MEAN NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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(Communicated by William D. Sudderth)

ABSTRACT. If $a_1, a_2, \dots, a_n$ are independent, normally distributed random variables with mean 0 and variance 1, and if ν_n is the mean value of the number of zeros on the interval $(0, 2\pi)$ of the trigonometric polynomial $a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx$, then

$$\nu_n = 3^{-1/2} \{ (2n+1) + D_1 + (2n+1)^{-1} D_2 + (2n+1)^{-2} D_3 \} + O\{(2n+1)^{-3}\},$$

in which $D_1 = 0.232423 \dots$, $D_2 = -0.25973 \dots$, and $D_3 = 0.2172 \dots$.

1. INTRODUCTION

Suppose that $a_1, a_2, \dots, a_n$ are independent, normally distributed random variables with mean 0 and variance 1, and that ν_n is the mean value of the number of zeros on the interval $(0, 2\pi)$ of the trigonometric polynomial $a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx$. Das [5] has shown that $\nu_n = (2n/3^{1/2}) + O(n^{1/2})$ for large n . Christensen and Sambandham [3] (cited by Bharucha-Reid and Sambandham [2, p. 114]) have shown that $|\nu_n - (2n/3^{1/2})| \leq 10n^{1/2}$ when $n \geq 25$. In this paper we will show that the error term $O(n^{1/2})$ in the Das result is actually $O(1)$. More precisely, we will prove the following theorem:

Theorem. *For large n , it is true that*

$$(1.1) \quad \nu_n = 3^{-1/2} (2n+1) \sum_{r=0}^3 (2n+1)^{-r} D_r + O\{(2n+1)^{-3}\},$$

in which $D_0 = 1$ and D_1, D_2 , and D_3 are explicitly defined constants (see Equations (3.57), (3.58), and (3.59)) whose numerical values are approximately 0.232423, -0.25973, and 0.2172, respectively.

After a statement of the basic formulas on which our analysis rests, we devote §2 to the derivation of a convergent series representation of ν_n (cf. Lemma 7). Asymptotic representations for the first few coefficients in that series are derived in §3 and are used to establish the Theorem. We have calculated 7S values of ν_n

Received by the editors November 20, 1989 and, in revised form, January 26, 1990.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 60G99.

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0002-9939/91 \$1.00 + \$.25 per page

when $n = 2(1)80$. In §4 we use these values to assess the accuracy of (1.1). In particular, the approximation to ν_n obtained from (1.1) if the $O\{(2n+1)^{-3}\}$ term is neglected produces 7S values that differ from the calculated values by at most 10^{-5} when $n \geq 12$ and by only about $(2/15)\%$ when n is as small as 2.

2. PRELIMINARY ANALYSIS

It is known [2, p. 105; 4, p. 285] that, when $n \geq 2$,

$$(2.1) \quad \nu_n = \pi^{-1} \int_0^{2\pi} F_n(x) dx,$$

where

$$(2.2) \quad F_n = A_n^{-1} (A_n C_n - B_n^2)^{1/2},$$

$$(2.3) \quad A_n = \sum_{j=1}^n \cos^2 jx, \quad B_n = \sum_{j=1}^n j \sin jx \cos jx, \quad C_n = \sum_{j=1}^n j^2 \sin^2 jx.$$

It is then obvious that

$$(2.4) \quad \nu_n = 4\pi^{-1} \int_0^{\pi/2} F_n(x) dx.$$

We will need the explicit representations of A_n , B_n , and C_n stated in the following lemma.

Lemma 1. *It is true that*

$$(2.5) \quad 4A_n = (2n+1)g_0 + g_1,$$

$$(2.6) \quad 8B_n = (2n+1)^2 h_0 + (2n+1)h_1 + h_2, \text{ and}$$

$$(2.7) \quad 48C_n = (2n+1)^3 k_0 + (2n+1)^2 k_1 + (2n+1)k_2 + k_3,$$

if the quantities $g_0, g_1, h_0, h_1, h_2, k_0, k_1, k_2$, and k_3 are defined so that

$$(2.8) \quad g_0 = 1 + z^{-1} \sin z = 4 \int_0^{1/2} \cos^2 zy \, dy,$$

$$(2.9) \quad g_1 = -2 + f(x) \sin z, \quad f(x) = \csc x - x^{-1},$$

$$(2.10) \quad h_0 = -dg_0/dz = 4 \int_0^{1/2} y \sin 2zy \, dy = -z^{-1} \cos z + z^{-2} \sin z,$$

$$(2.11) \quad h_1 = -f(x) \cos z, \quad h_2 = -f'(x) \sin z,$$

$$(2.12) \quad \begin{aligned} k_0 &= 1 - 3dh_0/dz = 48 \int_0^{1/2} y^2 \sin^2 zy \, dy \\ &= 1 - 3z^{-1} \sin z - 6z^{-2} \cos z + 6z^{-3} \sin z, \end{aligned}$$

$$(2.13) \quad k_1 = -3f(x) \sin z, \quad k_2 = 6f'(x) \cos z - 1, \quad k_3 = 3f''(x) \sin z, \text{ and}$$

$$(2.14) \quad z = (2n + 1)x.$$

It is a consequence of a known result [6, p. 82; 8, p. 177] (or of a simple mathematical induction) that

$$(2.15) \quad 4A_n = 2n - 1 + \csc x \sin z.$$

The validity of (2.5), together with (2.8) and (2.9), is now obvious. The validity of (2.6) together with (2.10) and (2.11), and of (2.7) together with (2.12) and (2.13), follows at once from (2.5) and the observations that

$$(2.16) \quad \begin{aligned} 2B_n &= -dA_n/dx, \\ 48C_n &= 24 \sum_{j=1}^n j^2 - 24dB_n/dx = (2n+1)^3 - (2n+1) - 24dB_n/dx. \end{aligned}$$

For future reference we record in the following lemma certain properties of the function $f(x)$ [1, pp. 75, 804, 805].

Lemma 2. *If β_{2m} is the Bernoulli number of order $2m$, then*

$$(2.17) \quad \begin{aligned} f(x) &= \sum_{m=1}^{\infty} (-1)^{m-1} (2^{2m} - 2) \beta_{2m} x^{2m-1} / (2m)! \\ &= (x/6) + (7x^3/360) + (31x^5/15120) + \cdots \end{aligned}$$

The power series in (2.17) converges when $|x| < \pi$, and its coefficients are positive.

We calculate the first factor A_n^{-1} in (2.2) in the following lemma:

Lemma 3. *It is true that*

$$(2.18) \quad (4A_n)^{-1} = \{(2n+1)g_0\}^{-1} \sum_{r=0}^{\infty} (2n+1)^{-r} (-g_1/g_0)^r.$$

The series in (2.18) converges absolutely and uniformly when $0 \leq x \leq \pi/2$, $n \geq 2$.

The absolute minimum value of $z^{-1} \sin z$ (when $0 < z < \infty$) is attained when [7, p. 4] $z = 4.4934095 \cdots$, and is greater than -0.2172337 . The maximum value of $f(x)$ (when $0 \leq x \leq \pi/2$) is $f(\pi/2)$, or $1 - 2\pi^{-1}$ (because the coefficients in the power series (2.17) are positive). We infer from (2.8) and (2.9) that $g_0 > 0.7827663$ and $0 < -g_1 \leq 3 - 2\pi^{-1} < 2.363381$. Lemma 3 is now a consequence of the observation that $0 < (2n+1)^{-1}(-g_1/g_0) < 0.604 < 1$ when $n \geq 2$.

A straightforward calculation, based on Equations (2.5) through (2.13), shows

the validity of the following lemma:

Lemma 4. *It is true that*

$$(2.19) \quad 192(A_n C_n - B_n^2) = (2n+1)^4 f_0 + (2n+1)^3 f_1 + (2n+1)^2 f_2 + (2n+1) f_3 + f_4,$$

if the quantities f_0, f_1, f_2, f_3 , and f_4 are defined so that

$$(2.20) \quad f_0 = g_0 k_0 - 3h_0^2 \\ = 1 - 2z^{-1} \sin z - 3z^{-2}(1 + 2 \cos z) + 6z^{-3} \sin z + 3z^{-4} \sin^2 z,$$

$$(2.21) \quad f_1 = g_0 k_1 + g_1 k_0 - 6h_0 h_1 \\ = -2\{k_0 + (\sin z + 3z^{-1} - 3z^{-3} \sin^2 z)f\},$$

$$(2.22) \quad f_2 = g_0 k_2 + g_1 k_1 - 3h_1^2 - 6h_0 h_2 \\ = -1 + 6f \sin z + 6f' \cos z - 3f^2 - z^{-1} \sin z + 6f' z^{-2} \sin^2 z,$$

$$(2.23) \quad f_3 = g_0 k_3 + g_1 k_2 - 6h_1 h_2 \\ = 2 - f \sin z - 12f' \cos z + 3f'' \sin z + 3f'' z^{-1} \sin^2 z,$$

and

$$(2.24) \quad f_4 = g_1 k_3 - 3h_2^2 = 3(ff'' - f'^2) \sin^2 z - 6f'' \sin z.$$

We calculate the second factor $(A_n C_n - B_n^2)^{1/2}$ in (2.2) in the following lemma:

Lemma 5. *There exists an integer n_0 such that $n_0 \geq 2$*

$$(2.25) \quad (192)^{1/2} (A_n C_n - B_n^2)^{1/2} = (2n+1)^2 f_0^{1/2} \sum_{r=0}^{\infty} (2n+1)^{-r} c_r$$

when $n \geq n_0$, in which $b_r = f_r/f_0$ ($r = 0, 1, \dots, 4$), and

$$(2.26) \quad \begin{aligned} c_0 &= 1, \\ c_1 &= b_1/2, \\ c_2 &= (b_2/2) - (b_1^2/8), \\ c_3 &= (b_3/2) - (b_1 b_2/4) + (b_1^3/16), \dots \end{aligned}$$

The series in (2.25) converges absolutely and uniformly when $0 \leq x \leq \pi/2$, $n \geq n_0$.

It is a consequence of the representations of g_0, h_0 , and k_0 as integrals in (2.8), (2.10), (2.12), and the Schwarz inequality that

$$(2.27) \quad f_0/192 = \int_0^{1/2} \cos^2 zy \, dy \int_0^{1/2} y^2 \sin^2 zy \, dy - \left\{ \int_0^{1/2} y \sin zy \cos zy \, dy \right\}^2 > 0$$

when $z > 0$. Moreover, $f_0 = (4z^2/15) + O(z^4)$ for small z and $f_0 = 1 + O(z^{-1})$ for large z . It follows from Lemma 2 that $x^{-1}f(x)$, $x^{-2}\{6f'(x) - 1\}$, and $x^{-1}f''(x)$ are bounded when $0 \leq x \leq \pi/2$. Because $x = z/(2n+1) \leq z/5$

when $n \geq 2$, an inspection of Equations (2.21) through (2.25) now shows that $f_r = O(z^2)$ when z is small, and that $f_r = O(1)$ for all z . This remark suffices to justify the following identity:

$$(2.28) \quad 192(A_n C_n - B_n^2) = (2n+1)^4 f_0 \sum_{r=0}^4 (2n+1)^{-r} b_r,$$

in which $b_r = f_r/f_0$. We may also conclude that $b_r = O(1)$, uniformly in x and n when $0 \leq x \leq \pi/2$, $n \geq 2$.

Let ρ be a fixed constant such that $0 < \rho < 1$. Then there surely exists an integer n_0 such that $n_0 \geq 2$ and

$$(2.29) \quad \sum_{r=1}^4 (2n+1)^{-r} |b_r| < \rho$$

when $n \geq n_0$ and $0 \leq x \leq \pi/2$. Lemma 5 is now a consequence of (2.28) and (2.29).

If we use (2.2) and Lemmas 3 and 5, we obtain the following lemma:

Lemma 6. *It is true that, when $n \geq n_0$,*

$$(2.30) \quad F_n(x) = (2n+1)(f_0/12)^{1/2} g_0^{-1} \sum_{r=0}^{\infty} (2n+1)^{-r} u_r,$$

in which

$$(2.31) \quad u_r = \sum_{s=0}^r (-g_1/g_0)^s c_{r-s}.$$

The series in (2.30) converges absolutely and uniformly when $0 \leq x \leq \pi/2$, $n \geq n_0$.

The following lemma is now a consequence of (2.4) and Lemma 6:

Lemma 7. *It is true that, when $n \geq n_0$,*

$$(2.32) \quad \nu_n = 3^{-1/2} (2n+1) \sum_{r=0}^{\infty} (2n+1)^{-r} v_r,$$

in which

$$(2.33) \quad v_r = 2\pi^{-1} \int_0^{\pi/2} G(z) u_r dx \text{ and}$$

$$(2.34) \quad G(z) = f_0^{1/2} g_0^{-1}.$$

3. PROOF OF THE THEOREM

In the next four lemmas we will exhibit constants S_{rm} ($0 \leq r+m \leq 3$) and S_r ($r = 0, 1, 2, 3$) such that

$$(3.1) \quad v_r = \sum_{m=0}^{3-r} (2n+1)^{-m} S_{rm} + (-1)^n (2n+1)^{r-3} S_r + O\{(2n+1)^{r-4}\}$$

when $r = 0, 1, 2, 3$. In view of (2.32), this will be sufficient to prove (1.1) if

$$(3.2) \quad D_r = \sum_{m=0}^r S_{r-m,m} (r = 0, 1, 2, 3), \quad \sum_{r=0}^3 S_r = 0.$$

Lemma 8. Equation (3.1) is true when $r = 0$ if

$$(3.3) \quad \begin{aligned} S_{00} &= 1, \\ S_{01} &= 2\pi^{-1} \int_0^\infty \{G(z) - 1\} dz, \\ S_{02} &= -3\pi^{-2}, \\ S_{03} &= 0, \\ S_0 &= -8\pi^{-3}. \end{aligned}$$

It follows from (2.33) and (2.31) when $r = 0$, and (2.26) that

$$(3.4) \quad \pi v_0/2 = (\pi/2) + (2n+1)^{-1} \int_0^\lambda \{G(z) - 1\} dz,$$

in which

$$(3.5) \quad \lambda = (2n+1)\pi/2.$$

If z is large, we deduce from the definitions (2.34), (2.8), and (2.20) of $G(z)$, g_0 , and f_0 , respectively, that

$$(3.6) \quad G(z) = 1 - 2z^{-1} \sin z - 0.75z^{-2}(1 + 4 \cos z + \cos 2z) + z^{-3}T_1(z) + O(z^{-4}),$$

in which $T_1(z)$ is a trigonometric sine polynomial of degree 3. It follows from (3.4), and (3.6) to ensure convergence, that

$$(3.7) \quad \begin{aligned} \pi v_0/2 &= (\pi/2) + (2n+1)^{-1} \int_0^\infty \{G(z) - 1\} dz \\ &\quad - (2n+1)^{-1} \int_\lambda^\infty \{G(z) - 1\} dz. \end{aligned}$$

Repeated integration by parts of the last term in (3.7) now shows, with the help of (3.6), that Lemma 8 is true.

Lemma 9. Equation (3.1) is true when $r = 1$ if

$$(3.8) \quad \begin{aligned} S_{10} &= 1, \\ S_{11} &= 2\pi^{-1} \int_0^\infty \{H_1(z) - 1\} dz - 3\pi^{-1} \int_0^{\pi/2} x^{-1} f(x) dx, \\ S_{12} &= 3\pi^{-2} - (3\pi)^{-1} \int_0^\infty z \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\} dz, \\ S_1 &= -16\pi^{-3}, \end{aligned}$$

in which

$$(3.9) \quad \begin{aligned} H_1(z) &= f_0^{-1/2} g_0^{-2} \{1 - 2z^{-1} \sin z - 1.5z^{-2}(3 + 4 \cos z + \cos 2z) \\ &\quad + 3z^{-3}(2 \sin z + \sin 2z)\} \end{aligned}$$

and

(3.10)

$$H_2(z) = f_0^{-1/2} g_0^{-2} \{2 \sin z + 0.5 z^{-1} (5 + \cos 2z) - 3 z^{-2} \sin 2z + 3 z^{-3} \sin^2 z\}.$$

It follows from (2.31) when $r = 1$, (2.26), the definition of b_r as f_r/f_0 , and the definitions (2.8), (2.9), (2.20), and (2.21) of g_0 , g_1 , f_0 , and f_1 , respectively, that

(3.11)

$$G(z)u_1 = H_1(z) - H_2(z)f(x),$$

if H_1 and H_2 are the functions defined in (3.9) and (3.10). We deduce from (3.9), (3.10) and the definitions (2.8) and (2.20) of g_0 and f_0 that, when z is large,

(3.12)

$$H_1(z) = 1 - 3z^{-1} \sin z - 0.75z^{-2}(1 + 4 \cos z + 5 \cos 2z) + z^{-3}T_2(z) + O(z^{-4}),$$

and

(3.13)

$$H_2(z) = 2 \sin z + 3z^{-1} \cos^2 z + 6z^{-2} \sin^3 z + O(z^{-3}),$$

in which $T_2(z)$ is a trigonometric sine polynomial of degree 3. With the help of an argument like that which led from (3.4) through (3.6) to the proof of Lemma 8, we see that

(3.14)

$$\int_0^{\pi/2} H_1(z) dx = (\pi/2) + (2n+1)^{-1} \int_0^\infty \{H_1(z) - 1\} dz + 1.5\pi^{-1}(2n+1)^{-2} + O\{(2n+1)^{-3}\}.$$

Moreover,

(3.15)

$$\int_0^{\pi/2} H_2(z)f(x)dx = I_1 + I_2 + I_3 + I_4 + I_5,$$

in which

(3.16)

$$I_1 = \int_0^{\pi/2} \{2 \sin z + 1.5z^{-1}(1 + \cos 2z)\}f(x) dx,$$

(3.17)

$$I_2 = \int_0^{\pi/2} \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\}(x/6)dx,$$

(3.18)

$$I_3 = \int_0^{\pi/2} \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z - 6z^{-2} \sin^3 z\}(7x^3/360) dx,$$

(3.19)

$$I_4 = 6 \int_0^{\pi/2} z^{-2} \sin^3 z \{f(x) - (x/6)\} dx,$$

and

(3.20)

$$I_5 = \int_0^{\pi/2} \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z - 6z^{-2} \sin^3 z\} \\ \times \{f(x) - (x/6) - (7x^3/360)\} dx.$$

It is obvious that

$$(3.21) \quad \int_0^{\pi/2} f(x) z^{-1} dx = (2n+1)^{-1} \int_0^{\pi/2} x^{-1} f(x) dx,$$

and repeated integration by parts shows that

$$(3.22) \quad \int_0^{\pi/2} f(x) \sin z dx = 4(-1)^n \pi^{-2} (2n+1)^{-2} + O\{(2n+1)^{-4}\}$$

and

$$(3.23) \quad \int_0^{\pi/2} f(x) z^{-1} \cos 2z dx = O\{(2n+1)^{-3}\}.$$

It follows from (3.16), (3.21), (3.22), and (3.23) that

$$(3.24) \quad I_1 = 1.5(2n+1)^{-1} \int_0^{\pi/2} x^{-1} f(x) dx \\ + 8(-1)^n \pi^{-2} (2n+1)^{-2} + O\{(2n+1)^{-3}\}.$$

We use (3.13), (3.5), (2.14), and (2.17) to deduce from (3.17), (3.18), and (3.20) that

$$(3.25) \quad 6I_2 = (2n+1)^{-2} \int_0^\lambda z \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\} dz \\ = (2n+1)^{-2} \int_0^\infty z \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\} dz + O\{(2n+1)^{-4}\},$$

$$(3.26) \quad 360I_3 = (2n+1)^{-4} \int_0^\lambda z^3 O(z^{-3}) dz = O\{(2n+1)^{-3}\},$$

and

$$(3.27) \quad I_5 = \int_0^{\pi/2} O(x^5) O(z^{-3}) dx = O\{(2n+1)^{-3}\}.$$

If we define $g(x)$ so that $f(x) = (x/6) + x^2 g(x)$, then $g(x)$ is an odd, infinitely differentiable function. The reasoning that led to (3.22) now enables us to deduce from (3.19) that

$$(3.28) \quad I_4 = 1.5(2n+1)^{-2} \int_0^{\pi/2} g(x) (3 \sin z - \sin 3z) dx = O\{(2n+1)^{-4}\}.$$

We conclude from (3.15), (3.24), (3.25), (3.26), (3.27), and (3.28) that

$$(3.29) \quad \int_0^{\pi/2} H_2(z) f(x) dx = 1.5(2n+1)^{-1} \int_0^{\pi/2} x^{-1} f(x) dx \\ + (2n+1)^{-2} \left[8(-1)^n \pi^{-2} + 6^{-1} \int_0^\infty z \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\} dz \right] \\ + O\{(2n+1)^{-3}\}.$$

If we combine (3.14) and (3.29) and use (3.11) and (2.33), we see that Lemma 9 is true.

Lemma 10. Equation (3.1) is true when $r = 2$ if

$$(3.30) \quad \begin{aligned} S_{20} &= 1 - 1.5\pi^{-1} \int_0^{\pi/2} f^2(x) dx, \\ S_{21} &= \pi^{-1} \int_0^\infty \{2J_1(z) - 2 - \cos z\} dz - 3\pi^{-1} \int_0^{\pi/2} x^{-1} f(x) dx, \\ S_2 &= 24\pi^{-3}, \end{aligned}$$

in which

$$(3.31) \quad \begin{aligned} 2J_1(z) &= f_0^{-1/2} g_0^{-1} (-1 + \cos z - z^{-1} \sin z + z^{-2} \sin^2 z) \\ &\quad - f_0^{-3/2} g_0^{-1} k_0^2 - 4f_0^{-1/2} g_0^{-2} k_0 + 8f_0^{1/2} g_0^{-3}. \end{aligned}$$

It follows from (2.31) when $r = 2$, (2.26), the definition of b_r as f_r/f_0 , and the definitions (2.8), (2.9), (2.20), (2.21), and (2.22) of g_0 , g_1 , f_0 , f_1 and f_2 , respectively, that

$$(3.32) \quad G(z)u_2 = J_1(z) + J_2(z)f(x) + J_3(z)f^2(x) + J_4(z)\{6f'(x) - 1\},$$

if $J_1(z)$ is the function defined in (3.31), and $J_2(z)$, $J_3(z)$, and $J_4(z)$ are explicitly definable functions concerning which we need to know, by analogy with (3.6), (3.12), and (3.13), only that for all z , $J_2(z)$ is bounded and

$$(3.33) \quad J_2(z) = -3 \sin z - 1.5z^{-1}(1 + 5 \cos 2z) + z^{-2}T_3(z) + O(z^{-3}),$$

$$(3.34) \quad J_3(z) = -0.75(1 + \cos 2z) - 6z^{-1} \sin^3 z + O(z^{-2}), \text{ and}$$

$$(3.35) \quad 2J_4(z) = \cos z + O(z^{-2}),$$

in which $T_3(z)$ is a trigonometric sine polynomial of degree 3. We also need to note that (3.31) implies that $J_1(z)$ is bounded for all z and, for large z , that

$$(3.36) \quad 2J_1(z) = 2 + \cos z - 13z^{-1} \sin z + O(z^{-2}).$$

If for large z we use (3.36), we find that

$$(3.37) \quad \begin{aligned} 2 \int_0^{\pi/2} J_1(z) dx &= \pi + (2n+1)^{-1} \left[(-1)^n + \int_0^\infty \{2J_1(z) - 2 - \cos z\} dz \right] \\ &\quad + O\{(2n+1)^{-2}\}. \end{aligned}$$

If we use (3.21), (3.22), and (3.23), it follows from (3.33) that

(3.38)

$$\begin{aligned}
 \int_0^{\pi/2} J_2(z) f(x) dx &= - \int_0^{\pi/2} \{3 \sin z + 1.5 z^{-1} (1 + 5 \cos 2z)\} f(x) dx \\
 &\quad + \int_0^{\pi/2} \{J_2(z) + 3 \sin z + 1.5 z^{-1} (1 + 5 \cos 2z)\} \{f(x) - (x/6)\} dx \\
 &\quad + \int_0^{\pi/2} \{J_2(z) + 3 \sin z + 1.5 z^{-1} (1 + 5 \cos 2z)\} (x/6) dx \\
 &= -1.5(2n+1)^{-1} \int_0^{\pi/2} x^{-1} f(x) dx + O\{(2n+1)^{-2}\} \\
 &\quad + \int_0^{\pi/2} O(z^{-2}) O(x^3) dx \\
 &\quad + 6^{-1} (2n+1)^{-2} \int_0^{\lambda} \{J_2(z) + 3 \sin z + 1.5 z^{-1} (1 + 5 \cos 2z)\} z dz \\
 &= -1.5(2n+1)^{-1} \int_0^{\pi/2} x^{-1} f(x) dx + O\{(2n+1)^{-2}\}.
 \end{aligned}$$

With the help of some integration by parts, we also find from (3.34) and (3.35) that

(3.39)

$$\begin{aligned}
 4 \int_0^{\pi/2} J_3(z) f^2(x) dx &= - \int_0^{\pi/2} \{3(1 + \cos 2z) + 6z^{-1} (3 \sin z - \sin 3z)\} f^2(x) dx \\
 &\quad + \int_0^{\pi/2} \{4J_3(z) + 3(1 + \cos 2z) + 24z^{-1} \sin^3 z\} f^2(x) dx \\
 &= -3 \int_0^{\pi/2} f^2(x) dx + O\{(2n+1)^{-2}\} + \int_0^{\pi/2} O(z^{-2}) O(x^2) dx \\
 &= -3 \int_0^{\pi/2} f^2(x) dx + O\{(2n+1)^{-2}\},
 \end{aligned}$$

(3.40)

$$\begin{aligned}
 2 \int_0^{\pi/2} J_4(z) \{6f'(x) - 1\} dx &= \int_0^{\pi/2} \{6f'(x) - 1\} \cos z dx \\
 &\quad + \int_0^{\pi/2} O(z^{-2}) O(x^2) dx \\
 &= (24\pi^{-2} - 1)(-1)^n (2n+1)^{-1} + O\{(2n+1)^{-2}\}.
 \end{aligned}$$

If we combine (3.37), (3.38), (3.39), and (3.40), and use (3.32) and (2.33), we find that Lemma 10 is true.

Lemma 11. Equation (3.1) is true when $r = 3$ if

$$(3.41) \quad S_{30} = 2 - 1.5\pi^{-1} \int_0^{\pi/2} f^2(x) dx, \quad S_3 = 0.$$

It follows from (2.31) when $r = 3$, (2.26), the definition of b_r as f_r/f_0 , and the definitions (2.8), (2.9), (2.20), (2.21), (2.22), and (2.23) of g_0, g_1, f_0, f_1, f_2 and f_3 , respectively, that

$$(3.42) \quad G(z)u_3 = K_1 + K_2f + K_3f'' + K_4f^2 + (K_5 + K_6f)(6f' - 1) + K_7f^3,$$

in which $K_1, \dots, K_7$ are explicitly definable functions concerning which we need to know only that K_1 is bounded and that

$$(3.43) \quad K_1 = 2 - 0.5 \cos z + z^{-1}T_4(z) + O(z^{-2}),$$

$$(3.44) \quad K_2 = T_5 \sin z + O(z^{-1}),$$

$$(3.45) \quad K_3 = T_6 \sin z + O(z^{-1}),$$

$$(3.46) \quad 4K_4 = -3 - 15 \cos 2z + O(z^{-1}),$$

$$(3.47) \quad 2K_5 = \cos z + O(z^{-1}),$$

$$(3.48) \quad K_6 = O(z^{-1}), \text{ and}$$

$$(3.49) \quad K_7 = -2 \sin^3 z + O(z^{-1}),$$

in which $T_4(z)$ is a trigonometric sine polynomial of degree 2 and T_5 and T_6 are constants. It then follows by familiar arguments that

$$(3.50) \quad \int_0^{\pi/2} K_1 dx = \pi + O\{(2n+1)^{-1}\},$$

$$(3.51) \quad \int_0^{\pi/2} K_4 f^2 dx = -0.75 \int_0^{\pi/2} f^2(x) dx + O\{(2n+1)^{-1}\},$$

and the remaining terms in (3.42) contribute only $O\{(2n+1)^{-1}\}$ to the value of v_3 . This remark completes the proof of Lemma 11.

We infer from Lemmas 8 through 11 that $S_0 + S_1 + S_2 + S_3 = 0$, and then from (3.2) that (1.1) is valid if $D_0 = 1$, and

$$(3.52) \quad D_1 = 2\pi^{-1} \int_0^\infty \{G(z) - 1\} dz + 1,$$

$$(3.53) \quad \begin{aligned} D_2 = & 3\pi^{-2} + 2\pi^{-1} \int_0^\infty \{H_1(z) - 1\} dz - 3\pi^{-1} \int_0^{\pi/2} x^{-1} f(x) dx + 1 \\ & - 1.5\pi^{-1} \int_0^{\pi/2} f^2(x) dx, \end{aligned}$$

and

$$\begin{aligned}
 D_3 = & 3\pi^{-2} - (3\pi)^{-1} \int_0^\infty z \{H_2(z) - 2 \sin z - 3z^{-1} \cos^2 z\} dz \\
 (3.54) \quad & + \pi^{-1} \int_0^\infty \{2J_1(z) - 2 - \cos z\} dz - 3\pi^{-1} \int_0^{\pi/2} x^{-1} f(x) dx \\
 & + 2 - 1.5\pi^{-1} \int_0^{\pi/2} f^2(x) dx.
 \end{aligned}$$

The conditionally convergent infinite integrals in (3.52), (3.53), and (3.54) can be replaced by absolutely convergent infinite integrals with the help of the identity,

$$(3.55) \quad 1 = 2\pi^{-1} \int_0^\infty z^{-1} \sin z \, dz,$$

and of the relations (3.6), (3.12), (3.13), and (3.36). In addition, it is clear that

$$(3.56) \quad \int_0^{\pi/2} \{f^2(x) + 2x^{-1} f(x)\} dx = \int_0^{\pi/2} (\csc^2 x - x^{-2}) dx = 2\pi^{-1}.$$

Therefore,

$$(3.57) \quad D_1 = 2\pi^{-1} \int_0^\infty \{G(z) - 1 + 2z^{-1} \sin z\} dz - 1,$$

$$(3.58) \quad D_2 = 2\pi^{-1} \int_0^\infty \{H_1(z) - 1 + 3z^{-1} \sin z\} dz - 2, \text{ and}$$

$$\begin{aligned}
 D_3 = & \pi^{-1} \int_0^\infty \{2J_1(z) - 2 - \cos z + 13z^{-1} \sin z\} dz - 5 \\
 (3.59) \quad & - (3\pi)^{-1} \int_0^\infty z \{H(z) - 2 \sin z - 3z^{-1} \cos^2 z - 6z^{-2} \sin^3 z\} dz.
 \end{aligned}$$

This completes the proof of the Theorem, except for the numerical values of D_1 , D_2 , and D_3 .

4. NUMERICAL RESULTS

We calculated the four integrals in (3.57), (3.58), and (3.59) over the interval $(0, 25\pi)$ by Simpson's rule. The integrals over the interval $(25\pi, \infty)$ were calculated using the asymptotic relations (3.6), (3.12), (3.13), and (3.36) for $G(z)$, $H_1(z)$, $H_2(z)$, and $J_1(z)$, respectively. In each case it was desirable to include additional terms in those relations. The results for the coefficients D_1 , D_2 , and D_3 are those recorded in the statement of the Theorem.

We have also numerically computed the integral in (2.4) when $n = 2(1)80$. The approximation $3^{-1/2}(2n+1+D_1)$ is in error (in excess) by about 0.00100% when $n = 80$, 0.00176% when $n = 60$, 0.00391% when $n = 40$, 0.0150% when $n = 20$, 0.0558% when $n = 10$, 0.192% when $n = 5$, and only

0.698% when $n = 2$. The more accurate approximation $3^{-1/2}\{2n + 1 + D_1 + (2n + 1)^{-1}D_2\}$ produces 7S results that differ from the calculated values by at most 10^{-5} (one unit in the last calculated decimal place) when $n \geq 47$, by at most 10^{-4} when $n \geq 18$, by at most 10^{-3} when $n \geq 6$, and by only 0.00905, or about 0.302%, when $n = 2$. The most accurate approximation

$$3^{-1/2}\{2n + 1 + D_1 + (2n + 1)^{-1}D_2 + (2n + 1)^{-2}D_3\}$$

produces 7S results that differ from the calculated results by at most 10^{-5} when $n \geq 12$, by at most 10^{-4} when $n \geq 7$, by at most 10^{-3} when $n \geq 3$, and by 0.00403, or about 0.134% when $n = 2$.

Although it is not really necessary to know the value of the integer n_0 in Lemma 5, we have found numerical bounds for $|b_r|$, when $0 \leq x \leq \pi/2$, $n \geq 2$, which imply that the inequality (2.29) is true if $\rho = 0.93$. Therefore, Lemma 5 is true when $n_0 = 2$.

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