THE FIRST SIGN CHANGE OF A COSINE POLYNOMIAL

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ABSTRACT. Nulton and Stolarsky [1] studied the first (i.e., the least positive) sign change of a real cosine polynomial as a function of its smallest frequency. In the present article we will study this problem further, especially to point out that their fundamental proposition is not correct, and that therefore their principal hypothesis is unreasonable. Moreover, various results of Nulton and Stolarsky are improved or corrected and two open questions set in their paper are solved.

1. INTRODUCTION

Let $a_1, \dots, a_N$ and $0 < \lambda_1 < \dots < \lambda_N$ be real numbers, and let

$$(1.1) \quad f(x) = \sum_{i=1}^N a_i \cos \lambda_i x, \quad a_1 \neq 0,$$

be the real cosine polynomial. It is well known (see, e.g., [2]) that the number of sign changes of $f(x)$ in the interval $(0, t)$ is $\lambda_1 t / \pi + O(1)$; therefore there exists the first (i.e., the least positive) sign change of $f(x)$. In 1982, Nulton and Stolarsky [1] studied the first sign change of a cosine polynomial as a function of its frequencies λ_i . In [1], the following proposition was given without proof:

Proposition 2.1. *If $0 < \lambda_1 < \lambda_2$ and a is real, the first sign change of*

$$(1.2) \quad \cos \lambda_1 x + a \cos \lambda_2 x$$

will not increase if λ_1 is replaced by λ' , where

$$(1.3) \quad \lambda_1 \leq \lambda' \leq \lambda_2.$$

From some preliminary work, especially the above proposition, Nulton and Stolarsky proposed the following hypothesis:

Hypothesis. *Increases in the lowest frequency tend to decrease the position of the first sign change.*

Although at the end of [1] Nulton and Stolarsky proved that this hypothesis is false in some cases which they thought unusual, they expected that in

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$(\lambda_1, \dots, \lambda_N, a_1, \dots, a_N)$ phase space, the set of points where their hypothesis fails is asymptotically very small in relative measure and raised the question of finding the smallest integer $N = N_0$ (difficult to compute by [1]) for which their Hypothesis 2 fails. In fact, there is an explicit counterexample of their Hypothesis 2. We see that the first sign change of $\cos x - 2 \cos 2x$ is $\arccos(1 + \sqrt{33}/8)$ and the first sign change of $\cos 2x - 2 \cos 3x$ is $\pi/4$ — i.e., $\arccos(\sqrt{32}/8)$. Therefore their Proposition 2.1 is false. Their Hypothesis 2 is consequently not valid even for $N = N_0 = 2$. We will give a corrected version of their Proposition 2.1 in our Remark 2 and show that the sign of $f(0)$ plays a crucial role in this problem.

The object of the present paper is to give a further study of the relation between the first sign change and the smallest frequency of a real cosine polynomial. Some necessary lemmas will be given in §2 and the main results in §3. Finally, in §4, a number of hypotheses are proposed and some special cases are discussed.

2. SOME LEMMAS

Lemma 1. *Let $m \geq 3n > 0$ be real. Then there exists a point $x \in (\pi/(2n), \pi/n]$ such that $\cos mx = -1$.*

Lemma 2. *Let $m \geq 3n > 0$ be real. Then there exists a point $x \in [\pi/(2n), \pi/n)$ such that $\cos mx = 0$.*

These facts are simple, so we omit their proofs here.

Lemma 3. *Let $a \geq 0$, $b \geq 0$, and $0 < \lambda_1 < \lambda_2 < \lambda_3$ be real. Then the first sign change of*

$$(2.1) \quad f(x) = \cos \lambda_1 x + a \cos \lambda_2 x + b \cos \lambda_3 x$$

belongs to $(0, \pi/\lambda_1)$.

Proof. At first, we have

$$(2.2) \quad f(0) = 1 + a + b > 0.$$

Therefore, to prove the lemma, it is sufficient to prove that there exists a point x_0 in $(0, \pi/\lambda_1]$ such that $f(x_0) < 0$. We will prove this fact by dividing the set $\{(a, b) | a \geq 0, b \geq 0\}$ into five cases.

Case 1. $0 \leq a \leq 1/2$ and $0 \leq b \leq 1/2$. Let $x_0 = \pi/\lambda_1$. Then

$$f(x_0) = -1 + a \cos \frac{\lambda_2}{\lambda_1} \pi + b \cos \frac{\lambda_3}{\lambda_1} \pi < -1 + \frac{1}{2} + \frac{1}{2} = 0.$$

Case 2. $a + 1/2 \leq b$. If $\lambda_3 < 3\lambda_1$, take $x_0 = \pi/\lambda_3$. Then

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_3} \pi + a \cos \frac{\lambda_2}{\lambda_3} \pi - b < \frac{1}{2} + a - b \leq 0.$$

If $\lambda_3 \geq 3\lambda_1$, by Lemma 1 there exists $x_0 \in (\pi/(2\lambda_1), \pi/\lambda_1]$ such that $\cos \lambda_3 x_0 = -1$. Hence

$$f(x_0) = \cos \lambda_1 x_0 + a \cos \lambda_2 x_0 - b < a - b < 0.$$

Case 3. $b \geq 1/2$ and $0 < b - a < 1/2$. If $\lambda_3 \leq 2\lambda_1$, take $x_0 = \pi/\lambda_3$. Then

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_3} \pi + a \cos \frac{\lambda_2}{\lambda_3} \pi - b \leq a - b < 0.$$

If $\lambda_3 \geq 3\lambda_1$, by Lemma 1 there exists $x_0 \in (\pi/(2\lambda_1), \pi/\lambda_1]$ such that $\cos \lambda_3 x_0 = -1$. Therefore,

$$f(x_0) = \cos \lambda_1 x_0 + a \cos \lambda_2 x_0 - b < a - b < 0.$$

If $2\lambda_1 < \lambda_3 < 3\lambda_1$ and $\lambda_3 \leq 2\lambda_2$, take $x_0 = \pi/\lambda_3$. Then

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_3} \pi + a \cos \frac{\lambda_2}{\lambda_3} \pi - b < \frac{1}{2} - b \leq 0.$$

If $2\lambda_1 < \lambda_3 < 3\lambda_1$ and $\lambda_3 > 2\lambda_2$, then

$$3\lambda_1 < \lambda_2 + \lambda_3 < \frac{9}{2}\lambda_1.$$

It follows that

$$\frac{2\pi}{3\lambda_1} < \frac{3\pi}{\lambda_2 + \lambda_3} < \frac{\pi}{\lambda_1}.$$

Let $x_0 = 3\pi/(\lambda_2 + \lambda_3)$. Then we have

$$f(x_0) = \cos \frac{3\lambda_1\pi}{\lambda_2 + \lambda_3} + (b - a) \cos \frac{3\lambda_3\pi}{\lambda_2 + \lambda_3} < -\frac{1}{2} + b - a < 0.$$

Case 4. $a \geq 1/2$ and $0 < a - b < 1/2$. If $\lambda_2 \leq 2\lambda_1$, let $x_0 = \pi/\lambda_2$. Then

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_2} \pi - a + b \cos \frac{\lambda_3}{\lambda_2} \pi \leq -a + b < 0.$$

If $\lambda_2 \geq 3\lambda_1$, by Lemma 1 there exists $x_0 \in (\pi/(2\lambda_1), \pi/\lambda_1]$ such that $\cos \lambda_2 x_0 = -1$. Therefore,

$$f(x_0) = \cos \lambda_1 x_0 - a + b \cos \lambda_3 x_0 \leq -a + b < 0.$$

If $2\lambda_1 < \lambda_2 < 3\lambda_1$ and $\lambda_3 \geq 4\lambda_1$, then

$$\frac{2\pi}{\lambda_2 + \lambda_3} < \frac{\pi}{3\lambda_1}.$$

So there exists $x_0 = (2n+1)\pi/(\lambda_2 + \lambda_3) \in [2\pi/(3\lambda_1), \pi/\lambda_1]$ such that $\cos \lambda_2 x_0 + \cos \lambda_3 x_0 = 0$, from which it follows that

$$f(x_0) = \cos \lambda_1 x_0 + (a - b) \cos \lambda_2 x_0 \leq -\frac{1}{2} + a - b < 0.$$

If $2\lambda_1 < \lambda_2 < 3\lambda_1$ and $\lambda_3 < 3\lambda_1$, then $\lambda_3 \leq 3\lambda_2/2$. Taking $x_0 = \pi/\lambda_2$, we have

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_2} \pi - a + b \cos \frac{\lambda_3}{\lambda_2} \pi < \frac{1}{2} - a \leq 0.$$

If $2\lambda_1 < \lambda_2 < 3\lambda_1$ and $3\lambda_1 \leq \lambda_3 < 4\lambda_1$, then

$$\frac{2\pi}{3\lambda_1} < \frac{5\pi}{\lambda_2 + \lambda_3} < \frac{\pi}{\lambda_1}.$$

Taking $x_0 = 5\pi/(\lambda_2 + \lambda_3)$, we have

$$f(x_0) = \cos \frac{5\lambda_1}{\lambda_2 + \lambda_3} \pi + (a - b) \cos \frac{5\lambda_2}{\lambda_2 + \lambda_3} \pi < -\frac{1}{2} + a - b \leq 0.$$

Case 5. $a \geq b + 1/2$. If $\lambda_2 < 3\lambda_1$, let $x_0 = \pi/\lambda_2$. Then

$$f(x_0) = \cos \frac{\lambda_1}{\lambda_2} \pi - a + b \cos \frac{\lambda_3}{\lambda_2} \pi < \frac{1}{2} - a + b \leq 0.$$

If $\lambda_2 \geq 3\lambda_1$, by Lemma 1 there exists $x_0 \in (\pi/(2\lambda_1), \pi/\lambda_1]$ such that $\cos \lambda_2 x_0 = -1$. Hence

$$f(x_0) = \cos \lambda_1 x_0 - a + b \cos \lambda_3 x_0 < -a + b < 0.$$

Combining these five cases and (2.2), the proof is completed. $\square$

Lemma 4. Let $a < 0$, $b < 0$, and $0 < \lambda_1 < \lambda_2 < \lambda_3$ be real. Then the first sign change of

$$(2.3) \quad f(x) = \cos \lambda_1 x + a \cos \lambda_2 x + b \cos \lambda_3 x$$

belongs to $(0, \pi/\lambda_1)$.

Since the proof is similar to that of Lemma 3, we omit it here.

3. MAIN RESULTS

Theorem 1. Let $0 < \lambda_1 < \lambda_2 < \lambda_3$. If a and b have the same sign, the first sign change of

$$(3.1) \quad f(x) = \cos \lambda_1 x + a \cos \lambda_2 x + b \cos \lambda_3 x$$

belongs to $(0, \pi/\lambda_1)$.

Proof. This proof follows from Lemmas 3 and 4. $\square$

Theorem 2. Under the same conditions as in Theorem 1, if $a + b \geq -1$, the first sign change of $f(x)$ is a decreasing function of λ_1 ; if $a + b < -1$, the first sign change of $f(x)$ is an increasing function of λ_1 .

Proof. Let

$$f(x; \lambda) = \cos \lambda x + a \cos \lambda_2 x + b \cos \lambda_3 x.$$

For $\lambda_1 < \lambda' < \lambda_2$, note x_0 (resp. x'_0) the first sign change of $f(x; \lambda_1)$ (resp. $f(x; \lambda')$). It follows from Theorem 1 that

$$0 < x_0 < \frac{\pi}{\lambda_1} \quad \text{and} \quad 0 < x'_0 < \frac{\pi}{\lambda'}.$$

Noting that $x'_0 < \pi/\lambda' < 2\pi/(\lambda_1 + \lambda')$ and $\cos \lambda_1 x > \cos \lambda' x$ for $x \in (0, 2\pi/(\lambda_1 + \lambda'))$, we have

$$(3.2) \quad f(x; \lambda_1) > f(x; \lambda') \quad \text{for } x \in (0, x'_0].$$

We will distinguish three cases according to the value of $a + b$:

Case 1. $a + b > -1$. Then $f(0; \lambda_1) = f(0; \lambda'_1) > 0$. By (3.2), we have $x'_0 < x_0$.

Case 2. $a + b = -1$. Then $f(0; \lambda_1) = f(0; \lambda'_1) = 0$ and $f'(0; \lambda_1) = f'(0; \lambda'_1) = 0$. But

$$f''(0; \lambda'_1) = -\lambda'^2 - a\lambda_2^2 - b\lambda_3^2 > \lambda_2^2 - \lambda'^2 > 0.$$

Hence $f(x; \lambda'_1) > 0$, for $x \in (0, x'_0)$. By (3.2), we have $x'_0 < x_0$.

Case 3. $a + b < -1$. Then $f(0; \lambda_1) = f(0; \lambda'_1) < 0$. By (3.2), we have $x_0 < x'_0$. $\square$

Remark 1. Let $a = b = 1$. Theorem 2 shows that Hypothesis 2 of [1] is true for all *trinomials* $\cos \lambda_1 x + \cos \lambda_2 x + \cos \lambda_3 x$, which is an open question in [1].

Remark 2. Let $b = 0$. Theorem 2 gives a corrected version of Proposition 2.1 of [1] (cf. §1).

Remark 3. If a and b do not have the same sign, the above theorems may not be valid. This can be seen by the fact that the first sign change of $g(x, 1)$ is not in $(0, \pi)$ and the first sign change increases with $\lambda \in (1, 1.005)$, where

$$g(x; \lambda) = \cos \lambda x - \frac{4}{3} \cos \frac{4}{3} x + \frac{1}{2} \cos 2x.$$

Finally, as a consequence of Theorem 2, we have the following result which improves the main theorem of [1].

Theorem 3. Let $m > 0$ and $1 < \lambda_1 < \lambda_2 < \lambda_3$ be real. Let

$$T(x; \lambda_1, \lambda_2, \lambda_3) = \cos \lambda_1 x + a \cos \lambda_2 x + b \cos \lambda_3 x,$$

with a and b nonpositive and $a + b < -1$. Then, whenever $0 < \varepsilon \leq \lambda_2 - \lambda_1$, there exists a point $\lambda_0 \in (0, 1]$ such that the first sign change of $T(x; \lambda_0 + \varepsilon, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$ exceeds the first sign change of $T(x; \lambda_0, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$ by more than m .

Proof. Let x_1 and x_2 be the first sign change of $T((m+1)x; 1, \lambda_2, \lambda_3)$ and $T((m+1)x; 1 + \varepsilon, \lambda_2, \lambda_3)$, respectively. By Theorem 2, we have $\delta = x_2 - x_1 > 0$. Let

$$\lambda_0 = \begin{cases} 1 & \text{if } \delta \geq 1, \\ \delta & \text{if } \delta < 1. \end{cases}$$

Clearly, $(m+1)x_1/\lambda_0$ and $(m+1)x_2/\lambda_0$ are, respectively, the first sign change of $T(x; \lambda_0, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$ and $T(x; \lambda_0 + \varepsilon \lambda_0, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$. By Theorem 2, the first sign change of $T(x; \lambda_0 + \varepsilon, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$, say x_3 , is not less than that of $T(x; \lambda_0 + \varepsilon \lambda_0, \lambda_0 \lambda_2, \lambda_0 \lambda_3)$. Therefore

$$x_3 - \frac{(m+1)x_1}{\lambda_0} \geq \frac{(m+1)x_2}{\lambda_0} - \frac{(m+1)x_1}{\lambda_0} = \frac{(m+1)\delta}{\lambda_0} > m. \quad \square$$

4. SOME CONJECTURES

It is plausible that the previous results are also valid in more general cases. From Remark 1, we obtain naturally the following two conjectures.

Hypothesis 1. Let $N \geq 1$, be an integer and $0 < \lambda_1 < \cdots < \lambda_N$. Then the first sign change of

$$\cos \lambda_1 x + \cos \lambda_2 x + \cdots + \cos \lambda_N x$$

belongs to $(0, \pi/\lambda_1)$.

Hypothesis 2. Let $N \geq 1$ be an integer and $0 < \lambda_1 < \cdots < \lambda_N$. Then the first sign change of

$$\cos \lambda_1 x + \cos \lambda_2 x + \cdots + \cos \lambda_N x$$

is a decreasing function of λ_1 .

It is easy to see that if Hypothesis 1 turns out to be true, Hypotheses 2 is true also. So far, the above two hypotheses are verified only for $N \leq 4$. Indeed, for $N = 4$ we have the following propositions:

Proposition 1. Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and

$$f(x) = \cos \lambda_1 x + \cos \lambda_2 x + \cos \lambda_3 x + \cos \lambda_4 x.$$

Then the first sign change of $f(x)$ is in $(0, \pi/\lambda_1)$.

Proposition 2. Hypothesis 2 is true for $N = 4$.

Here we only sketch the proof of Proposition 1 and leave the details to interested readers.

Proof. Since

$$(4.1) \quad f(0) = 4 > 0,$$

we need to show only that there exists a point $x_0 \in (0, \pi/\lambda_1]$ such that $f(x_0) < 0$. Note that if $\lambda_3 < \lambda_1 + \lambda_2$, then

$$f\left(\frac{\pi}{\lambda_3}\right) = 2 \cos \frac{\lambda_1 + \lambda_2}{2\lambda_3} \pi \cos \frac{\lambda_2 - \lambda_1}{2\lambda_3} \pi + \cos \frac{\lambda_4}{\lambda_3} \pi - 1 < 0;$$

if $\lambda_4 \leq \lambda_2 + \lambda_3$, then

$$f\left(\frac{\pi}{\lambda_4}\right) = \cos \frac{\lambda_1}{\lambda_4} \pi + 2 \cos \frac{\lambda_2 + \lambda_3}{2\lambda_4} \pi \cos \frac{\lambda_3 - \lambda_2}{2\lambda_4} \pi - 1 < 0.$$

Therefore we can assume in the sequel that

$$\lambda_3 \geq \lambda_1 + \lambda_2 \quad \text{and} \quad \lambda_4 > \lambda_2 + \lambda_3.$$

We will distinguish among six cases. In every case, we only indicate the point x_0 we wish but omit the details.

Case 1. $\lambda_2 \geq 8\lambda_1$. It is not difficult to see that there exist two integers $n \geq 1$ and $k \geq 1$ such that

$$\frac{2n+1}{\lambda_3 + \lambda_4} \pi \in \left(\frac{4k+1}{2\lambda_2} \pi, \frac{4k+3}{2\lambda_2} \pi \right) \subset \left(\frac{\pi}{2\lambda_1}, \frac{\pi}{\lambda_1} \right).$$

Take $x_0 = (2n+1)\pi/(\lambda_3 + \lambda_4)$.

Case 2. $10\lambda_1/3 < \lambda_2 < 8\lambda_1$. We show that there exists an integer $m \geq 1$ such that

$$\frac{2m+1}{\lambda_3+\lambda_4}\pi \in \left(\frac{8\pi}{3\lambda_2}, \frac{10\pi}{3\lambda_2}\right) \subset \left(\frac{\pi}{3\lambda_1}, \frac{\pi}{\lambda_1}\right).$$

Take $x_0 = (2m+1)\pi/(\lambda_3+\lambda_4)$.

Case 3. $3\lambda_1 < \lambda_2 \leq 10\lambda_1/3$. We show that there exists an integer $s \geq 1$ such that

$$\frac{2s+1}{\lambda_3+\lambda_4}\pi \in \left(\frac{7\pi}{3\lambda_2}, \frac{3\pi}{\lambda_2}\right) \subset \left(\frac{2\pi}{3\lambda_1}, \frac{\pi}{\lambda_1}\right).$$

Take $x_0 = (2s+1)\pi/(\lambda_3+\lambda_4)$.

Case 4. $3\lambda_1/2 \leq \lambda_2 \leq 3\lambda_1$ and $\lambda_3+\lambda_4 \geq 4\lambda_2$. We show that there exists an integer $t \geq 1$ such that

$$\frac{2t+1}{\lambda_3+\lambda_4}\pi \in \left(\frac{\pi}{\lambda_2}, \frac{3\pi}{2\lambda_2}\right) \subset \left(\frac{\pi}{3\lambda_1}, \frac{\pi}{\lambda_1}\right).$$

Take $x_0 = (2t+1)\pi/(\lambda_3+\lambda_4)$.

Case 5. $3\lambda_1/2 \leq \lambda_2 \leq 3\lambda_1$ and $\lambda_3+\lambda_4 < 4\lambda_2$. Take $x_0 = 3\pi/(\lambda_3+\lambda_4)$.

Case 6. $0 < \lambda_2 < 3\lambda_1/2$. We show that there exists an integer $i \geq 1$ such that

$$\frac{2i+1}{\lambda_3+\lambda_4}\pi \in \left(\frac{\pi}{2\lambda_1}, \frac{\pi}{\lambda_1}\right) \subset \left(\frac{\pi}{2\lambda_2}, \frac{3\pi}{2\lambda_2}\right).$$

Take $x_0 = (2i+1)\pi/(\lambda_3+\lambda_4)$.

The conclusion follows by combining the above six cases and (4.1). $\square$

Some other special cases of the above hypotheses are also verified. For example, it is not difficult to establish the following more general result, which both improves and corrects Proposition 3.1 of [1].

Proposition 3. Let $g(x)$ be an even, real-valued function of period 2π that is strictly decreasing on $[0, \pi]$. Assume that

$$(4.2) \quad g(0) = 1, \quad g\left(\frac{\pi}{2}\right) = 0.$$

Let $0 < \lambda_1 < \lambda_2 < \dots < \lambda_N$ and

$$(4.3) \quad G(x) = g(\lambda_1 x) + a_2 g(\lambda_2 x + c_2) + \dots + a_N g(\lambda_N x + c_N),$$

where a_i, c_i are real numbers. Then whenever

$$0 < 4\lambda_j < \lambda_{j+1}, \quad 1 \leq j \leq N-1,$$

if $G(0) > 0$, the first sign change of $G(x)$ is a decreasing function of λ_1 ; if $G(0) < 0$, the first sign change of $G(x)$ is an increasing function of λ_1 ; if $G(0) = G'(0) = \dots = G^{(n-1)}(0) = 0$ but $G^{(n)}(0) \neq 0$ for some integer $n \geq 1$ (only here we assume that $g(x)$ is differentiable up to the n th derivative), then if $G^{(n)}(0) \geq 0$, that sign change is a decreasing function of λ_1 ; if $G^{(n)}(0) < 0$, that sign change is an increasing function of λ_1 .

More ambitiously, the problems are the following:

Hypothesis 3. Let $a_1, \dots, a_N$ and $0 < \lambda_1 < \dots < \lambda_N$ be real numbers. If all a_i have the same sign, then the first sign change of

$$f(x) = \cos \lambda_1 x + a_2 \cos \lambda_2 x + \dots + a_N \cos \lambda_N x$$

belongs to $(0, \pi/\lambda_1)$.

Hypothesis 4. Under the same condition as Hypothesis 3, if $f(0) \geq 0$ (resp. $f(0) < 0$), the first sign change of $f(x)$ is a decreasing (resp. increasing) function of λ_1 .

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