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CONVERGENCE OF ARGUMENTS OF BLASCHKE PRODUCTS IN L_p-METRICS

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ABSTRACT. It is shown that the naturally defined argument of a Blaschke product is a function which is the harmonic conjugate of an integrable function of constant sign. A direct construction of this function is obtained. This fact allows us to investigate the relation between conditions on the zeros of a Blaschke product and the convergence of the arguments of its partial finite subproducts in L_p-metrics, $0 < p \leq \infty$.

1. INTRODUCTION

Let $\mathbb{D} \equiv \{z : |z| < 1\}$, $\mathbb{T} \equiv \partial\mathbb{D}$. Given $\{z_k\} \equiv \sigma \subset \mathbb{D}$ (the points $\{z_k\}$ are enumerated taking into account their multiplicity), we construct the Blaschke product

$$B(z) = \prod_k b_k(z), \qquad b_k(z) = \frac{\overline{z}_k}{|z_k|} \cdot \frac{z_k - z}{1 - \overline{z}_k z}, \qquad z \in \mathbb{D}. \tag{1.1}$$

We suppose for the sake of simplicity that $0 \notin \sigma$. It is well known that the product (1.1) converges in $\mathbb{D}$ iff the Blaschke condition

$$\sum_k (1 - |z_k|) < \infty \qquad \left(\text{or } \sum_k \log \frac{1}{|z_k|} < \infty\right) \tag{1.2}$$

is satisfied.

When investigating the spectral shift function and the trace formula for nonselfadjoint operators [8] and in some problems of complex analysis, it is important to consider the $\log B(z)$. It is natural to define $\log B(z)$ on the disc $\mathbb{D}$ with radial cuts from z_k to $\xi_k \equiv \frac{z_k}{|z_k|}$ in order to have

$$\arg(B_1 B_2) = \arg B_1 + \arg B_2. \tag{1.3}$$

We are going to consider the behavior of $\log B(z)$ on $\mathbb{T}$. This function can be represented in the form of a series (see (2.2)). The direct investigation of

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this series is difficult. The basis for our analysis is that this series represents a function which is the harmonic conjugate of some integrable function and which can be exactly constructed from σ (Theorem 2). This fact allows us to study the convergence of the series constructed in the different spaces L_p, $0 < p \leq \infty$, in terms of σ.

Let $f(\xi) = f(e^{i\varphi}) \equiv f(\varphi)$ be the boundary values of $f(z)$, $z \in \mathbb{D}$ on $\mathbb{T}$, and let $\tilde{f}$ be the harmonic conjugate of f. The symbol C will be used to denote nonessential constants. In what follows we are going to use the standard notation of [3, 5].

2. The choice of the value of the argument: the basic theorem

Let $\nu_k(\varphi) \equiv \arg b_k(\varphi)$ be the branch of the argument of the Blaschke factor, which is fixed by the condition $\nu_k(\varphi_k \pm 0) = \mp\pi$, where $\varphi_k \equiv \arg z_k$. Then (1.1) leads to the formula

$$\tan \nu_k(\varphi) = -\frac{(1-|z_k|^2)\sin(\varphi-\varphi_k)}{2|z_k| - (1+|z_k|^2)\cos(\varphi-\varphi_k)}.$$

Hence,

$$\nu_k(\varphi) = 2\arctan\left\{-\frac{1-|z_k|}{1+|z_k|}\cot\frac{\varphi-\varphi_k}{2}\right\}. \tag{2.1}$$

Definition. The formal series

$$\begin{aligned}\nu(\varphi) = \arg B(\varphi) &= \sum_k \nu_k(\varphi)\\ &= 2\sum_k \arctan\left\{-\frac{1-|z_k|}{1+|z_k|}\cot\frac{\varphi-\varphi_k}{2}\right\}\end{aligned} \tag{2.2}$$

is called the principal value $\nu(\varphi)$ of the argument of the Blaschke product B on $\mathbb{T}$.

If the series (2.2) converges in some sense, then the natural condition (1.3) is obviously satisfied.

We now consider the product

$$G(z) = \left(\prod_k g_k(z)\right)^2, \qquad g_k(z) = \overline{z}_k\frac{\xi_k - z}{1-\overline{z}_k z}, \qquad \xi_k = \frac{z_k}{|z_k|} \tag{2.3}$$

in $\mathbb{D}$.

Theorem 1. *The product $G(z)$ converges in $\mathbb{D}$ iff the Blaschke condition is satisfied. The function $G(z)$ is an outer function and $\|G\|_\infty < 1$.*[1]

Proof. The first part of the theorem is a direct consequence of the equality

$$1 - g_k(z) = 1 - \overline{z}_k\frac{\xi_k - z}{1-\overline{z}_k z} = \frac{1-|z_k|}{1-\overline{z}_k z}. \tag{2.4}$$

[1]Henceforth, $\|f\|_p = (\frac{1}{2\pi}\int_0^{2\pi}|f|^p\,d\varphi)$, $0 < p < \infty$, $\|f\|_\infty = \sup|f(\varphi)|$.

The factors $g_k^2(z)$ are obviously outer functions, and consequently

$$g_k^2(z) = e^{u_k(z)+i\tilde{u}_k(z)}, \qquad z \in \mathbb{D},$$

where $u_k(z)$ is the Poisson integral of the function

$$u_k(\varphi) \equiv \log|g_k^2(z)| = -\log\left\{\frac{1}{|z_k|}\left(1+\frac{(1-|z_k|)^2}{4|z_k|\sin^2\frac{\varphi-\varphi_k}{2}}\right)\right\} < 0. \tag{2.5}$$

By the Cauchy theorem,

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\{u_k(\xi)+i\tilde{u}_k(\xi)\}\frac{d\xi}{\xi} = \log g_k^2(0) = -2\log\frac{1}{|z_k|}. \tag{2.6}$$

But the left-hand side of (2.6) is equal to $\int_0^{2\pi} u_k(\varphi)\frac{d\varphi}{2\pi} = -\|u_k\|_1$ i.e. $\|u_k\|_1 = 2\log 1/|z_k|$. Consequently, the series $\sum_k u_k(\varphi)$ converges in L_1 to some integrable function $u(\varphi)$ and

$$\begin{aligned} G(z) &= \exp\left\{\sum_k u_k(z) + \widetilde{i\sum_k u_k(\varphi)}\right\} \\ &= \exp\{u(z) + i\widetilde{u(z)}\} \end{aligned}$$

is an outer function. In view of (2.5) we have $\|G\|_\infty < 1$. Q.E.D.

The utility of G is demonstrated by

Theorem 2. *The following factorization*

$$B(\xi) = \frac{G(\xi)}{|G(\xi)|} \tag{2.7}$$

holds a.e. on $\mathbb{T}$, *i.e.*

$$\begin{aligned} &\arg B(\varphi) \equiv \nu(\varphi) = \tilde{u}(\varphi), \\ &u(\varphi) = -\sum_k \log\left\{\frac{1}{|z_k|}\left(1+\frac{(1-|z_k|)^2}{4|z_k|\sin^2\frac{\varphi-\varphi_k}{2}}\right)\right\}. \end{aligned} \tag{2.8}$$

Proof. It follows from

$$\begin{aligned} \log g_k^2(\varphi) &= \log|g_k(\varphi)|^2 + 2i\arg g_k(\varphi) \\ &= u_k(\varphi) + i\tilde{u}_k(\varphi), \end{aligned}$$

that

$$\begin{aligned} \tilde{u}_k(\varphi) &= 2\arg g_k(\varphi) = 2\arg\frac{1-e^{i(\varphi-\varphi_k)}}{1-|z_k|e^{i(\varphi-\varphi_k)}} \\ &= 2\arctan\left\{-\frac{1-|z_k|}{1+|z_k|}\cot\frac{\varphi-\varphi_k}{2}\right\} \\ &= v_k(\varphi). \end{aligned}$$

Hence, by M. Riesz's theorem about conjugate functions,

$$\left\| \tilde{u} - \sum_{k=1}^{n} \nu_k \right\|_p \le C_p \left\| \sum_{k=n+1}^{\infty} u_k \right\|_1, \qquad 0 < p < 1.$$

Thus, the series (2.2) converges in L_p, $0 < p < 1$, to the function $\tilde{u}$ and (2.8) is satisfied a.e. Next, according to a theorem by J. Walsh (see [2]), we have $\ell.i.m. \prod_{k=1}^{n} b_k(\varphi) = B(\varphi)$ in the sense of convergence in L_2 where $B(\varphi) = \lim_{r \to 1^-} B(re^{i\varphi})$, from which (2.7) easily follows. Q.E.D.

Note that the possibility of factorization (2.7) with some $G \in H^1$ follows from the Adamjan-Arov-Krein theorem [3, 5].

Formula (2.8) plays an important role in the analysis of series (2.2).

3. Conditions for convergence of the argument

Theorem 3. (i) *The series* (2.2) *converges in measure to a function* ν, $\nu \in \bigcap_{p<1} L_p$, *and*

$$\|\nu\|_p \le C_p \sum_k \log \frac{1}{|z_k|}, \qquad 0 < p < 1. \tag{3.1}$$

(ii) *The series* (2.2) *converges a.e. to* ν *and* $\nu \in L_1$ *if*

$$\sum_k (1 - |z_k|) \log \frac{1}{1 - |z_k|} < \infty. \tag{3.2}$$

In this case

$$\|\nu\|_1 \le C \sum_k (1 - |z_k|) + 2 \sum_k (1 - |z_k|) \log \frac{1}{1 - |z_k|}.$$

Proof. The statement (i) is already proved. The statement (ii) can be obtained by the direct calculation of $\|\nu_k\|_1$:

$$\|\nu_k\|_1 \le (2 + \log 2\pi^2)(1 - |z_k|) + 2(1 - |z_k|) \log \frac{1}{1 - |z_k|}$$

Q.E.D.

Note that if condition (3.2) fails [6], it is possible to select $\{\varphi_k\}$ so that the resulting series (2.2) diverges at every point of $\mathbb{T}$.

The following statement can be obtained by the direct calculation of $\|\nu_k\|_p$.

Theorem 4. *The series* (2.2) *converges in* L_p, $p > 1$, *if* $\sum_k (1 - |z_k|)^{1/p} < \infty$. *In this case,*

$$\|\nu\|_p \le C_p \sum_k (1 - |z_k|)^{1/p}, \qquad p > 1.$$

Theorem 5. *Let* $\xi \in \mathbb{T}$ *and let* $\Gamma_\theta(\xi) = \{z \in \mathbb{D} \colon |\xi - z| \cos\theta \le 1 - |z|\}$ *be the Stolz angle. If*

$$\sup_{\xi \in \mathbb{T}} \operatorname{card}\{z_k \in \Gamma_\theta(\xi)\} = N < \infty \tag{3.3}$$

for some $0 < \theta < \pi/2$, *then* $\nu \in \bigcap_{p>0} L_p$.

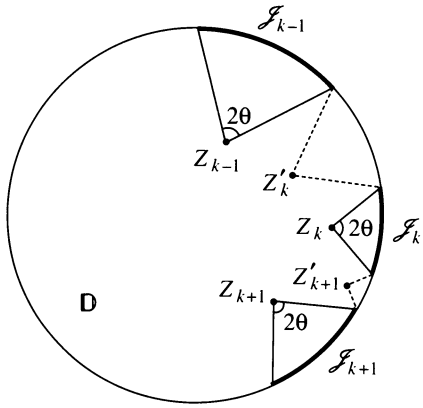

FIGURE 1.

Proof. Condition (3.3) means that the set σ can be decomposed into N subsequences such that each of them has at most one point in any angle $\Gamma_\theta(\xi)$. These subsequences are interpolating [4]. Therefore it is sufficient to prove Theorem 5 in the case $N = 1$. Let us complete our set σ with points $\{z'_k\}$ so that $\operatorname{card}\{z_k, z'_k \in \Gamma_\theta(\xi)\} = 1$ (see Figure 1). Obviously it is sufficient to prove the statement for $\sigma' \equiv \sigma \cup \{z'_k\}$, which we simply denote again as $\sigma(= \{z_k\})$.

Let us represent the function $u(\varphi) = \sum_k u_k(\varphi)$ in the form

$$u(\varphi) = u_{j\varphi}(\varphi) + \sum_{k \neq j_\varphi} u_k(\varphi), \tag{3.4}$$

where j_φ is the index of the arc J_k for which $\varphi \in J_k$. We have

$$u_{j_\varphi}(\varphi) = \log\left\{\frac{1}{|z_k|}\left(1 + \frac{(1-|z_{j_\varphi}|)^2}{4|z_{j_\varphi}|\sin^2\frac{1}{2}(\varphi - \varphi_{j_\varphi})}\right)\right\} \in L_p, \qquad \forall p > 0.$$

The sum in (3.4) can be estimated as follows:

$$\sum_{k \neq j_\varphi} u_k(\varphi) \leq \sum_k \log\frac{1}{|z_k|} + \sum_{k \neq j_\varphi} \frac{(1-|z_k|)^2}{4|z_k|\sin^2\frac{\varphi-\varphi_k}{2}}.$$

Figure 1 shows that for $\varphi \notin J_k$, $(\varphi - \varphi_k)^2 \geq \frac{1}{4}J_k^2 = C_\theta(1-|z_k|)^2$, where $C_\theta \simeq \tan^2\theta$. That is why (for the sufficient choice of the constant C, which depends on θ only)

$$4|z_k|\sin^2\frac{\varphi-\varphi_k}{2} \geq \frac{1}{C}\left(4|z_k|\sin^2\frac{\varphi-\varphi_k}{2} + (1-|z_k|)^2\right).$$

Hence

$$\sum_{k\neq j_\varphi} u_k(\varphi) \le \sum_k \log\frac{1}{|z_k|} + C\sum_{k\neq j_\varphi}(1-|z_k|)P_{z_k}(\varphi), \tag{3.5}$$

where $P_z(\varphi) = \frac{1-|z|^2}{|1-\overline{z}e^{i\varphi}|^2}$ is the Poisson kernel. Now it is sufficient to note that

$$\begin{aligned}\sum_{k\neq j_\varphi}(1-|z_k|)P_{z_k}(\varphi) &\le \sum_k (1-|z_k|)P_{z_k}(\varphi)\\ &= \int_{\mathbb{D}} P_z(\varphi)\,d\mu(z),\end{aligned} \tag{3.6}$$

where μ is the measure on $\mathbb{D}$ given by $\mu(\Omega) = \sum_{z_k\in\Omega}(1-|z_k|)$. Since $\{z_k\}$ is an interpolating sequence, the measure μ is of Carleson type [3, 5]. So [3] the integral in (3.6) determines a function in BMO $\subset \bigcap_{p>0} L_p$. That is why the right-hand side of (3.5) also belongs to $\bigcap_{p>0} L_p$. The theorem is proved, because $\nu = \tilde{u}$.

Let us represent one more statement which, in fact, is the Frostman's theorem [1].

Theorem 6. *The series* (2.2) *converges absolutely and uniformly to a function* $\nu \in L_\infty$ *iff the Frostman condition,*

$$\sup_{\xi\in\mathbb{T}} \sum_k \frac{1-|z_k|}{|1-\overline{z}_k\xi|} < \infty,$$

is satisfied.

Proof. Since $|\arg g_k| \le \frac{\pi}{2}$, we have

$$\begin{aligned}|\nu_k(\varphi)| &= 2|\arg g_k(\varphi)| \le \pi\sin|\arg g_k(\varphi)|\\ &= \frac{1-|z_k|}{|1-\overline{z}_k\xi|}\pi\cos\frac{\varphi-\varphi_k}{2} \le \pi\frac{1-|z_k|}{|1-\overline{z}_k\xi|}.\end{aligned}$$

Now for the converse. Obviously, the behavior of the function ν near the point φ is determined only by those points $z_k \in \sigma$, for which $\cos\frac{\varphi-\varphi_k}{2} \ge C > 0$. Then for such a φ

$$\begin{aligned}|\nu_k(\varphi)| &\ge 2\sin|\arg g_k(\varphi)| = \frac{2(1-|z_k|)}{|1-\overline{z}_k\xi|}\cos\frac{\varphi-\varphi_k}{2}\\ &\ge 2C\frac{1-|z_k|}{|1-\overline{z}_k\xi|}. \quad \text{Q.E.D.}\end{aligned}$$

4. Converse statements

Theorem 5 shows that condition (3.2) does not follow from the summability of $\arg B$ because the assumptions of Theorem 5 pose no other restriction on $\{|z_k|\}_{k\ge1}$ except for the Blaschke condition (1.2) (the lemma of Naftalevich [7]). However, the following theorem then holds.

Theorem 7. *If* $\nu = \arg B \in L_1$, *then* $\forall \Gamma_\theta(\xi)$

$$\sum_{z_k \in \Gamma_\theta(\xi)} (1 - |z_k|) \log \frac{1}{1 - |z_k|} < \infty. \tag{4.1}$$

Proof. From the obvious chain of inequalities,

$$\begin{aligned}
\operatorname{Re} \frac{1 - |z_k|}{1 - \overline{z}_k e^{i\varphi}} &= (1 - |z_k|) \frac{1 - |z_k| + 2|z_k| \sin^2 \frac{\varphi - \varphi_k}{2}}{(1 - |z_k|)^2 + 4|z_k| \sin^2 \frac{\varphi - \varphi_k}{2}} \\
&= 1 - |z_k| \left(1 - \frac{1 - |z_k|}{2}\right) e^{u_k(\varphi)} \\
&\le 1 - \frac{1}{|z_k|} \left(1 - \frac{1 - |z_k|}{2}\right) (1 + u_k(\varphi)) \\
&\le - \frac{1 - |z_k| + (1 + |z_k|) u_k(\varphi)}{2|z_k|},
\end{aligned}$$

we can conclude that the sum of rational fractions,

$$F(z) = \sum_k \frac{1 - |z_k|}{1 - \overline{z}_k z}, \tag{4.2}$$

belongs to the space H^1 together with $\log G(z)$.

We use the duality of the spaces H^1 and BMO (see [3, 5]), which provides that for every function $f \in \mathrm{BMOA} = \mathrm{BMO} \cap H^2$ and $F \in H^1$, there exists a finite limit

$$\lim_{r \to 1^-} \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\varphi}) \overline{F(re^{i\varphi})}\, d\varphi,$$

which determines a linear continuous functional on H^1. For our function (4.2) this limit is equal to

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_{\mathbb{T}} \sum_k \frac{1 - |z_k|}{\xi - r z_k} f(\xi)\, d\xi = \sum_k (1 - |z_k|) f(z_k).$$

That is, for every $f \in \mathrm{BMOA}$,

$$\left| \sum_k (1 - |z_k|) f(z_k) \right| < \infty. \tag{4.3}$$

Because $\nu \in L_1$, $\nu = \tilde{u}$, $u \in L_1$, $u < 0$, every term of the series ν also belongs to L_1 (the direct consequence of Zygmund's theorem about the $L \log L$ class [3, 5]). That is why inequality (3.4) is valid for every subsequence $\sigma \cap \Gamma_\theta(\xi)$. Let us take $f(z) = \log \frac{2}{\xi - z} \in \mathrm{BMOA}$. Then (4.3) leads to the inequality

$$\sum_{z_k \in \Gamma_\theta(\xi)} (1 - |z_k|) \log \left| \frac{2}{\xi - z_k} \right| < \infty,$$

which is equivalent to the original one in the Stolz angle. Q.E.D.

Remark. Duality allows us also to obtain uniformly the necessary condition for $\arg B \in L_1$:

$$\sup_{\xi\in\mathbb{T}} \sum_k (1-|z_k|)\log\left|\frac{2}{1-\overline{z}_k\xi}\right| < \infty.$$

Now we obtain the converse statement to Theorem 4. As in the previous theorem, using the duality $H^p - H^q$, $\frac{1}{p}+\frac{1}{q}=1$ and the test function $(\varepsilon > 0)$

$$f(z) = (\xi - z)^{-1/q}\left(\log\frac{2}{\xi - z}\right)^{-(1+\varepsilon)/q} \in H^q \qquad (\notin H^{q'},\ q' > q),$$

one can prove the following

Theorem 8. *If* $\arg B \in L_p$, $p > 1$, *then* $\forall\Gamma_\theta(\xi)$,

$$\sum_{z_k\in\Gamma_\theta(\xi)} (1-|z_k|)^{1/p}\left(\log\frac{1}{1-|z_k|}\right)^{-(1+\varepsilon)/q} < \infty, \qquad q = \frac{p}{p-1}.$$

Remark 1. One can see that in Theorems 7 and 8, the Stolz angle $\Gamma_\theta(\xi)$ can be replaced by the orocycle

$$C_\alpha(\xi) \equiv \{z \in \mathbb{D}\colon |\xi - z|^2 < \alpha(1-|z|^2),\ \alpha > 0\}$$

as well as by any domain of the form

$$C_\alpha^\gamma(\xi) \equiv \{z \in \mathbb{D}\colon |\xi - z|^\gamma < \alpha(1-|z|)\}, \qquad \gamma > 1.$$

Remark 2. If $\exists\theta$ such that $\sigma \in \bigcup_{k=1}^N \Gamma_\theta(\xi_k)$, $N < \infty$, then Theorems 7 and 8 are valid for all σ. However, for $N = \infty$ this is obviously not true (a counterexample is provided by Theorem 5).

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References

1. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Monographs Math. Phys. no. 56, 1966.
2. P. Colwell, *Blaschke products*, Univ. Michigan Press, Ann Arbor, MI, 1985.
3. J. B. Garnett, *Bounded analytic functions*, Academic Press, Orlando, FL, 1981.
4. S. V. Hruščev and S. A. Vinogradov, *Inner functions and multipliers of Cauchy type integrals*, Ark. Math. **19** (1981), 23–42.
5. P. Koosis, *Introduction to H_p spaces*, Cambridge Univ. Press, New York, 1980.
6. C. Linden, *On Blaschke products diverging everywhere on the boundary of the unit disk*, Proc. Amer. Math. Soc. **55** (1976), 62–64.
7. N. K. Nikol'skii, *Treatise on the shift operator*, Springer-Verlag, Berlin and New York, 1985.
8. A. V. Rybkin, *The spectral shift function for a dissipative and selfadjoint operator, and trace formulas for resonances*, Math. USSR-Sb. **53** (1986), 421–431.

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