FREE SUBALGEBRAS OF ENVELOPING FIELDS

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ABSTRACT. Let K be a field of characteristic zero and L a nonabelian Lie algebra over K . We show that the skew field of fractions of the enveloping algebra of L over K contains a free noncommutative K -algebra when L is solvable or finite-dimensional.

The first author has announced a belief [M-L] that all reasonable skew fields (i.e., those not locally finite-dimensional over their centers) contain free noncommutative algebras. We pursue this by showing that the skew field of fractions of an enveloping algebra of a Lie algebra contains a free algebra when the Lie algebra is solvable or finite-dimensional over a field of characteristic zero. Here the main tool is the result of the first author that the skew field of quotients of the Weyl algebra contains a free algebra.

For a Lie algebra L over a (commutative) field K , the enveloping algebra UL is the (universal) associative K -algebra such that the Lie operation on (a copy of) L coincides with the additive commutator $[x, y] = xy - yx$ in UL . Thus we will use the notation $[x, y]$ for both of these without confusion. The enveloping algebra UL is always a domain; in some cases it is an Ore domain and thus has a skew field of fractions QUL , called the *enveloping field* of L . We reproduce a fairly general result of Lichtman on the existence of QUL .

Proposition (4.1 of [Li]). *Let L be a Lie algebra which has an ascending (possibly transfinite) series of subalgebras, each an ideal of the next, such that all the factors are either locally finite-dimensional or locally solvable. Then the enveloping algebra UL is an Ore domain.*

To find a free algebra inside a skew field, we need only find two elements satisfying no (noncommutative) polynomial relation. Are the coefficients of a possible relation to be in the prime field or in the central field? The following lemma makes the question moot.

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Lemma 1. *Let R be a domain with prime subfield K_0 (generated by 1). Let C be any central subfield of R and let $y, z \in R$. Then y and z are free over K_0 if and only if they are free over C .*

Proof. If y, z are free over C (i.e., satisfy no polynomial with C -coefficients), then they are free over K_0 trivially. Conversely, suppose y, z are free over K_0 . Then they generate a free algebra $K_0\langle y, z \rangle \subseteq R$. Consider the multiplication map $\mu: C \otimes K_0\langle y, z \rangle \rightarrow R$, where $\otimes$ is over K_0 (equivalently, over the integers $\mathbb{Z}$). Now y, z are free over C if and only if μ has no kernel. So assume that

$$\mu \left(\sum_{i=1}^n c_i \otimes f_i \right) = 0$$

where $n > 0$ is minimal. If $n = 1$, then $c_1 f_1 = 0$ so that $c_1 \otimes f_1 = 0$ (recall R is a domain). If $n > 1$, let $g \in K_0\langle y, z \rangle$ be arbitrary. Then

$$\mu \left(\sum c_i \otimes (f_i g f_1 - f_1 g f_i) \right) = 0$$

also (recall the c_i are central). Since this element of the kernel has $n-1$ terms, we must have $f_i g f_1 = f_1 g f_i$ for every i (and every g). Examining leading terms (say lexicographically) shows that each f_i has the same leading monomial as f_1 . Then for some $m_i \in K_0$, $f_i - m_i f_1$ has a different leading monomial than f_1 , but we still have $(f_i - m_i f_1) g f_1 = f_1 g (f_i - m_i f_1)$ for each g . Thus $f_i = m_i f_1$ for each i and

$$\sum_{i=1}^n c_i \otimes f_i = \left(\sum_{i=1}^n c_i m_i \right) \otimes f_1,$$

reducing to the case of $n = 1$. This shows that $\text{Ker } \mu = 0$.

We now assume that the characteristic of K is zero. For the case of L solvable we can get the slightly more general result that the skew field need only contain L . We will use the following lemma.

Lemma 2. *Let D be a skew field, K a central subfield of D . Suppose that $u, v \in D$ and $[u, v] \neq 0$, $[[u, v], v] = 0$. Then D contains a copy of the Weyl algebra over K and hence a free noncommutative K -algebra.*

Proof. Let $w = [u, v]$; then $[uw^{-1}, v] = 1$. Thus uw^{-1}, v generate a copy of the Weyl algebra $A_1(K)$ over K inside D (recall $A_1(K)$ is simple). Since D is a skew field and $A_1(K)$ is Ore, D also contains the skew field of quotients $QA_1(K) = D_1(K)$. But Makar-Limanov [M-L'] has shown that the latter contains a free noncommutative K -algebra.

Theorem A. *Let D be a division algebra over K which contains a solvable non-abelian Lie algebra over K (under $[\ , \]$). Then D contains a noncommutative free K -algebra.*

Proof. Let L be the Lie algebra, and write $L_0 = L$, $L_i = [L_{i-1}, L_{i-1}]$ inductively. Solvability implies that some $L_n = 0$ (say with minimal n), but L is

nonabelian so $L_1 \neq 0$, $n > 1$. If $[L_{n-1}, L_{n-2}] = 0$, then pick $u, v \in L_{n-2}$ so that $0 \neq [u, v] \in L_{n-1}$. Then $[[u, v], v] \in [L_{n-1}, L_{n-2}] = 0$ so we may apply Lemma 2. If $[L_{n-1}, L_{n-2}] \neq 0$, pick $u \in L_{n-2}$, $v \in L_{n-1}$ so that $[u, v] \neq 0$. Then $[[u, v], v] \in [L_{n-1}, L_{n-1}] = 0$ so we may again apply Lemma 2.

For the case of finite-dimensional L , the result is more difficult. The proof will proceed by a series of claims and will make up most of the rest of the paper.

Theorem B. *Let L be a nonabelian finite-dimensional Lie algebra over a field K of zero characteristic. Then the enveloping field of L contains a noncommutative free k -algebra.*

Proof. Suppose L is a counterexample of minimal dimension. We will proceed by a series of claims to get information on L before we obtain a contradiction.

First we claim that we may assume that L is semisimple. Otherwise L contains a solvable ideal $T \neq 0$. If T is nonabelian then the enveloping field $QUT \subset QUL$ contains a free algebra by Theorem A. Otherwise $[T, T] = 0$, and we use a result of Lichtman ([Li, Theorem 5.1]) which states that there is a specialization from QUL to $QU(L/T)$. This specialization is essentially a homomorphism from a subring of QUL onto $QU(L/T)$. If L/T is abelian then L is solvable so QUL contains a free algebra. If L/T is nonabelian then it is not a counterexample (being of smaller dimension than L) and so $QU(L/T)$ contains a free algebra. Pulling back along the specialization we obtain a free subalgebra of QUL .

Now L is assumed semisimple, so choose a Cartan subalgebra H of L . This is necessarily abelian by general theory [J], or we could even remark that H cannot be a counterexample and that $QUH \subset QUL$ contains no free algebra.

For each $x \in L$ we have the linear map $ad_x: L \rightarrow L$ defined by $ad_x(y) = [x, y]$. We say H is *split* over K if the characteristic polynomial of ad_h factors completely over K for all $h \in H$. As this may not be true in our situation we wish to extend the field K to obtain a field $\bar{K}$ over which $H \otimes \bar{K}$ is a split Cartan subalgebra of $L \otimes \bar{K}$. According to general theory [J], by extending the field we can find finitely many roots $\alpha \in H^* \otimes \bar{K}$ and root space vectors $u_\alpha \in L \otimes \bar{K}$ so that $[h, u_\alpha] = \alpha(h)u_\alpha$ for $h \in H$. By picking bases of H and L , we can see that all the coefficients involved lie in a finite extension of K . We therefore may assume that $\bar{K}$ is finite Galois over K .

Now $L \otimes \bar{K}$ is semisimple with $H \otimes \bar{K}$ split, so there is a set of simple roots $\gamma_1, \dots, \gamma_n$ such that every root α is either a nonnegative or nonpositive integral linear combination of the γ_i and so that the set of root vectors u_α as above forms a basis for $L \otimes \bar{K}$ over $H \otimes \bar{K}$. The Cartan subalgebra and the set of simple roots decompose into summands, each corresponding to a simple summand of $L \otimes \bar{K}$ and each of type $A_k, B_k, C_k, D_k, E_k, F_4$, or G_2 ([J, p. 121–135]). We recall the notion of the *level* of a root, the absolute value of the sum of the coefficients in terms of the γ_i .

Furthermore, the Galois group of $\bar{K}/K$ acts on $L \otimes \bar{K}$ and the corresponding roots in $H^* \otimes \bar{K}$. (In particular, if we regard $\alpha: H \otimes \bar{K} \rightarrow \bar{K}$ and $\sigma \in \text{Gal}(\bar{K}/K)$,

then $\alpha^\sigma = (1 \otimes \sigma)\alpha(1 \otimes \sigma^{-1})$.) Similarly, the Killing form extends to $L \otimes \bar{K}$ and the Galois group acts appropriately on this form.

We wish to assume that $L \otimes \bar{K}$ has simple summands not of type C_k . Subalgebras of type C_k have “long” and “short” roots, as determined by their lengths under the Killing form. Thus the Galois group of $\bar{K}/K$ will map long roots to long roots and short roots to short roots. If only summands of type C_k appear in $L \otimes \bar{K}$, consider the subalgebra generated by the Cartan subalgebra $H \otimes \bar{K}$ and the root space vectors u_α of summands of type C_k corresponding to long roots α . It is straightforward to check that this subalgebra does not include the root space vectors u_α corresponding to short roots α of C_k -summands, so its dimension is less than that of $L \otimes \bar{K}$. Also, this subalgebra is clearly invariant under all elements of the Galois group, so that by taking traces we can get a proper subalgebra L' of L , defined over K . Furthermore, L' is still nonabelian. Thus if $L \otimes \bar{K}$ had only simple summands of type C_k we would restrict our attention to L' ; or equivalently, we may assume that $L \otimes \bar{K}$ has simple summands not of type C_k .

The point of eliminating C_k is the following lemma.

Lemma 3. *Suppose M is a split simple Lie algebra of dimension greater than three and not of type C_k , and let $\gamma_1, \dots, \gamma_m$ be the simple roots of M and β the highest level root of M . Then there is an integer i such that $\beta - \gamma_i$ is a root but $\beta - 2\gamma_i$ is not a root.*

Proof. The conclusion may be restated in terms of the Killing form $(\ , \)$ as $2(\gamma_i, \beta) = (\gamma_i, \gamma_i)$ ([J, p. 116]). This may be checked case-by-case ([BMP, Chapter 3]); in fact i is unique except for the A_k type.

If $L \otimes \bar{K}$ contains a summand of the kind in Lemma 3, let β be the highest level root of this summand and γ_1 the simple root provided by the lemma. Renumber the simple roots of $L \otimes \bar{K}$ as $\gamma_1, \gamma_2, \dots, \gamma_n$, where $\gamma_1, \dots, \gamma_m$ ($m \leq n$) are the simple roots of the simple component of β . We will refer to this as the “normal” case. Otherwise, $L \otimes \bar{K}$ contains only summands of dimension three, so let β be one of the highest level roots and set $\gamma_1 = 2\beta$. We will (inaccurately) refer to this as the “three-dimensional” case. In either case, $\beta - \gamma_1$ is a root but $\beta - 2\gamma_1$ is not a root.

Now choose $a \in H$ (over K) so that $\gamma_1(a \otimes 1) \neq 0$. Choose $b \in L$ (over K) so that each of the root space vectors u_α (satisfying $[h, u_\alpha] = \alpha(h)u_\alpha$ for $h \in H \otimes \bar{K}$) appears in $b \otimes 1$ with nonzero coefficient (a simple linear algebra argument will assure this is possible). We will obtain a contradiction to $a^{-1}, a^{-1}b^{-1}$ algebraically dependent in $QU(L \otimes \bar{K})$. (Here and from now on we identify $a \otimes 1, b \otimes 1$ with a, b .)

To do this we will examine “leading terms” of these elements. We will recast the algebra involved to make this notion precise.

Note that elements of $U(H \otimes \bar{K})$ are just commutative polynomials in the basis elements $h_1, \dots, h_k$ of H . For each root α and corresponding root vector u_α , we have $[h_i, u_\alpha] = \alpha(h_i)u_\alpha$, which may be rewritten as $u_\alpha h_i = (h_i - \alpha(h_i))u_\alpha$. So we see that if we let the root α act additively on $f(h_1, \dots, h_k) \in U(H \otimes \bar{K})$ by $(f + \alpha)(h_1, \dots, h_k) = f(h_1 + \alpha(h_1), \dots, h_k + \alpha(h_k))$, then we get $u_\alpha f = (f - \alpha)u_\alpha$. (This applies even if u_α is an element of H , since α acts trivially.) Now we claim that in $U(L \otimes \bar{K})$ the nonzero elements of $U(H \otimes \bar{K})$ form a left Ore set. This follows from [BR] or can be worked out with less difficulty, in this case, by finding a common left denominator for several monomials $u_1 \cdots u_l f^{-1}$.

Thus by inverting the nonzero elements of $U(H \otimes \bar{K})$ we get a subring $UL_H^\otimes$ of $QU(L \otimes \bar{K})$ in which every element is a linear combination of terms of the form $fu_1 \cdots u_l$, where f is in the field $QU(H \otimes \bar{K})$ and each u_i is a root space vector over $H \otimes \bar{K}$. We can also see (by the Poincaré-Birkhoff-Witt theorem or its analog) that $UL_H^\otimes$ is a vector space over $QU(H \otimes \bar{K})$ with basis consisting of the monomials $u_1^{e_1} \cdots u_n^{e_n}$. Finally, $UL_H^\otimes$ is still an Ore ring with skew field of fractions $QU(L \otimes \bar{K})$, since $U(L \otimes \bar{K})$ was.

Now define a degree function on $UL_H^\otimes$ as follows. In the normal case, for each root space vector u corresponding to root α , write $\alpha = d_1\gamma_1 + \cdots + d_n\gamma_n$ with the d_i nonnegative or nonpositive integers. Then define the degree of u to be the integer $d_2 + \cdots + d_m$. For the three-dimensional case, define the degree of u to be 1 if it corresponds to either of the roots β or $-\beta = \beta - \gamma_1$, and 0 otherwise. In both cases the degree will be zero if α is not in the simple component of β in $L \otimes \bar{K}$. Also, define the degree of a nonzero element of $QU(H \otimes \bar{K})$ to be zero. Extend this by defining the degree of a term of the form $fu_1 \cdots u_l$ to be the sum of the degrees of the factors. Finally, the degree of the sum of such terms will be the largest of the degrees of the terms.

This is well-defined by the following considerations. For u, u' root vectors corresponding to roots α, α' and for $f \in QU(H \otimes \bar{K})$, we have $uf = (f - \alpha)u$ and $[u, u'] = u''$, with u'' corresponding to $\alpha + \alpha'$ (or $u'' = 0$ if $\alpha + \alpha'$ is not a root). Both of these relations can easily be seen to be homogeneous with respect to the degree function, except in the three-dimensional case, when the latter may not be. However, in the exceptional case, $uu' = u'u + [u, u']$, where the leading terms have degree 2 and the term $[u, u']$ has lower degree 0. Since the monomials $u_1^{e_1} \cdots u_n^{e_n}$ form a basis for $UL_H^\otimes$ over $QU(H \otimes \bar{K})$, the degree function is well-defined on $UL_H^\otimes$. The degree function then extends uniquely to the quotient field $QU(L \otimes \bar{K})$ of $UL_H^\otimes$.

By our original assumption, a^{-1} and $a^{-1}b^{-1}$ do not generate a free algebra over K . Therefore the monomials $m_i = a^{-i_0}b^{-1}a^{-i_1}b^{-1} \cdots b^{-1}a^{-i_k}$ are linearly dependent over K (for various multi-indices $i = (i_0, \dots, i_k)$). This dependence certainly also occurs over $\bar{K}$ in $QU(L \otimes \bar{K})$. Then the leading terms of these m_i with respect to the degree function must also be linearly dependent. By construction, the only roots corresponding to root vectors of highest degree

are β and $\beta - \gamma_1$ (since $\beta - 2\gamma_1$ is not a root and others have lower degree). Let u_1, u_2 be the root vectors corresponding to $\beta, \beta - \gamma_1$ respectively. Then the leading term of b is $c_1 u_1 + c_2 u_2$, where c_1, c_2 are nonzero elements of $\bar{K}$. So the leading term of m_i is $\bar{m}_i = a^{-i_0}(c_1 u_1 + c_2 u_2)^{-1} \cdots a^{-i_k}$.

These monomials $\bar{m}_i$ lie in the skew subfield generated over $QU(H \otimes \bar{K})$ by u_1, u_2 . In the normal case the u_1, u_2 commute ($2\beta - \gamma_1$ is not a root), so this is simply the skew field of fractions of a skew polynomial ring in two variables over $QU(H \otimes \bar{K})$. In the three-dimensional case the skew subfield is only slightly more complicated. By considering the smallest k in $\iota = (i_0, i_1, \dots, i_k)$, we see that the $\bar{m}_i$ with this fixed k must be linearly dependent over $\bar{K}$. We may rewrite

$$\begin{aligned} \bar{m}_i &= c_1^{-k} u_1^{-k} (a - k\beta(a))^{-i_0} (1 + c_1^{-1} c_2 u_1^{-1} u_2)^{-1} (a - (k-1)\beta(a))^{-i_1} \\ &\quad (1 + c_1^{-1} c_2 u_1^{-1} u_2)^{-1} \cdots (a - \beta(a))^{-i_{k-1}} (1 + c_1^{-1} c_2 u_1^{-1} u_2)^{-1} a^{-i_k}. \end{aligned}$$

(This is not quite an equality in the three-dimensional case, but the leading terms are the same.)

So the monomials $M_i = c_1^k u_1^k \bar{m}_i$ are linearly dependent and lie in the subskew field generated over $\bar{K}$ by a and $t = -c_1^{-1} c_2 u_1^{-1} u_2$, with $at = t(a - \gamma_1(a))$ and $\gamma_1(a) \neq 0$.

If $\beta(a) = 0$ we could proceed as in [Lo], but in this case we follow the method of [M-L'] to derive a contradiction. For economy of notation let us put $\beta = \beta(a), \gamma = \gamma_1(a)$.

As in [M-L'], we embed the skew field generated by a, t in the skew field of Laurent series $\sum_{p=l}^{\infty} t^p g_p(a)$, where $l \in \mathbb{Z}$ and $g_p(a)$ are rational functions of a over $\bar{K}$, still with $at = t(a - \gamma)$. Then, expanding $(1 - t)^{-1}$ as usual, we get

$$\begin{aligned} M_i &= \sum_j (a - k\beta)^{-i_0} t^{j_1} (a - (k-1)\beta)^{-i_1} t^{j_2} \cdots (a - \beta)^{-i_{k-1}} t^{j_k} a^{-i_k} \\ &= \sum_j t^{|j|} (a - k\beta - |j|\gamma)^{-i_0} (a - (k-1)\beta - (|j| - j_1)\gamma)^{-i_1} \\ &\quad \cdots (a - \beta - j_k \gamma)^{-i_{k-1}} a^{-i_k}, \end{aligned}$$

where $j = (j_1, \dots, j_k)$ is a multi-index with $|j| = j_1 + \cdots + j_k$. Now set $n = |j|$, $n_1 = |j| - j_1, \dots, n_p = j_{p+1} + \cdots + j_k, \dots, n_{k-1} = j_k$, so that

$$M_i = \sum_{n=0}^{\infty} t^n f_i(a, n) (a - k\beta(a) - n\gamma)^{-i_0} a^{-i_k},$$

where $f_i(a, n) = \sum_{n \geq n_1 \geq \cdots \geq n_{k-1} \geq 0} (a - (k-1)\beta - n_1 \gamma)^{-i_1} \cdots (a - \beta - n_{k-1} \gamma)^{-i_{k-1}}$. Here the $\iota = (i_1, \dots, i_{k-1})$ has been shortened from its original length.

Before proceeding we follow the lead of [Lo] to clarify the functions involved here. We wish to regard n as an indeterminate in the functions $f_i(a, n)$. So let $\mathcal{F}$ be the set of all equivalence classes of functions from the nonnegative

integers to the field $\overline{K}(a)$, where two functions are equivalent if they agree except on a finite set. We have a subfield of constant functions in $\mathcal{F}$, which we will again denote by $\overline{K}(a)$. Also, since $\mathbf{Z} \subset \overline{K}$, we have the class of the identity function, which we will denote by n . Then n is transcendental over $\overline{K}(a)$, and any nonzero polynomial in n over $\overline{K}(a)$ is even a unit. Thus $\mathcal{F}$ contains the purely transcendental rational function field $\overline{K}(a, n)$. Because of the summation, $f_i(a, n)$ is not in $\overline{K}(a, n)$, but the assumption that the M_i are linearly dependent over $\overline{K}$ while the t^n are independent implies that the $f_i(a, n)$ are linearly dependent over $\overline{K}(a, n)$. This we will contradict.

What follows is a recasting of the lemma of [M-L']. Consider the set of all $f_i(a, n)$ where $\iota = (i_1, i_2, \dots, i_{k-1})$ and k are allowed to vary, and suppose they are linearly dependent over $\overline{K}(a, n)$. Denote by $w(\iota)$ the width $k-1$ of ι and choose a dependence which is "minimal" in the following sense: the maximal width $w(\iota)$ of any ι for which $f_i(a, n)$ has nonzero coefficient is the smallest possible, and furthermore the number of summands having this maximal width is also the smallest possible. Let us write this relation as

$$(1) \quad \sum_{w(\iota)=u} b_\iota f_\iota = \sum_{w(\iota)<u} c_\iota f_\iota$$

by collecting the "widest" f_ι on the left. Here $b_\iota, c_\iota \in \overline{K}(a, n)$, and we may assume that one b_ι is the constant 1.

Consider now the equation

$$(2) \quad \Delta \left(\sum b_\iota f_\iota \right) = \Delta \left(\sum c_\iota f_\iota \right),$$

where $\Delta g(a, n) = g(a, n+1) - g(a, n)$. It is easy to see that $\Delta f_\iota(a, n)$ is a linear combination over $\overline{K}(a, n)$ of the $f_{\iota_s}(n, a)$, where $\iota_s = (i_s, i_{s+1}, \dots, i_{k-1})$ for each $s > 1$ (take $f_\iota = 1$ if $w(\iota) = 0$). Now (2) becomes

$$(3) \quad \sum (\Delta b_\iota) f_\iota + \sum b_\iota (\Delta f_\iota) = \sum (\Delta c_\iota) f_\iota + \sum c_\iota (\Delta f_\iota),$$

which is a "smaller" relation than (1) and hence trivial. Thus all $\Delta b_\iota = 0$ so that all b_ι are functions of a alone. Picking one f_ι from the left side of (1), we may collect the coefficients of f_{ι_2} in (3) to obtain

$$(4) \quad \sum_{j_2=\iota_2} b_j(a) (a - (k-1)\beta - (n+1)\gamma)^{-j_1} = \Delta c_{\iota_2}(a, n),$$

where the sum is over all $j = (j_1, i_2, i_3, \dots, i_{k-1})$ for which $b_j \neq 0$. This is an equality in $\overline{K}(a, n)$, a purely transcendental function field. Thus $c_{\iota_2}(a, n)$ must have a pole at $a = (k-1)\beta + (n+1)\gamma$ or at $a = (k-1)\beta + n\gamma$. If the former, then the $c_{\iota_2}(a, n)$ also must have poles at $a = (k-1)\beta + (n+2)\gamma$, $a = (k-1)\beta + (n+3)\gamma$, etc., because the left side of (4) has no other poles. If the latter, then similarly $c_{\iota_2}(a, n)$ must have poles at $a = (k-1)\beta + (n-1)\gamma$, etc. In either case we have an infinite number of poles (in terms of a) and hence a contradiction to $c_{\iota_2} \in \overline{K}(a, n)$.

This completes the proof of Theorem B.

The referee has pointed out that the two theorems can be combined.

Theorem C. *Let L be a nonabelian solvable-by-finite Lie algebra over a field K of characteristic zero. Then the enveloping field of L contains a noncommutative free algebra.*

Proof. Let L have a solvable ideal Z such that L/Z is finite-dimensional. If L/Z is nonabelian we may pull back a free algebra from $QU(L/Z)$ by Theorem B as before. If L/Z is abelian then L is solvable and we apply Theorem A.

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