SINGULARLY CONTINUOUS MEASURES IN NEVAI'S CLASS M

D. S. LUBINSKY

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ABSTRACT. Let $d\nu$ be a nonnegative Borel measure on $[-\pi, \pi]$, with $0 < \int_{-\pi}^{\pi} d\nu < \infty$ and with support of Lebesgue measure zero. We show that there exist $\{\eta_j\}_{j=1}^{\infty} \subset (0, \infty)$ and $\{t_j\}_{j=1}^{\infty} \subset (-\pi, \pi)$ such that if

$$d\mu(\theta) := \sum_{j=1}^{\infty} \eta_j d\nu(\theta + t_j), \quad \theta \in [-\pi, \pi],$$

(with the usual periodic extension $d\nu(\theta \pm 2\pi) = d\nu(\theta)$), then the leading coefficients $\{\kappa_n(d\mu)\}_{n=0}^{\infty}$ of the orthonormal polynomials for $d\mu$ satisfy

$$\lim_{n \rightarrow \infty} \kappa_n(d\mu)/\kappa_{n+1}(d\mu) = 1.$$

As a consequence, we obtain pure singularly continuous measures $d\alpha$ on $[-1, 1]$ lying in Nevai's class M .

1. INTRODUCTION

Suppose that $d\alpha$ is a nonnegative measure on $\mathbb{R}$ with all power moments finite and with support containing infinitely many points. Let $\{p_j(d\alpha, x)\}_{j=0}^{\infty}$ denote the sequence of orthonormal polynomials for $d\alpha$, satisfying

$$\int p_j p_k d\alpha = \delta_{jk}.$$

The orthonormal polynomials satisfy the recurrence relation

$$xp_n(d\alpha, x) = a_{n+1}p_{n+1}(d\alpha, x) + b_n p_n(d\alpha, x) + a_n p_{n-1}(d\alpha, x),$$

where $a_n > 0$, $b_n \in \mathbb{R}$. We say that $d\alpha \in M = M(0, 1)$ if

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = 0.$$

For the significance of Nevai's class M and references to the numerous investigations thereof, see [4], [7], [9], and [10].

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The most general simple criterion for a measure $d\alpha$ on $[-1, 1]$ to belong to M is due to Rahmanov [12]: If $\alpha' > 0$ a.e. in $[-1, 1]$, then $d\alpha \in M$. However, there are measures $d\alpha$ on $[-1, 1]$ with $\alpha' = 0$ a.e. that belong to M , and in fact *pure jump measures*. Explicit examples were first given by the author in [5] and subsequently by Van Assche and Magnus [6]. Probabilistic existence of such measures was established earlier in [1].

At the NATO conference on orthogonal polynomials held at Columbus (May–June, 1989), the question was raised whether M contains *pure singularly continuous* measures: That is, $\alpha : [-1, 1] \rightarrow \mathbb{R}$ is continuous, but the set of points of increase of α has Lebesgue measure zero. We answer this question affirmatively in this paper, with a fairly explicit example. It is noteworthy that orthogonal polynomials associated with Julia sets often lead to pure singularly continuous measures. See the papers [2], [3] of the group working with J. S. Geronimo.

Associated with any measure $d\alpha$ on $[-1, 1]$ is the measure $d\mu$ on $[-\pi, \pi]$, defined by

$$(1.1) \quad \mu(\theta) := \begin{cases} \alpha(1) - \alpha(\cos \theta), & \theta \in [0, \pi], \\ \alpha(\cos \theta) - \alpha(1), & \theta \in [-\pi, 0]. \end{cases}$$

Note that μ is odd, and so $\mu(0) = 0$. We shall also say that $d\mu$ is even. We extend μ to $\mathbb{R}$ by the periodicity relation

$$(1.2) \quad \mu(s + 2\pi) - \mu(t + 2\pi) = \mu(s) - \mu(t),$$

for all $s, t \in \mathbb{R}$. Equivalently,

$$(1.3) \quad d\mu(s \pm 2\pi) = d\mu(s).$$

We shall adopt the above normalization and extension to $\mathbb{R}$, for all measures on $[-\pi, \pi]$, without further mention. Note that conversely, any even $d\mu$ (or, equivalently, odd μ) generates a measure $d\alpha$ on $[-1, 1]$ via the relation (1.1).

One can define orthonormal polynomials

$$\varphi_n(d\mu, z) = \kappa_n(d\mu)z^n + \dots, \quad \kappa_n(d\mu) > 0, \quad n = 0, 1, 2, \dots,$$

satisfying

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(d\mu, z) \overline{\varphi_m(d\mu, z)} d\mu(\theta) = \delta_{mn}, \quad z := e^{i\theta}.$$

For $d\alpha$ and $d\mu$ related by (1.1), it is well known [9], [10], [12] that

$$(1.4) \quad d\alpha \in M \text{ iff } \lim_{n \rightarrow \infty} \kappa_n(d\mu)/\kappa_{n+1}(d\mu) = 1.$$

Our main result is the following:

Theorem 1. *Let $d\nu$ be a nonnegative Borel measure on $[-\pi, \pi]$ with $0 < \int_{-\pi}^{\pi} d\nu < \infty$ and with the support of $d\nu$ having Lebesgue measure zero. Then there exist $\{\eta_j\}_{j=1}^{\infty} \subset (0, \infty)$, and $\{t_j\}_{j=1}^{\infty} \subset (-\pi, \pi)$ such that*

$$\sum_{j=1}^{\infty} \eta_j < \infty$$

and such that if

$$(1.5) \quad d\mu(\theta) := \sum_{j=1}^{\infty} \eta_j d\nu(\theta + t_j), \quad \theta \in [-\pi, \pi],$$

then

$$(1.6) \quad \lim_{n \rightarrow \infty} \kappa_n(d\mu) / \kappa_{n+1}(d\mu) = 1.$$

Corollary 2. *There exist pure singularly continuous measures $d\alpha$ on $[-1, 1]$, lying in M .*

Remarks. (a) Note that if $d\nu$ is a pure, singularly continuous measure, then clearly so is $d\mu$. Likewise, a pure jump $d\nu$ leads to a pure jump $d\mu$.

(b) We can give a more explicit description for $d\mu$. For $n \geq 1$, let

$$(1.7) \quad \theta_{nl} := -\pi + (l - \frac{1}{2}) \frac{2\pi}{n}, \quad 1 \leq l \leq n,$$

and let $d\rho_n$ be the measure placing a mass of $1/n$ at each θ_{nl} ; that is,

$$(1.8) \quad d\rho_n := n^{-1} \sum_{l=1}^n d\delta_{\theta_{nl}},$$

where $d\delta_t$ is a Dirac delta (unit mass) at t . This is just the measure associated with the usual Chebyshev quadrature for $[-\pi, \pi]$, with shifted abscissas. We shall choose $\{n_j\}_{j=1}^{\infty}$ to be a sequence of positive integers, increasing sufficiently rapidly, and set

$$(1.9) \quad d\rho := \sum_{j=1}^{\infty} j^{-2} d\rho_{n_j}.$$

Note that $d\rho$ is a pure jump measure and is even, as each $d\rho_{n_j}$ is. Finally, for continuous f from the unit circle to $\mathbb{C}$, we set

$$(1.10) \quad \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{is} e^{it}) d\rho(t) d\nu(s).$$

Since $d\rho$ is a pure jump measure, we shall see in §2 that $d\mu$ can be written in the form (1.5). Essentially, $d\mu(\theta)$ is the convolution of $d\nu(\theta)$ with $d\rho(\theta)$, and the latter in a certain sense approximates $d\theta/(2\pi)$.

(c) As a consequence of Szegő's theory [15],

$$\lim_{n \rightarrow \infty} \kappa_n(d\mu) = \infty.$$

However, the proof of Theorem 1 shows that we can ensure that $\kappa_n(d\mu)$ increases to ∞ arbitrarily slowly.

(d) To obtain the measures of Corollary 2, one starts with a pure singularly continuous measure on $[-1, 1]$, for example, the equilibrium measure for the

usual Cantor ternary set. Then we transform to $[-\pi, \pi]$, using (1.1). Then $d\mu$ of Theorem 1 is clearly singularly continuous, and we shall see in §2, that it is even and so generates a corresponding measure on $[-1, 1]$.

(e) For results on the distribution of zeros of orthogonal polynomials on the unit circle and $[-1, 1]$, see [3, 7, 8, 10, 11, 13, 16, 17].

2. PROOFS

Throughout, P_n denotes the set of polynomials of degree at most n , with complex coefficients, and P_n^0 denotes the monic polynomials of degree n . Recall first the well-known identities [15, p. 289]

$$(2.1) \quad \kappa_n^{-2}(d\mu) = \min_{P \in P_n^0} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\mu(\theta)$$

and [5, p. 519, 521 and Equations (2.3), (2.12)] for $P \in P_{n-1}$ and $s \in \mathbb{R}$,

$$(2.2) \quad \int_{-\pi}^{\pi} |P(e^{is} e^{i\theta})|^2 d\rho_n(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta.$$

In the proof of Theorem 1, we shall use the following lemma:

Lemma 2.1. *Let $0 < \delta < 1$ satisfy*

$$(2.3) \quad 0 < \delta \log \frac{1}{\delta} < \frac{1}{2}.$$

Let S be a compact subset of $[-\pi, \pi]$ of Lebesgue measure less than δ , and let l be a positive integer. Then there exists an integer $n > l$, and $P \in P_n^0$, such that

$$(2.4) \quad |P(e^{i\theta})| < 2\delta, \quad \theta \in S,$$

and

$$(2.5) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta < 1 + 3\delta \log \frac{1}{\delta}.$$

Proof. Since S is compact, we can choose a (relatively) open subset T of $[-\pi, \pi]$, of measure less than 2δ , containing S , and consisting of finitely many (relatively) open intervals. We may assume that $\pm\pi \in T$. Choose T_1 to be a (relatively) open subset of $[-\pi, \pi]$ containing $\bar{T}$, of Lebesgue measure less than 4δ and also consisting of finitely many intervals. Define

$$w(\theta) := \begin{cases} \delta^{-2}, & \theta \in T, \\ 1, & \theta \in [-\pi, \pi] \setminus T_1. \end{cases}$$

In the finitely many intervals between T and $[-\pi, \pi] \setminus T_1$, define w to be piecewise linear. Thus w is continuous on $[-\pi, \pi]$, with $w(\pi) = w(-\pi) (= \delta^{-2})$, and it satisfies there a Lipschitz condition. Furthermore,

$$1 \leq w(\theta) \leq \delta^{-2} \quad \text{in } [-\pi, \pi].$$

By Szegő's theory [15, p. 302, Equation (12.3.15)],

$$\lim_{n \rightarrow \infty} \kappa_n(w) = \exp \left[-\frac{1}{4\pi} \int_{-\pi}^{\pi} \log w(\theta) d\theta \right] =: A,$$

and the orthonormal polynomials $\varphi_n(w, z)$ satisfy [15, p. 297, Theorem 12.1.3]

$$\lim_{n \rightarrow \infty} |\varphi_n(w, e^{i\theta})| w(\theta)^{1/2} = 1$$

uniformly for $\theta \in [-\pi, \pi]$. Now as $\log w = 0$ in $[-\pi, \pi] \setminus T_1$,

$$A \geq \exp \left[-\frac{4\delta}{4\pi} \log(\delta^{-2}) \right] \geq 1 - \frac{2}{\pi} \delta \log \frac{1}{\delta},$$

and hence, as (2.3) holds,

$$(2.6) \quad A^{-1} < 1 + \delta \log \frac{1}{\delta}.$$

Then the monic polynomial $\Phi_n(z) := \varphi_n(w, z)/\kappa_n(w)$ satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Phi_n(e^{i\theta})|^2 d\theta \\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(\theta)^{-1} d\theta / A^2 \leq 1/A^2 < 1 + 3\delta \log \frac{1}{\delta}, \end{aligned}$$

as $w \geq 1$ in $[-\pi, \pi]$. Then, for n large enough, $P := \Phi_n$ satisfies (2.5). Furthermore, uniformly for $\theta \in T \supset S$,

$$\lim_{n \rightarrow \infty} |\Phi_n(e^{i\theta})| = \delta/A < 2\delta,$$

in view of (2.3) and (2.6). So we can satisfy (2.4) as well. $\square$

Proof of Theorem 1. We shall define $d\mu$ by (1.10), where $d\rho$ is as at (1.7–9), and $\{n_j\}_{j=1}^{\infty}$ is an increasing sequence of positive integers, chosen inductively. We remark that the definition (1.10) does have the form (1.5); more precisely,

$$(2.7) \quad d\mu(\theta) = \sum_{j=1}^{\infty} j^{-2} n_j^{-1} \sum_{l=1}^{n_j} d\nu(\theta - \theta_{n_j, l}), \quad \theta \in [-\pi, \pi].$$

One has to take a little care with the definition of integrals with respect to $d\mu$, because of possible jumps of $\nu(\theta - \theta_{n_j, l})$ at $\theta = \pm\pi$. However, this can be avoided by using (1.10). To prove (2.7), note that, for continuous f on the unit circle, and $n \geq 1$,

$$\begin{aligned} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{is} e^{it}) d\rho_n(t) d\nu(s) \\ = \frac{1}{n} \sum_{l=1}^n \int_{-\pi}^{\pi} f(e^{i(s+\theta_{nl})}) d\nu(s) \\ = \frac{1}{n} \sum_{l=1}^n \int_{-\pi+\theta_{nl}}^{\pi+\theta_{nl}} f(e^{i\theta}) d\nu(\theta - \theta_{nl}) \\ = \int_{-\pi}^{\pi} f(e^{i\theta}) \frac{1}{n} \sum_{l=1}^n d\nu(\theta - \theta_{nl}), \end{aligned}$$

where we have also used the 2π -periodicity of f and $d\nu$. Thus (2.7) is valid and $d\mu$ has the same form as (1.5).

Let us set $n_1 := 1$. Assume now, as an induction hypothesis, that for some $k \geq 1$, we have already chosen $n_1, n_2, \dots, n_k$. Then we have already constructed the measure

$$d\mu_k(\theta) := \sum_{j=1}^k j^{-2} n_j^{-1} \sum_{l=1}^{n_j} d\nu(\theta - \theta_{n_j, l}),$$

which is the k th partial sum of the first series in (2.7). Since $d\nu$ has support of Lebesgue measure zero, so does $d\mu_k$. Then we can choose a compact set S of measure $< (k+1)^{-2}$ containing this latter support and, say, consisting of finitely many intervals. The endpoints of those intervals in $(-\pi, \pi)$ may be assumed to lie outside the support. According to the lemma, with $\delta := (k+1)^{-2}$, we can choose an integer $n_{k+1} > n_k$, and $P \in P_{n_{k+1}-1}^0$, satisfying

$$(2.8) \quad |P(e^{i\theta})| < 2(k+1)^{-2}, \quad \theta \in S,$$

and

$$(2.9) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta < 1 + 6(k+1)^{-2} \log(k+1).$$

Then, irrespective of how the $\{n_j\}_{j=k+2}^{\infty}$ will be chosen, their monotonicity, (1.10), and (2.2) ensure that

$$\begin{aligned} & \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\mu(\theta) \\ &= \int_S |P(e^{i\theta})|^2 d\mu_k(\theta) + \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta \left\{ \sum_{j=k+1}^{\infty} j^{-2} \right\} \int_{-\pi}^{\pi} d\nu(s) \\ &\leq 4(k+1)^{-4} \left\{ \sum_{j=1}^k j^{-2} \right\} \int_{-\pi}^{\pi} d\nu(s) + \{1 + 6(k+1)^{-2} \log(k+1)\} k^{-1} \\ &\quad \times \int_{-\pi}^{\pi} d\nu(s) \\ &\quad \quad \quad \text{[by (2.8) and (2.9) and choice of } d\mu_k] \\ &\leq k^{-1} \int_{-\pi}^{\pi} d\nu(s) \{1 + 8(k+1)^{-1}\}. \end{aligned}$$

Hence, from (2.1),

$$(2.10) \quad \kappa_{n_{k+1}-1}^{-2}(d\mu) \leq (2\pi k)^{-1} \int_{-\pi}^{\pi} d\nu(s) \{1 + 8(k+1)^{-1}\}.$$

On the other hand, since for $P \in P_n^0$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta \geq 1,$$

we see that

$$\begin{aligned}
 \kappa_{n_{k+1}-1}^{-2}(d\mu) &\geq \inf_{P \in P_{n_{k+1}-1}^0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |P(e^{is}e^{it})|^2 \left\{ \sum_{j=k+1}^{\infty} j^{-2} d\rho_{n_j}(t) \right\} d\nu(s) \\
 &= \inf_{P \in P_{n_{k+1}-1}^0} \left(\frac{1}{2\pi} \right)^2 \int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta \left\{ \sum_{j=k+1}^{\infty} j^{-2} \right\} \int_{-\pi}^{\pi} d\nu(s) \\
 &\quad \text{[by (2.2)]} \\
 (2.11) \quad &\geq [2\pi(k+1)]^{-1} \int_{-\pi}^{\pi} d\nu(s).
 \end{aligned}$$

In this way, proceeding inductively, we can choose $\{n_j\}_{j=1}^{\infty}$ satisfying (2.10) and (2.11) for $k \geq 1$. Note that, then, for $k \geq 2$, $n_k \leq l < n_{k+1}$ implies that

$$\begin{aligned}
 1 &\leq \kappa_l(d\mu)/\kappa_{l-1}(d\mu) \\
 &\leq \kappa_{n_{k+1}-1}(d\mu)/\kappa_{n_k-1}(d\mu) \\
 &\leq (k+1)^{1/2}/\{(k-1)^{-1}(1+8/k)\}^{-1/2} \quad \text{[by (2.10) and (2.11)]} \\
 &\rightarrow 1,
 \end{aligned}$$

as $k \rightarrow \infty$. So (1.6) is valid. $\square$

Proof of Corollary 2. Let $d\beta$ denote a pure singularly continuous measure on $[-1, 1]$. For example, we can choose, on $[0, 1]$, $d\beta$ to be the equilibrium measure for the usual Cantor ternary set, and then extend $d\beta$ to $[-1, 1]$ so as to be even. Define a measure $d\nu$ on $[-\pi, \pi]$ by

$$\nu(\theta) := \begin{cases} \beta(1) - \beta(\cos \theta), & \theta \in [0, \pi], \\ \beta(\cos \theta) - \beta(1), & \theta \in [-\pi, 0]. \end{cases}$$

Obviously, $d\nu$ is singularly continuous, and so $d\mu$ of Theorem 1 inherits this property. It remains to show that μ is an odd function, so that there is a measure $d\alpha$ on $[-1, 1]$ corresponding to $d\mu$. Recall that we established (2.7) in the proof of Theorem 1. Since ν is continuous, integrating (2.7) from 0 to x yields [recalling the normalization $\mu(0) = 0$ from the introduction]

$$\begin{aligned}
 \mu(x) &= \sum_{j=1}^{\infty} j^{-2} n_j^{-1} \sum_{l=1}^{n_j} \{\nu(x - \theta_{n_j, l}) - \nu(-\theta_{n_j, l})\} \\
 &= \sum_{j=1}^{\infty} j^{-2} \int_{-\pi}^{\pi} \{\nu(x - s) - \nu(-s)\} d\rho_{n_j}(s) \\
 &= \int_{-\pi}^{\pi} \{\nu(x - s) - \nu(-s)\} d\rho(s).
 \end{aligned}$$

This expresses μ as the convolution of ν and $d\rho$. Further, since ν is odd and $d\rho$ is even, we can also write

$$\mu(x) = \int_{-\pi}^{\pi} \nu(x - s) d\rho(s).$$

Then

$$\begin{aligned}\mu(-x) &= \int_{-\pi}^{\pi} \nu(-x-s) d\rho(s) \\ &= \int_{-\pi}^{\pi} -\nu(x+s) d\rho(s) \\ &= \int_{-\pi}^{\pi} -\nu(x-s) d\rho(s) = -\mu(x).\end{aligned}$$

So μ is odd, and we can define α by (1.1), normalized, for example, by $\alpha(0) = 0$. In view of (1.4), $d\alpha$ will belong to M . $\square$

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