m has singularities at $\mathcal{R}_1 \cup \mathcal{R}_2$.

Of course one cannot expect in general that each value in $\mathcal{R}_1 \cup \mathcal{R}_2$ corresponds to a solution which is subordinate at ∞ . For this to occur one needs much more than the matching of the frequencies g' and 2μ at infinity. But $\mathcal{R}_1$ and $\mathcal{R}_2$ describe just that. Thus, if g' and 2μ differ sufficiently strongly, no subordinate solution will exist even if $\lim g'(t) - 2\mu(t, \lambda) = 0$ (Example 1). Optimal matching of the potential and wavefunction will occur if $2\sigma = g$. If this and related conditions hold, subordinate solutions, which may be square integrable even, will exist in general. Not every solution which is square integrable at infinity defines a bound state of the Hamiltonian $\mathcal{H}$. For this to occur, this solution has to belong to the domain of $\mathcal{H}$. Thus, if 0 is regular or quasi regular, the boundary conditions at 0 have to be satisfied. The same remarks also hold for subordinate solutions.

In order to simplify the examples as much as possible, while still retaining the characteristic behaviour, we assume

$$(12) \quad \mathcal{S} = 0, \quad \mathcal{P} = \mathcal{P}(\infty), \quad W_2 = 0 \quad \text{and} \quad W_i = d_i W.$$

Moreover we assume

$$(13) \quad W = c_1 t^{-\alpha_1} \sin g_1 + c_2 t^{-\alpha_2} \sin g_2, \quad 1/3 < \alpha_i \leq 1.$$

Then $\mu = \mu(\infty)$, $\sigma(t) = \mu \cdot (t - c)$ and, with an appropriate constant b , (5) and (6) can be written as

$$(14) \quad x' = W \begin{pmatrix} i & be^{-2i\sigma} \\ be^{2i\sigma} & -i \end{pmatrix} x.$$

Here the Schrödinger case corresponds to $b = 1$, while the usual Dirac case with $W_1 = W_3$ leads to $b = (1 - a^2)/(1 + a^2)$.

Example 1. Assume, above, $\alpha_1 = \alpha_2 = 1$ and $g_i = 2\mu(\lambda_i)t + t^{\beta_i}$ with $1/2 < \beta_i < 1$ then $\mathcal{R}_1 = \{\lambda_1, \lambda_2\}$, but (4), respectively (1), has no solution, which is subordinate at infinity.

Example 2. If we assume $g_1 = 2\mu(\lambda)t$ and $g_2 = et$ and $e \neq 2\mu(\lambda)$, $4\mu(\lambda)$, then $\lambda \in \mathcal{R}_1 \setminus \mathcal{R}_2$. By successive Harris and Lutz transformations, (14) can be

brought into the form

$$(15) \quad z' = \left[\begin{pmatrix} il & s \\ \bar{s} & -il \end{pmatrix} + R \right] z$$

with $l = b^2[-(1/16\mu)f_1^2 + (\mu/(e^2 - 4\mu^2))f_2^2]$ and $s = (i/2)bf_1$. This system can be solved approximately by diagonalizing it, if Levinson's dichotomy condition, in the form $-l^2 + s\bar{s}$ does not change sign far out, holds. This is obviously fulfilled in our situation. Thus (15) has a fundamental matrix

$$Z(t) = C(t) \text{diag}(\exp k, \exp -k)(1 + o(1)),$$

where $C(t)$ is the diagonalizing matrix and where $k = \int_c^t (-l^2 + s\bar{s})^{1/2} ds$.

If now $2\alpha_2 > \alpha_1$, then $l = o(f_1)$ and the matrix in (15) is essentially skew if $c_1, c_2, b \neq 0$. The subordinate solution behaves in both components approximately like $\exp -(1/2) \int_c^t |f_1(s)| ds$. It is thus always square integrable if $\alpha_1 < 1$. If $\alpha_1 = 1$ the subordinate solution decays like a power. By choosing the constants appropriately, one obtains square integrable or arbitrarily slowly decaying solutions.

If we choose $2\alpha_2 < \alpha_1$ the matrix is already approximately diagonal. No subordinate solution exists in this case. Thus interference from the second wave causes the first resonance to disappear.

Example 3. Of all subcases $\lambda \in \mathcal{R}_2$, $\lambda \notin \mathcal{R}_1$ we shall only discuss $g_1 = (1+t)^{1+\gamma} = g$, $\gamma > 0$, and $g_2 = g_1 + 2\sigma(t, \lambda)$. Then $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \{\lambda\}$. Proceeding as above we obtain (15) with

$$l = b^2 \left[f_1^2 \frac{\mu}{g'^2 - 4\mu^2} + f_2^2 \frac{\mu}{g'^2 + 4g'\mu} \right] \quad \text{and} \quad s = -bf_1f_2 \frac{\mu}{g'(g' + 2\mu)}.$$

If $\alpha_1 \neq \alpha_2$ a subordinate solution will not exist. For $f_1 = f_2 = f$ and $|b| < 1/4$, subordinate solutions will exist. These are square integrable even if $\alpha_1 = \alpha_2 < 1/2$ or if the coefficients are chosen appropriately. Such solutions correspond to a resonance in beats. As such they lead to a new type of bound states embedded in the continuum.

The same phenomenon can also be observed for the Schrödinger equation, even if one chooses $g_1 = e_1 t$ and $g_2 = e_2 t$ such that $g_1 + g_2 = 2\sigma$.

It is obvious that corresponding higher order resonance states can be constructed for potentials decaying more weakly.

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