INVARIANT SIGNED MEASURES AND THE CANCELLATION LAW

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ABSTRACT. Let X be a set, and let the group G act on X . We show that, for every $A, B \subset X$, the following are equivalent: (i) A and B are G -equidecomposable; and (ii) $\vartheta(A) = \vartheta(B)$ for every G -invariant finitely additive signed measure ϑ . If the sets and the pieces of the decompositions are restricted to belong to a given G -invariant field $\mathcal{A}$, then (i) $\Leftrightarrow$ (ii) if and only if the cancellation law ($n[A] = n[B] \Rightarrow [A] = [B]$) holds in the space $(X, G, \mathcal{A})$. We show that the cancellation law may fail even if the transformation group G is Abelian.

1. A CRITERION FOR EQUIDECOMPOSABILITY

Let X be a nonempty set and G be a group of bijections of X onto itself. The sets $A, B \subset X$ are said to be G -equidecomposable, if there are decompositions $A = \bigcup_{i=1}^n A_i$, $B = \bigcup_{i=1}^n B_i$ and maps $g_i \in G$ such that $A_i \cap A_j = B_i \cap B_j = \emptyset$, $1 \leq i < j \leq n$, and $B_i = g_i(A_i)$, $i = 1, \dots, n$. We shall denote this fact by $A \stackrel{G}{\sim} B$.

Obviously, if $A \stackrel{G}{\sim} B$ then $\mu(A) = \mu(B)$ holds for every G -invariant finitely additive measure μ . However, this condition is not sufficient for $A \stackrel{G}{\sim} B$, as the following simple example shows.

Let $X = \mathbb{Q}$ be the set of rationals, and let G denote the group of all translations by rational numbers. Let μ be any G -invariant finitely additive measure on $\mathbb{Q}$. If $\mu([0, 1) \cap \mathbb{Q}) = \infty$, then obviously $\mu([0, 1] \cap \mathbb{Q}) = \infty$. If $\mu([0, 1) \cap \mathbb{Q}) < \infty$, then $\mu(\{x\}) = 0$ for every $x \in \mathbb{Q}$ and hence $\mu([0, 1] \cap \mathbb{Q}) = \mu([0, 1) \cap \mathbb{Q})$. That is, $\mu([0, 1] \cap \mathbb{Q}) = \mu([0, 1) \cap \mathbb{Q})$ holds for every G -invariant finitely additive measure μ . On the other hand, it is easy to see that $[0, 1] \cap \mathbb{Q}$ and $[0, 1) \cap \mathbb{Q}$ are not equidecomposable [3, Theorem 17, p. 48].

We shall prove that if $\vartheta(A) = \vartheta(B)$ holds for every G -invariant finitely additive *signed* measure, then necessarily $A \stackrel{G}{\sim} B$. However, in this criterion we cannot restrict ϑ to finite valued signed measures. Indeed, let $X = \mathbb{Z}$ be the

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set of integers and let G be the group of translations by integers. Let $\mathbf{N}$ and $\mathbf{N}^+$ denote the sets of nonnegative integers and positive integers, respectively. If ϑ is a finite valued G -invariant signed measure on $\mathbf{Z}$ then $\vartheta(\mathbf{N}) = \vartheta(\mathbf{N}^+)$ and hence $\vartheta(\{0\}) = 0 = \vartheta(\emptyset)$. On the other hand, $\{0\}$ and $\emptyset$ are not equidecomposable.

Therefore, by a G -invariant finitely additive signed measure we shall mean a map ϑ from the subsets of X into $\mathbf{R} \cup \{\infty\}$ such that

$$(1) \quad \vartheta(g(A)) = \vartheta(A), \quad A \subset X, g \in G,$$

and

$$(2) \quad \vartheta(A \cup B) = \vartheta(A) + \vartheta(B), \quad A, B \subset X, A \cap B = \emptyset,$$

where we adopt the convention

$$\infty + \infty = \infty + a = \infty, \quad a \in \mathbf{R}.$$

Theorem 1. For every $A, B \subset X$ we have $A \stackrel{G}{\sim} B$ if and only if $\vartheta(A) = \vartheta(B)$ holds for every G -invariant finitely additive signed measure ϑ .

We shall also consider the more general situation where the sets A, B , and the pieces used in the decompositions are restricted to be in a prescribed ring of subsets of X .

By a *space* we shall mean a triple $(X, G, \mathcal{A})$, where X is a non-empty set, G is a group of bijections of X onto itself, and $\mathcal{A}$ is a G -invariant ring of subsets of X . We say that the sets $A, B \in \mathcal{A}$ are G -equidecomposable in $\mathcal{A}$, if they are G -equidecomposable in such a way that the pieces used in the decompositions belong to $\mathcal{A}$. A map $\vartheta : \mathcal{A} \rightarrow \mathbf{R} \cup \{\infty\}$ is said to be a G -invariant finitely additive signed measure on $\mathcal{A}$, if (1) and (2) hold with A, B restricted to be elements of $\mathcal{A}$.

Our aim is to characterize those spaces in which the conditions (i) $A \stackrel{G}{\sim} B$ in $\mathcal{A}$; and (ii) $\vartheta(A) = \vartheta(B)$ whenever ϑ is a G -invariant finitely additive signed measure on $\mathcal{A}$ are equivalent. Obviously, (i) $\Rightarrow$ (ii) in every space. We shall prove that (ii) $\Rightarrow$ (i) if and only if the cancellation law holds in $(X, G, \mathcal{A})$. In order to describe this condition, we shall need the notion of type semigroup.

Roughly speaking, the elements of the type semigroup are the equivalence classes under the equivalence relation $\stackrel{G}{\sim}$ in $\mathcal{A}$, and if a, b are the classes containing the sets A and B , respectively, then $a + b$ is defined as the class containing the set $A' \cup B'$, where A', B' are "disjoint copies" of A and B . However, as disjoint copies of A and B do not necessarily exist in X , we have to enlarge X and the action G as follows (cf. [7, Chapter 8, pp. 109–110]). Let $X^* = X \times \mathbf{N}$, and let $\mathcal{A}^*$ be the ring of sets $\bigcup_{i=0}^n A_i \times \{i\}$, $n \in \mathbf{N}$, $A_i \in \mathcal{A}$, $i = 0, \dots, n$. If $g \in G$ and π is a permutation of $\mathbf{N}$, then we define the map (g, π) by

$$(g, \pi)(x, n) = (g(x), \pi(n)), \quad x \in X, n \in \mathbf{N}.$$

Obviously, the set G^* of all these maps (g, π) forms a group of bijections of X^* onto itself, and $\mathcal{A}^*$ is a G^* -invariant ring. The semigroup types of

the space $(X, G, \mathcal{A})$ are defined as the equivalence classes with respect to the equivalence relation $\overset{G}{\sim}$ in $\mathcal{A}^*$. If $[A]$ and $[B]$ denote the classes containing the sets $A, B \in \mathcal{A}^*$, respectively, then $[A] + [B]$ is defined as the class containing the set $g_1(A) \cup g_2(B)$, where $g_1, g_2 \in G^*$ are such that $g_1(A) \cap g_2(B) = \emptyset$. It is easy to check that this operation is well-defined and makes the set of types a commutative semigroup. If we identify X with $X \times \{0\}$ then, for every $A, B \in \mathcal{A}$, we have $A \overset{G}{\sim} B$ in $\mathcal{A}$ if and only if $A \overset{G^*}{\sim} B$ in $\mathcal{A}^*$. That is, $[A] = [B]$ if and only if $A \overset{G}{\sim} B$ in $\mathcal{A}$.

In this section our aim is to prove the following theorem.

Theorem 2. *Let $(X, G, \mathcal{A})$ be a space and $A, B \in \mathcal{A}$ be arbitrary. Then $\vartheta(A) = \vartheta(B)$ holds for every G -invariant finitely additive signed measure on $\mathcal{A}$ if and only if there is a positive integer n such that $n[A] = n[B]$.*

We say that the cancellation law holds in the space $(X, G, \mathcal{A})$ if, for every $A, B \in \mathcal{A}$ and $n \in \mathbf{N}^+$, $n[A] = n[B]$ implies $[A] = [B]$. By Theorem 2, if the cancellation law holds in $(X, G, \mathcal{A})$, then conditions (i) and (ii) above are equivalent.

On the other hand, if the cancellation law does not hold, then (ii) does not imply (i). Indeed, let $A, B \in \mathcal{A}$ and $n \in \mathbf{N}^+$ be such that $n[A] = n[B]$ but $[A] \neq [B]$. It is easy to see that $n[A] = n[B]$ implies (ii) and hence (ii) $\Rightarrow$ (i) does not hold. We have proved the following.

Corollary. *In every space, conditions (i) and (ii) are equivalent if and only if the cancellation law holds.*

It is well-known that the cancellation law holds if $\mathcal{A}$ is the field of all subsets of X [7, Theorem 8.7, p. 112], and thus Theorem 1 follows.

Now we turn to the proof of Theorem 2. If ϕ is a homomorphism from the type semigroup into the additive semigroup $\mathbf{R} \cup \{\infty\}$, then

$$\vartheta(A) \stackrel{\text{def}}{=} \phi([A]) \quad A \in \mathcal{A},$$

defines a G -invariant finitely additive signed measure on $\mathcal{A}$. Therefore the statement of Theorem 2 is an immediate consequence of the following result.

Theorem 3. *Let $(S, +)$ be a commutative semigroup, and let $a, b \in S$ be such that $\phi(a) = \phi(b)$ for every homomorphism $\phi : S \rightarrow \mathbf{R} \cup \{\infty\}$. Then there is a positive integer n such that $na = nb$.*

Proof. Let the relation Θ be defined by $x\Theta y$ if $nx = ny$ for some $n \in \mathbf{N}^+$. It is easy to see that Θ is a congruence; that is, Θ is an equivalence relation on S such that $x\Theta y$ implies $(x+z)\Theta(y+z)$ for every $z \in S$. Let $S_1 = S/\Theta$ be the factor semigroup and ψ be the natural homomorphism from S into S_1 . Obviously, the cancellation law holds in S_1 ; i.e., if $x, y \in S_1$ and $nx = ny$ for some $n \in \mathbf{N}^+$ then $x = y$. If ϕ is any homomorphism from S_1 into $\mathbf{R} \cup \{\infty\}$ then the composition of ψ and ϕ will be a homomorphism from S into $\mathbf{R} \cup \{\infty\}$.

Therefore, replacing S by S_1 if necessary, we may assume that the cancellation law holds in S . We have to prove that, under this condition, distinct elements of S can be separated by homomorphisms mapping into $\mathbf{R} \cup \{\infty\}$.

We define the relation $\leq$ on S by putting $x \leq y$ if there is a $z \in S$ and $n \in \mathbf{N}^+$ such that $x + z = ny$. (This relation is transitive and reflexive but, in general, is not antisymmetric.)

Lemma 1. (i) If $x, y, z \in S$, $n \in \mathbf{N}^+$ and $x + nz = y + nz$, then $x + z = y + z$.

(ii) If $x, y, z \in S$, $z \leq x$, $z \leq y$ and $x + z = y + z$, then $x = y$.

Proof. If $x + 2z = y + 2z$ then

$$\begin{aligned} 2(x + z) &= x + (x + 2z) \\ &= x + (y + 2z) = y + (x + 2z) \\ &= y + (y + 2z) = 2(y + z) \end{aligned}$$

and hence $x + z = y + z$. This proves (i) for $n = 2$. The general statement follows by induction.

In order to prove (ii), let $u, v \in S$; $k, n \in \mathbf{N}^+$ be such that $z + u = nx$, $z + v = ky$. Putting $N = nk$, $u_1 = (k - 1)z + ku$, $v_1 = (n - 1)z + nv$ we have $z + u_1 = Nx$, $z + v_1 = Ny$. Hence

$$\begin{aligned} u_1 + (N + 1)z &= Nx + Nz \\ &= N(x + z) = N(y + z) \\ &= Ny + Nz = v_1 + (N + 1)z. \end{aligned}$$

By (i), this implies $u_1 + z = v_1 + z$. Therefore $Nx = Ny$ and hence $x = y$. $\square$

We shall say that a subsemigroup $G \subset S$ is *dense* in S if, for every $x \in S$, there is a $y \in G$ such that $x + y \in G$.

Lemma 2. If G is a dense subsemigroup of S and ϕ is a homomorphism from G into the additive semigroup of the reals, then ϕ can be extended to S as a homomorphism.

Proof. By Zorn's lemma, it is enough to show that if $c \in S \setminus G$ then ϕ can be extended to the semigroup H generated by G and c .

First we show that if $x, y, u, v \in G$, $n, k \in \mathbf{N}^+$ and $x = u + nc$, $y = v + kc$, then

$$(3) \quad \frac{\phi(x) - \phi(u)}{n} = \frac{\phi(y) - \phi(v)}{k}.$$

Indeed, we have $kx + nv = ku + nv + knc = ku + ny$ and hence $k\phi(x) + n\phi(v) = k\phi(u) + n\phi(y)$, which gives (3).

If there are $x, u \in G$ and $n \in \mathbf{N}^+$ such that $x = u + nc$ then we define $\phi(c) = (\phi(x) - \phi(u))/n$. By (3), in this case $\phi(c)$ is well-defined. If $x = u + nc$ is not solvable in G then we define $\phi(c)$ arbitrarily.

We may assume that S has a zero element which is also contained in G . Then every element of H is of the form $x + nc$, $x \in G$, $n \in \mathbf{N}$. We define

$\phi(x + nc) = \phi(x) + n\phi(c)$. In order to show that this definition makes sense we have to prove that

$$x + nc = y + kc, \quad x, y \in G; n, k \in \mathbb{N},$$

implies

$$(4) \quad \phi(x) + n\phi(c) = \phi(y) + k\phi(c).$$

Suppose $n \leq k$. Since G is dense in S there are elements $d \in S$, $e \in G$ such that $c + d = e$. Then we have $d + x + nc = d + y + kc$. By (i) of Lemma 1, this implies $d + x + c = d + y + (k - n)c + c$, or $e + x = e + y + (k - n)c$. Since $e, x, y \in G$, it follows from the definition of $\phi(c)$ that

$$\phi(e) + \phi(x) = \phi(e) + \phi(y) + (k - n)\phi(c),$$

from which (4) is obtained. In this way we have extended ϕ to H , and a similar argument shows that ϕ is a homomorphism from H to $\mathbb{R}$. $\square$

Lemma 3. *Let C denote the semigroup generated by the distinct elements $a, b \in S$. If $a \leq b$ and $b \leq a$ then there is a homomorphism $\phi : C \rightarrow (\mathbb{R}, +)$ such that $\phi(a) \neq \phi(b)$.*

Proof. We suppose first that the equation

$$(5) \quad na + kb = ma$$

has no solution in positive integers n, k, m . In this case we define $\phi(ia) = 0$ for every $i \in \mathbb{N}^+$. Then ϕ is a homomorphism from the subsemigroup $A = \{ia : i \in \mathbb{N}^+\}$ into $\mathbb{R}$. Since $b \leq a$, A is dense in C and hence, as we saw in the proof of Lemma 2, ϕ can be extended to C such that $\phi(b)$ is chosen arbitrarily.

Next suppose that there are positive integers n, k, m satisfying (5). Then the order of a is infinite; that is, $i \neq j$ implies $ia \neq ja$. Suppose this is not true, and let $pa = (p + q)a$, where $p, q \in \mathbb{N}^+$. Then $pa = (p + iq)a$ for every $i \in \mathbb{N}$. Multiplying (5) by a large integer we may assume that $n, m > p$ and $k > q$. Then we have

$$(6) \quad (n + iq)a + kb = (m + jq)a$$

for every $i, j \in \mathbb{N}^+$. We choose i and j such that

$$0 < (m + jq) - (n + iq) = N \leq q$$

holds. Since $a \leq b$ and $a \leq a$, (6) implies, by (ii) of Lemma 1, that $kb = Na$. Therefore $skb = (sN + tq)a$ whenever $s, t \in \mathbb{N}^+$ and $sN \geq p$. We can choose s, t such that $sk = sN + tq$, since $k > q \geq N$. This implies $b = a$, which is impossible.

We have proved that the order of a is infinite. Then $\phi(na) = n$ defines a homomorphism of $A = \{na : n \in \mathbb{N}^+\}$ into $\mathbb{R}$. As we saw in the proof of Lemma 2, ϕ can be extended to C such that

$$(7) \quad n\phi(a) + k\phi(b) = m\phi(a).$$

We claim that $\phi(b) \neq 1$. Indeed, if $\phi(b) = 1$ then (7) gives $k = m - n > 0$. Thus, by (5) and by (ii) of Lemma 1, we have $kb = ka$ and $b = a$, which is impossible. $\square$

Now we turn to the proof of Theorem 3. We have to show that if $a \neq b$ then there is a homomorphism ϕ of S into $\mathbf{R} \cup \{\infty\}$ which separates a and b (recall that the cancellation law holds in S).

Suppose first that $a \leq b$ does not hold. Then

$$\phi(x) = \begin{cases} 0, & \text{if } x \leq b, \\ \infty, & \text{otherwise} \end{cases}$$

defines a homomorphism such that $\phi(b) = 0$ and $\phi(a) = \infty$. If $b \leq a$ does not hold then we can find a separating homomorphism in the same way. Therefore we may assume that $a \neq b$, $a \leq b$ and $b \leq a$. Then, by Lemma 3, there is a homomorphism ϕ of C into $\mathbf{R}$ which separates a and b . Let $S' = \{x \in S : x \leq a\}$. Then S' is a subsemigroup of S in which C is dense. By Lemma 2, we can extend ϕ to S' . Finally we extend ϕ from S' to S by putting $\phi(x) = \infty$ for every $x \in S \setminus S'$. It is easy to check that ϕ is a homomorphism into $\mathbf{R} \cup \{\infty\}$, and this completes the proof. $\square$

Remark. As E. W. Kiss pointed out, Theorem 3 can be deduced from a theorem of Hewitt and Zuckerman stating that the characters of a commutative and separative semigroup S with identity separate the elements of S [1, Vol. I; Theorem 5.59, p. 198]. Our proof given above is somewhat longer, but self-contained and elementary. The statements of Lemma 1, together with the implication

$$(8) \quad nx + kz = ny + kz \Rightarrow x + z = y + z, \quad n, k \in \mathbf{N}^+,$$

were proved by Tarski in the special case when S is the semigroup of equidecomposability types with unrestricted pieces [4, pp. 221–222; 5, Theorem 16.9, p. 223]. Tarski proved these statements by generalizing König's proof of the cancellation law (see note 9 at the end of [4]). As our Lemma 1 shows, these assertions are direct algebraic consequences of the cancellation law. As for (8), we can argue as follows. By (i) of Lemma 1, the condition of (8) implies $nx + z = ny + z$. This gives $n(x + z) = (nx + z) + (n - 1)z = (ny + z) + (n - 1)z = n(y + z)$, and hence we have $x + z = y + z$ by the cancellation law.

2. SPACES WITHOUT THE CANCELLATION LAW

In [7, Problem 14, p. 231] Stan Wagon asked for a space in which the cancellation law fails. Examples with this property were constructed in [2] and [6]. The transformation groups of the spaces given in [2] and [6] are non-commutative. In this section we present three spaces with Abelian transformation groups, in which the cancellation law fails. These spaces are closely related to each other, but exhibit different properties. In the second example the underlying set X

is a compact Abelian group (the torus) and the transformation group is a subgroup of the translations. The third example is constructed on $\mathbf{R}$ such that the transformation group consists of *all* translations.

In the following examples α and β will denote fixed positive numbers such that α/β is irrational.

Example 1. Let $X_1 = \mathbf{R}^2$, let $\mathcal{R}$ denote the set of rectangles $[a, b) \times [c, d)$, $a < b$, $c < d$, and let $\mathcal{A}_1$ denote the field generated by $\mathcal{R}$. Let $T_1 = \{(x + n\alpha + k\beta, x) : x \in \mathbf{R}; n, k \in \mathbf{Z}\}$, and let G_1 denote the group of translations by vectors from T_1 .

We claim that the cancellation law fails in the space $(X_1, G_1, \mathcal{A}_1)$. Namely, if $A = [0, \alpha) \times [0, \beta/2)$ and $B = [0, \alpha/2) \times [0, \beta)$, then $2[A] = 2[B]$ but $[A] \neq [B]$.

Let $A' = [0, \alpha) \times [\beta/2, \beta)$. First we show that $A \stackrel{G_1}{\sim} A'$. Indeed, let n be an integer such that $n\alpha < \beta/2 < (n+1)\alpha$. Then $A = A_1 \cup A_2$ and $A' = A'_1 \cup A'_2$, where

$$\begin{aligned} A_1 &= [0, (n+1)\alpha - \beta/2) \times [0, \beta/2), & A_2 &= [(n+1)\alpha - \beta/2, \alpha) \times [0, \beta/2), \\ A'_1 &= [\beta/2 - n\alpha, \alpha) \times [\beta/2, \beta), & A'_2 &= [0, \beta/2 - n\alpha) \times [\beta/2, \beta). \end{aligned}$$

Since $A'_1 = A_1 + t_1$, where $t_1 = (\beta/2 - n\alpha, \beta/2) \in T_1$, and $A'_2 = A_2 + t_2$, where $t_2 = (\beta/2 - (n+1)\alpha, \beta/2) \in T_1$, we have $A \stackrel{G_1}{\sim} A'$ in $\mathcal{A}_1$. A similar argument shows $B \stackrel{G_1}{\sim} B'$ in $\mathcal{A}_1$, where $B' = [\alpha/2, \alpha) \times [0, \beta)$. Since $A \cup A' = B \cup B' = [0, \alpha) \times [0, \beta)$, this proves $2[A] = 2[B]$.

Next we show $[A] \neq [B]$. We shall prove that there is a G_1 -invariant additive interval function ϕ such that $\phi(A) \neq \phi(B)$. This will prove $[A] \neq [B]$, since the bounded elements of $\mathcal{A}_1$ are finite unions of pairwise disjoint rectangles, and hence $[X] = [Y]$ implies $\phi(X) = \phi(Y)$ for every $X, Y \in \mathcal{R}$. As we saw above, $2[A] = 2[B]$, and hence $2\phi(X) = 2\phi(Y)$ must hold for every G -invariant additive interval function ϕ . This shows that, if we want to choose ϕ such that $\phi(A) \neq \phi(B)$, then ϕ cannot map $\mathcal{R}$ into $\mathbf{R}$ (or into a group in which $2x = 0$ implies $x = 0$). The interval function we are going to construct will map $\mathcal{R}$ into the circle group $\mathbf{R}/[0, 1)$. That is, we shall construct a *real valued* additive interval function ϕ such that

$$(9) \quad \phi(A) \not\equiv \phi(B) \pmod{1},$$

and

$$(10) \quad \phi(R + t) \equiv \phi(R) \pmod{1}$$

for every $R \in \mathcal{R}$ and $t \in T_1$.

Let $F : \mathbf{R}^2 \rightarrow \mathbf{R}$ be an arbitrary function. It is well-known that

$$\phi([a, b) \times [c, d)) = F(a, c) - F(b, c) + F(b, d) - F(a, d)$$

defines an additive interval function on $\mathcal{R}$. Our aim is to choose F in such a way that (9) and (10) are satisfied.

Since $\alpha/\beta \notin \mathbf{Q}$, there are functions $f, g: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x+y) = f(x)+f(y)$, $g(x+y) = g(x)+g(y)$, for every $x, y \in \mathbf{R}$, and $f(\alpha) = g(\beta) = 1$, $f(\beta) = g(\alpha) = 0$. (Choose a Hamel basis containing α and β , and define $f(x)$ and $g(x)$ as the coefficients of α and β in the representation $x = \sum r_u \cdot u$, where u runs through the elements of H , $r_u \in \mathbf{Q}$ for every u , and $r_u = 0$ for all but finitely many $u \in H$.)

We define

$$F(x, y) = f(x-y)[g(x-y)], \quad x, y \in \mathbf{R},$$

where $[\cdot]$ denotes the integer part. Easy calculation shows

$$\begin{aligned} F(0, 0) = F(\alpha/2, 0) = F(\alpha, 0) = F(0, \beta/2) = F(0, \beta) = 0, \\ F(\alpha/2, \beta) = -1/2, \quad F(\alpha, \beta/2) = -1, \end{aligned}$$

and hence $\phi(A) = -1$ and $\phi(B) = -1/2$. This shows (9). In order to prove (10), it is enough to check that (10) holds in the following cases: $t = (z, z)$ ($z \in \mathbf{R}$), $t = (\alpha, 0)$, and $t = (\beta, 0)$.

The case of $t = (z, z)$ is obvious, since $F(x+z, y+z) = F(x, y)$ holds for every $x, y, z \in \mathbf{R}$, and hence we have (10) with equality in this case.

If $t = (\alpha, 0)$, then

$$\begin{aligned} F(x+\alpha, y) &= f(x-y+\alpha)[g(x-y+\alpha)] \\ &= (f(x-y)+1)[g(x-y)] \\ &\equiv f(x-y)[g(x-y)] \\ &= F(x, y) \pmod{1}, \end{aligned}$$

from which (10) easily follows.

Finally, let $t = (\beta, 0)$. Then

$$\begin{aligned} F(x+\beta, y) &= f(x-y+\beta)[g(x-y+\beta)] \\ &= f(x-y)[g(x-y)+1] \\ &= f(x-y)[g(x-y)] + f(x-y) \\ &= F(x, y) + f(x-y) \end{aligned}$$

for every $x, y \in \mathbf{R}$.

If $R = [a, b) \times [c, d)$ then this implies

$$\phi(R + (\beta, 0)) = \phi(R) + (f(a-c) - f(b-c) + f(b-d) - f(a-d)) = \phi(R),$$

which completes the proof.

Example 2. Let $X_2 = \{(x, y) : 0 \leq x < \alpha, 0 \leq y < \beta\}$ be the torus, where addition is defined mod α in the first coordinate and mod β in the second coordinate. Let $T_2 = \{(x, y) \in X_2 : y = x + n\alpha + k\beta, n, k \in \mathbf{Z}\}$; then T_2 is a subgroup of X_2 . Let G_2 denote the group of translations by elements of T_2 , and let $\mathcal{A}_2$ be the field generated by the rectangles $[a, b) \times [c, d)$, $0 \leq a < b \leq \alpha, 0 \leq c < d \leq \beta$.

Then the cancellation law fails in the space $(X_2, G_2, \mathcal{A}_2)$, since, for the sets $A = [0, \alpha) \times [0, \beta/2)$ and $B = [0, \alpha/2) \times [0, \beta)$, we have $2[A] = 2[B]$ and $[A] \neq [B]$. This follows directly from the considerations of Example 1.

Example 3. Let $X_3 = \mathbf{R}$, and let G_3 be the group of all translations of $\mathbf{R}$. Let

$$A = \bigcup_{n=-\infty}^{\infty} \left[n\alpha, \left(n + \frac{1}{2}\right)\alpha \right), \quad B = \bigcup_{n=-\infty}^{\infty} \left[n\beta, \left(n + \frac{1}{2}\right)\beta \right),$$

and let $\mathcal{A}_3$ be the translation-invariant field generated by A and B . We claim that the cancellation law fails in the space $(X_3, G_3, \mathcal{A}_3)$, because $2[A] = 2[B]$ but $[A] \neq [B]$.

Since $\mathbf{R} = A \cup (A + \alpha/2) = B \cup (B + \beta/2)$, we have $2[A] = 2[B]$. For the proof of $[A] \neq [B]$ we shall need a lemma.

Lemma 4. Let $\mathcal{B}$ and $\mathcal{C}$ be fields of subsets of a set X . Then the field generated by $\mathcal{B}$ and $\mathcal{C}$ consists of the sets $\bigcup_{i=1}^n (B_i \cap C_i)$, where $n \in \mathbf{N}^+$, $B_i \in \mathcal{B}$, $C_i \in \mathcal{C}$ for every $i = 1, \dots, n$, and the sets $B_i \cap C_i$, $i = 1, \dots, n$, are pairwise disjoint.

Proof. The family of sets $\bigcup_{i=1}^n (B_i \cap C_i)$, $B_i \in \mathcal{B}$, $C_i \in \mathcal{C}$, is closed under union and complementation, and hence it is the field generated by $\mathcal{B}$ and $\mathcal{C}$. Successive applications of the identity

$$(B \cap C) \setminus (B' \cap C') = ((B \setminus B') \cap (C \setminus C')) \cup ((B \setminus B') \cap (C \cap C')) \cup ((B \cap B') \cap (C \setminus C'))$$

transform these sets into unions in which the terms $B_i \cap C_i$ are pairwise disjoint. $\square$

Now we turn to the proof of $[A] \neq [B]$. For every positive real number c we shall denote by $\mathcal{F}_c$ the family of sets of the form $\bigcup_{n=-\infty}^{\infty} (H + nc)$, where H is the union of finitely many pairwise disjoint intervals $[x, y) \subset [0, c)$. It is easy to see that $\mathcal{F}_c$ is a translation invariant field for every $c > 0$. Obviously, $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$. Let $\mathcal{F}$ denote the field generated by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$. Since $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are translation invariant, so is $\mathcal{F}$ and hence $\mathcal{A}_3 \subset \mathcal{F}$.

Suppose that $A \stackrel{G_3}{\sim} B$ in $\mathcal{A}_3$. Then there are decompositions $A = \bigcup_{i=1}^m A_i$, $B = \bigcup_{i=1}^m B_i$ and real numbers x_i such that $A_i, B_i \in \mathcal{F}$ and $B_i = A_i + x_i$ for every $i = 1, \dots, m$. By Lemma 4, each A_i is a finite union of pairwise disjoint sets of the form $C \cap D$, where $C \in \mathcal{F}_\alpha$ and $D \in \mathcal{F}_\beta$. We may assume that each A_i itself is of this form, i.e., $A_i = C_i \cap D_i$, where

$$C_i = \bigcup_{n=-\infty}^{\infty} (H_i + n\alpha), \quad D_i = \bigcup_{n=-\infty}^{\infty} (K_i + n\beta),$$

H_i is a finite union of half-open subintervals of $[0, \alpha)$ and K_i is a finite union of half-open subintervals of $[0, \beta)$. We may also suppose that $C_i \cap D_i \neq \emptyset$ for every i . Our next aim is to show

$$(11) \quad \bigcup_{i=1}^m (H_i \times K_i) = [0, \alpha/2) \times [0, \beta).$$

Let $x \in [0, \alpha/2)$ and $y \in [0, \beta)$ be arbitrary. Since α/β is irrational, the set $\{x + n\alpha + k\beta : n, k \in \mathbf{Z}\}$ is everywhere dense in $\mathbf{R}$. Thus there is a sequence $x + n_j\alpha + k_j\beta$, $j \in \mathbf{N}$, converging to y from above. Since $x \in [0, \alpha/2)$, we have $x + n_j\alpha \in A$ for every j . Hence there is an $i \in \{1, \dots, m\}$ such that $x + n_j\alpha \in C_i \cap D_i$ for infinitely many j . Then $x \in C_i$ and, as $x \in [0, \alpha/2)$, we have $x \in H_i$. Also, $x + n_j\alpha + k_j\beta \in D_i$ for infinitely many j and hence, as $y \in [0, \beta)$ and K_i consists of half-closed intervals, we have $y \in K_i$. Therefore $(x, y) \in H_i \times K_i$, and thus the left hand side of (11) contains the right hand side.

To prove the reverse inclusion let $1 \leq i \leq m$, $x \in H_i$ and $y \in K_i$ be arbitrary. We have to prove that $x \in [0, \alpha/2)$.

There is a $\delta > 0$ such that $[x, x + \delta) \subset H_i$ and $[y, y + \delta) \subset K_i$. Since the set $\{n\alpha - k\beta : n, k \in \mathbf{Z}\}$ is everywhere dense, there are integers n, k such that

$$(12) \quad ([x, x + \delta) + n\alpha) \cap ([y, y + \delta) + k\beta) \neq \emptyset.$$

Let z be a point of this intersection. Then $z \in C_i \cap D_i \subset A$ and hence $z - n\alpha \in [0, \alpha/2)$. Since $0 \leq x \leq z - n\alpha$, this implies $x < \alpha/2$, proving (11).

Next we show that the sets $H_i \times K_i$ are pairwise disjoint. Suppose this is not true, and let $i \neq j$, $x \in H_i \cap H_j$, $y \in K_i \cap K_j$. Then there is a $\delta > 0$ such that $[x, x + \delta) \subset H_i \cap H_j$ and $[y, y + \delta) \subset K_i \cap K_j$. Using the argument above, we find integers n, k such that (12) holds. Let z be an element of the left-hand side of (12). Then

$$z \in [x, x + \delta) + n\alpha \subset (H_i + n\alpha) \cap (H_j + n\alpha) \subset C_i \cap C_j$$

and

$$z \in [y, y + \delta) + k\beta \subset (K_i + k\beta) \cap (K_j + k\beta) \subset D_i \cap D_j.$$

Hence $z \in (C_i \cap C_j) \cap (D_i \cap D_j)$, which contradicts the fact that the sets $C_i \cap D_i$ are pairwise disjoint.

Therefore (11) gives a decomposition of $[0, \alpha/2) \times [0, \beta)$ into pairwise disjoint rectangles. (Note that H_i and K_i are disjoint unions of intervals.)

By our assumption, $B = \bigcup_{i=1}^m [(C_i \cap D_i) + x_i]$. It follows, by repeating the proof of (11), that

$$(13) \quad \bigcup_{i=1}^m (H'_i \times K'_i) = [0, \alpha) \times [0, \beta/2),$$

where $H'_i = (C_i + x_i) \cap [0, \alpha)$ and $K'_i = (D_i + x_i) \cap [0, \beta)$. It is easy to see that the sets H_i and H'_i , respectively K_i and K'_i , are equidecomposable with two pieces and using translations of the form $x_i + n\alpha$ and $x_i + k\beta$. Thus $H_i \times K_i$ and $H'_i \times K'_i$ are G_1 -equidecomposable in $\mathcal{A}_1$, and hence, by (11) and (13), so are $[0, \alpha/2) \times [0, \beta)$ and $[0, \alpha) \times [0, \beta/2)$. As we saw in Example 1, this is not true, and this contradiction shows $[A] \neq [B]$.

Two problems. The phenomenon showed by Example 2 suggests the following question. Let X be a compact Abelian group, let G be the group of all translations in X and let $\mathcal{A}$ be an arbitrary G -invariant field in X . Does the cancellation law hold in $(X, G, \mathcal{A})$?

S. Wagon asked whether the cancellation law holds for Borel equidecomposability in every locally compact group [7, Problem 14, p. 231]. We remark that the sets A and B of Example 3 are Borel equidecomposable. This fact supports an affirmative answer to Wagon's question at least for Abelian locally compact groups.

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