ON PRODUCTS OF CHARACTERS IN A_n

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(Communicated by Warren J. Wong)

ABSTRACT. It is shown that every power with exponent greater than 1 of a nonprincipal irreducible character of A_n has at least 3 irreducible constituents. It is shown that, if the exponent is greater than 3, then the number of irreducible constituents is at least 4.

1. INTRODUCTION AND PRELIMINARY RESULTS

In this paper we discuss powers of the irreducible characters of the non-Abelian simple alternating groups A_n , $n \geq 5$. Let χ be an irreducible character of such a group. For each natural number k , χ^k is a character of A_n defined by $\chi^k(g) = [\chi(g)]^k$. Usually, χ^k is not irreducible. If ϕ is any class function of a finite group G , then $\text{Irr}(\phi)$ denotes the set of the irreducible constituents of ϕ . Our main target in this paper is the following theorem:

Theorem 1. *If $n \geq 5$ and $1_{A_n} \neq \chi \in \text{Irr}(A_n)$, then $|\text{Irr}(\chi^k)| \geq 3$ for every integer $k \geq 2$.*

After some preliminary results in §1 Theorem 1 is proved in §2 for $k = 2$ and in §3 for $k = p$, where p is an odd prime dividing $|A_n|$. The proof for arbitrary k is in §4. In §5 it is shown that the result of Theorem 1 can be improved if $k \geq 4$. The improved result is

Theorem 4. *If $n \geq 5$ and $1_{A_n} \neq \chi \in \text{Irr}(A_n)$, then $|\text{Irr}(\chi^k)| \geq 4$, for every integer $k \geq 4$. Furthermore, 1_{A_n} is a constituent of χ^4 .*

Theorem 1 is the best possible, in some sense. In A_5 we may find two irreducible characters, with degree 3, that satisfy $|\text{Irr}(\chi^2)| = 3$. If $n \geq 6$, there is no example of an irreducible character χ such that $|\text{Irr}(\chi^2)| = 3$. On the other hand, for $6 \leq n \leq 8$ there is an irreducible character of A_n with $|\text{Irr}(\chi^2)| = 4$. The smallest known value of $|\text{Irr}(\chi^3)|$, where $\chi \in \text{Irr}(A_n)$, is

Received by the editors October 6, 1989 and, in revised form, March 26, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20C30, 20C15.

These results are part of the author's Ph.D. thesis written at the Tel Aviv University under the supervision of Prof. M. Herzog.

5. This is the case in A_5 . In every known example we have $|\text{Irr}(\chi^k)| \geq 5$ for $k \geq 4$. This might suggest that Theorem 4 may be improved.

Blau and Chillag prove in [BC] the following theorem:

Theorem 2. *Let G be a finite group and $\chi \in \text{Irr}(G)$. If $|\text{Irr}(\chi^k)| = 1$ for some positive integer $k > 1$, then χ vanishes on $G - Z(\chi)$.*

Since A_n is simple for $n \geq 5$, Theorem 2 implies that $|\text{Irr}(\chi^k)| > 1$ for every nonprincipal χ and for every natural number $k > 1$. In order to prove Theorem 1, it is therefore sufficient to deal with the case $|\text{Irr}(\chi^k)| = 2$. One of our main tools in dealing with powers of characters is the class function $\chi^{(k)}$ defined by $\chi^{(k)}(g) = \chi(g^k)$. We need the following lemma [BC]:

Lemma 1. *If G is a finite group and $\chi \in \text{Irr}(G)$, then $\text{Irr}(\chi^{(k)}) \subseteq \text{Irr}(\chi^k)$.*

If G is a finite group and $\chi \in \text{Irr}(G)$, then χ^σ denotes an algebraic conjugate of χ . It may happen, of course, that $\chi^\sigma = \chi$.

Lemma 2. *Let G be a finite group and $\chi \in \text{Irr}(G)$ such that $Z(\chi) = 1$. Suppose that*

$$(1) \quad \chi^k = a\chi^\sigma + b\phi,$$

where a, b , and k are natural numbers such that k is not prime to $|G|$ and $\phi \in \text{Irr}(G)$. If $g \neq 1$ is an element of G such that $(o(g), k) = 1$ and $\chi(g)$ is rational, then $\chi(g) = 0$.

Proof. It follows from Lemma 1 that

$$(2) \quad \chi^{(k)} = a_1\chi^\sigma + b_1\phi,$$

where $a_1, b_1 \in \mathbb{Z}$. It should be noticed that $b_1 \neq 0$. The reason is that $a_1\chi^\sigma$ is faithful, while $\chi^{(k)}$ is not. Evaluation of (1) and (2) at 1, and elimination of $\phi(1)$ yield

$$\chi^{k-1}(1) = a + \frac{b(1-a_1)}{b_1}.$$

Since $\chi(g)$ is rational, $\chi^\sigma(g) = \chi(g)$. Moreover, g and g^k generate the same cyclic group. It follows from [I, 5.22] that $\chi^{(k)}(g) = \chi(g)$. Using (1) and (2), we get that

$$\begin{aligned} \chi^k(g) &= a\chi(g) + b\phi(g), \\ \chi(g) &= a_1\chi(g) + b_1\phi(g), \end{aligned}$$

which leads to $\chi^k(g) = [a + b(1-a_1)/b_1]\chi(g)$. Therefore, $\chi(g) = 0$ (otherwise $\chi^{k-1}(g) = \chi^{k-1}(1)$ in contradiction to $Z(\chi) = 1$). $\square$

Lemma 3. *Let a be an integer different from 1 or 3. If $\alpha = \pm 1 \pm i\sqrt{a}$ is not rational, then α^k is not rational for every positive integer k .*

Proof. If $a < 0$, let $b = -a$; hence, $\alpha = \pm 1 \pm \sqrt{b}$, $b > 0$. If α is not rational, then b is not a perfect square. The binomial theorem applied to $(1 \pm \sqrt{b})^k$

shows that this number is not rational. It follows that $(-1 \pm \sqrt{b})^k$ is not rational. This finishes the proof in the case $a < 0$. Assume now that $a > 0$. Let $|\alpha| = \sqrt{1+a}$ and $\beta = \alpha/|\alpha|$. If there exists a natural number k such that α^k is rational, then $\beta^{2k} = 1$; i.e., β is a root of unity. Let m be the smallest positive integer such that $\beta^m = 1$. Since β is a primitive root of unity of order m , then $\mathbb{Q}(\beta)/\mathbb{Q}$ is a field extension of degree $\Phi(m)$, where Φ is the Euler function. On the other hand, β is a root of an equation of degree 4 at the most. Therefore $\Phi(m) \leq 4$ and m is one of the following numbers: 3, 4, 5, 6, 8, 10, or 12. It is easy to see that if a is not 1 or 3, then β is not a root of unity of one of these orders. The claim of the Lemma follows. $\square$

Corollary 4. *Let $\chi \in \text{Irr}(A_n)$, $n \geq 5$. If $\chi(g)$ is not rational, then $\chi^k(g)$ is not rational for every natural number k .*

Proof. In [JK, p. 67] it is shown that if $\chi(g)$ is not rational, then its value is equal to $\alpha/2$, where α satisfies the conditions of Lemma 3. The result follows from Lemma 3. $\square$

Let us now pay attention to some special properties of the irreducible characters of A_n . Every irreducible character of A_n has at the most 2 irrational values. If $\chi(g)$ is not rational for some element g , then there is exactly one more character of A_n which is not rational on g . This character is an algebraic conjugate of χ . For a proof, see [JK, p. 67].

Corollary 5. *If $\chi \in \text{Irr}(A_n)$, $n \geq 5$, and χ is not rational-valued, then for each natural number k there is an algebraic conjugate of χ which is a constituent of χ^k . In particular, if χ is not real-valued, then χ or $\bar{\chi}$ is a constituent of χ^k .*

Proof. Let $g \in A_n$ be an element such that $\chi(g)$ is not rational. According to Corollary 4, $\chi^k(g)$ is not rational. If all the constituents of χ^k are not conjugate to χ , then all of them are rational on g . This contradicts the irrationality of $\chi^k(g)$. $\square$

2. THE CASE $k = 2$

In this section, we assume that $1_{A_n} \neq \chi \in \text{Irr}(A_n)$, $n \geq 5$. We show that χ^2 has at least 3 irreducible constituents. In order to do this we assume the contrary. Thus, let $\chi^2 = a\phi + b\psi$, where a and b are positive integers and $\phi, \psi \in \text{Irr}(A_n)$. By Theorem 2, we may assume that $\phi \neq \psi$. This relation will lead us to a contradiction. The proof is in the next six steps.

Step 1. χ is not real-valued.

Proof. If χ is real valued, then $\chi = \bar{\chi}$ and $[\chi, \bar{\chi}] = [\chi^2, 1_{A_n}] = 1$. Therefore $\chi^2 = 1_{A_n} + b\psi$. Let $g \in A_n$ such that $\chi(g) = 0$. Substitution of g in the last equation shows us that $\psi(g) = -1/b$, so $b = 1$ and $\chi^2 = 1_{A_n} + \psi$. Now put $m = \chi(1)$. The character χ^2 has a canonical decomposition into two characters

of degrees $(m^2+m)/2$ and $(m^2-m)/2$ [I, 4.5]. It follows that $(m^2-m)/2 = 1$. Hence $m = 2$, which is impossible in a simple group. $\square$

Step 2. ϕ is an algebraic conjugate of χ .

Proof. According to Corollary 5, ϕ or ψ must be conjugate to χ . Without loss of generality we may assume that $\phi = \chi^\sigma$. $\square$

Step 3. ψ is not conjugate to χ .

Proof. Suppose, to the contrary, that ψ is conjugate to χ . Let u be an involution in A_n . It follows that $\psi(u) = \chi^\sigma(u) = \chi(u)$, and $\psi(1) = \chi(1)$. Since, by Lemma 1, $\chi^{(2)} = a_1\chi^\sigma + b_1\psi$, we get that

$$\begin{aligned}(a_1 + b_1)\chi(1) &= a_1\chi^\sigma(1) + b_1\psi(1) = \chi^{(2)}(1) = \chi(1) \\ &= \chi^{(2)}(u) = a_1\chi(u) + b_1\psi(u) = (a_1 + b_1)\chi(u).\end{aligned}$$

It follows from $\chi(1) = (a_1 + b_1)\chi(1)$ that $a_1 + b_1 \neq 0$. Hence, the last chain of equalities show us that $\chi(1) = \chi(u)$, which is nonsense. $\square$

Step 4. $\chi^2 = \chi^\sigma + b\psi$ (i.e., $a = 1$).

Proof. Since χ is not real-valued, then $\bar{\chi}$ is its unique nontrivial algebraic conjugate. It follows that we have only two cases to consider:

Case (i). $\chi^2 = a\chi + b\psi$. Let $g \in A_n$ be an element such that $\chi(g)$ is not real. Denote $X = \chi(g)$. It follows from Step 3 that X is a root of the equation $X^2 - aX + m = 0$ (where $m = -b\psi(g)$). Notice that m is rational, by Step 3 and the remarks following Corollary 4. The roots of this equation are $\chi(g)$ and $\bar{\chi}(g)$. Since $\chi(g) + \bar{\chi}(g) = \pm 1$ [JK, 2.5.13] we have that $a = 1$.

Case (ii). $\chi^2 = a\bar{\chi} + b\psi$. In terms of the first case we get $X^2 = a\bar{X} - m$. Therefore $\bar{X}^2 = aX - m$. Subtraction of these two equations leads us to $-a = \chi(g) + \bar{\chi}(g) = \pm 1$. Once again, we get $a = 1$. $\square$

Step 5. $\text{Rank}_2(A_n) \leq 2$.

Proof. The last step implies $\chi^2 = \chi^\sigma + b\psi$. We know that $\chi^{(2)} = a_1\chi^\sigma + b_1\psi$. Since both $(\chi^2 \pm \chi^{(2)})/2$ are characters of A_n [I, 4.5], $a_1 = \pm 1$ and $b_1 \leq b$. If $a_1 = 1$ then $b_1\psi(1) = 0$ which is impossible. Therefore $\chi^{(2)} = -\chi^\sigma + b_1\psi$, where $b_1 > 0$. In particular, we get $\psi(1) = 2\chi(1)/b_1$. It follows that

$$\chi^2(1) = \chi(1) + \frac{2b}{b_1}\chi(1).$$

Therefore $\chi(1) = 1 + 2b/b_1$. Let u be an involution in A_n . The equations

$$\begin{aligned}\chi^2(u) &= \chi(u) + b\psi(u), \\ \chi(1) &= -\chi(u) + b_1\psi(u)\end{aligned}$$

lead us to the following quadratic equation:

$$\chi^2(u) - \left(1 - \frac{b}{b_1}\right)\chi(u) - \frac{b}{b_1}\chi(1) = 0.$$

The roots of this equation are $1 + 2b/b_1$ and $-b/b_1$. Since χ is faithful, $\chi(u) = -b/b_1$.

Let E be an elementary Abelian 2-group contained in A_n . Then

$$0 \leq |E|[\chi_E, 1_E] = \chi(1) + (|E| - 1)\chi(u) = 1 + \frac{3b}{b_1} - |E|\frac{b}{b_1}.$$

Therefore, we get $|E| \leq 3 + b_1/b \leq 4$. In particular, $\text{Rank}_2(A_n) \leq 2$. $\square$

Step 6. Final contradiction.

Proof. If $n \geq 8$, then $\text{Rank}_2(A_n) \geq 3$, so we have to deal only with the cases $n = 5, 6, 7$. Since A_5 and A_6 are real groups, then by Step 1, all we have to do is take care of A_7 . This group has only 2 (conjugate) nonreal irreducible characters. A simple check of the character table show that the square of each of these characters has 6 irreducible constituents. This fact completes the proof of the claim of this section. $\square$

3. THE CASE $k = p$

In this section, we assume that p is an odd prime which divides $|A_n|$. We show that χ^p has at least 3 irreducible constituents for each $1_{A_n} \neq \chi \in \text{Irr}(A_n)$. The proof is by contradiction. We assume that $\chi^p = a\phi + b\psi$, $a, b \in \mathbb{N}$, $\phi, \psi \in \text{Irr}(A_n)$. The contradiction is derived in the next 4 steps.

Step 1. ϕ is an algebraic conjugate of χ .

Proof. We have to separate the proof into two cases.

(i) χ is real-valued. In this case

$$\chi^2 = 1_{A_n} + \{\text{other characters}\},$$

so

$$\chi^{p-1} = 1_{A_n} + \{\text{other characters}\}.$$

Therefore

$$\chi^p = \chi + \{\text{other characters}\}.$$

It follows that χ is a constituent of χ^p , and we may assume $\phi = \chi$.

(ii) χ is not real-valued. According to Corollary 5, ϕ or ψ must be conjugate to χ . $\square$

By Step 1 we may assume that $\chi^p = a\chi^\sigma + b\psi$.

Step 2. χ is of 2-defect 0.

Proof. Suppose g is a 2-element in A_n . Since only elements of odd order can be irrational elements in A_n , $\chi(g)$ is rational. Lemma 2 implies that $\chi(g) = 0$ provided that $g \neq 1$. Our claim follows. $\square$

Step 3. There exists an odd prime $q \neq p$ such that χ is not of q -defect 0.

Proof. Let χ be a character of A_n of 2-defect 0. We have to separate the discussion into two cases.

Case (i). There exists $\phi \in \text{Irr}(S_n)$ such that $\phi|_{A_n} = \chi + \chi^\sigma$. In this case $\phi(1) = 2\chi(1)$, so ϕ is of 2-defect 0. By [JK, 2.3.21] ϕ is associated to a partition of n of the form $[k-1, k-2, \dots, 2, 1]$, where $n = \binom{k}{2}$ and $k \geq 4$. The Young diagram of such a partition contains hooks of every odd length, which is $\leq 2k-3$. In particular, it contains a hook of length q , with $p \neq q \leq 2k-3$ an odd prime. According to [JK, 2.3.21] χ is not of q -defect 0.

Case (ii). There exists $\phi \in \text{Irr}(S_n)$ such that $\phi|_{A_n} = \chi$. In this case, $\phi(1) = \chi(1)$, so ϕ is of 2-defect 1. The associated partition of the character ϕ must contain exactly one hook of length 2. The only possible partitions are $[k+1, k-2, k-3, \dots, 2, 1]$ and its adjoint partition. Here $k \geq 3$ and $n = \binom{k}{2} + 2$. The Young diagrams of these partitions contain hooks of every odd length, which is $\leq 2k-1$. Our result follows as in case (i). $\square$

Step 4. Final contradiction.

Proof. Here we show that the q -defect of χ is 0, where q is an odd prime different from p . This is in contradiction to Step 3. Let $g \in A_n$ be a q -element. We show that $\chi(g)$ is rational (apart from one exception which does not violate our theorem). We have to separate the discussion into the two cases of Step 3.

Case (i). Since g is a q -element, its cycle structure must contain only cycles of lengths which are powers of q . If g is not rational, then these lengths must be the lengths of the main hooks of the corresponding diagram [JK, 2.5.13]. The only possible case is $q = 5$, $k = 4$, $n = 6$, and ϕ is the character of S_6 which is associated to the partition $[3, 2, 1]$. If χ is one of the constituents of $\phi|_{A_6}$ then, using a character table of A_6 , we get that $\text{Irr}(\chi^3) = \text{Irr}(A_6)$. Hence χ cannot satisfy $|\text{Irr}(\chi^p)| = 2$. In any other case, $\chi(g)$ is rational.

Case (ii). This is a trivial case since χ is rational-valued. In any event, Lemma 2 implies that the q -defect of χ is 0. This contradicts Step 3. $\square$

4. A PROOF OF THEOREM 1

We are now able to prove our main result.

Theorem 1. *If $n \geq 5$ and $1_{A_n} \neq \chi \in \text{Irr}(A_n)$, then $|\text{Irr}(\chi^k)| \geq 3$ for every integer $k \geq 2$.*

Proof. First of all, by Theorem 2, $|\text{Irr}(\chi^k)| \geq 2$. We use induction on k in order to prove that $|\text{Irr}(\chi^k)| \neq 2$. By §2 the result is true for $k = 2$. The result of §3 shows the assertion for $k = 3$. Suppose now that $k \geq 4$ and that the claim is true for smaller values of k .

As always, we have to separate the proof into two cases:

Case (i). χ is a real-valued character. In this case, $[\chi^2, 1_{A_n}] = 1$ and

$$\chi^k = \chi^2 \chi^{k-2} = (1_{A_n} + \{\text{other characters}\}) \cdot \chi^{k-2}.$$

Since $k \geq 4$, χ^{k-2} has at least 3 irreducible constituents. It follows that χ^k has the same 3 irreducible constituents (at least).

Case (ii). χ is not a real-valued character. We treat first the case $k = 4$. According to Corollary 5, χ or $\bar{\chi}$ are constituents of χ^2 . It follows that χ^2 or $\bar{\chi}^2$ are constituents of χ^4 ; hence, our claim is proved in this case. If $k = 5$, the theorem is a direct consequence of §3. Assume now that $k \geq 6$. In this case, we divide the rest of the proof into subcases.

(A) There exists a natural number l , $2 \leq l \leq k-2$, such that $[\chi^{l-1}, \bar{\chi}] \neq 0$. Here we have

$$\chi^l = 1_{A_n} + \{\text{other characters}\}.$$

Since $\chi^k = \chi^l \chi^{k-l}$ and $k-l \geq 2$, the result follows in this subcase.

(B) For every natural number l , $2 \leq l \leq k-2$, we have $[\chi^{l-1}, \bar{\chi}] = 0$. It follows from Corollary 5 that for every such l we have $[\chi^{l-1}, \chi] > 0$. In particular, χ is a constituent of both χ^3 and χ^{k-3} . Therefore χ^2 is a constituent of χ^k . It follows that χ^k has at least 3 irreducible constituents. $\square$

5. RESULTS FOR $k \geq 4$

In this section we show that our previous results may be slightly improved.

Lemma 6. Let χ be a character of a finite group G such that $\text{Irr}(\chi) = \text{Irr}(G)$. For each character ϕ of G , $\text{Irr}(\phi\chi) = \text{Irr}(G)$.

Proof. The irreducible constituents of a character do not change if the character is multiplied by an integer. We may therefore assume that the regular character ρ is a constituent of χ . Hence $\chi = \rho + \{\text{other characters}\}$. Therefore $\phi\chi = \phi(1)\rho + \{\text{other characters}\}$. It follows that ρ is a constituent of $\phi\chi$. This proves our claim. $\square$

Lemma 7. Let χ be a faithful character of a finite group G such that $1_G \in \text{Irr}(\chi)$. If ϕ is a character of G and $\text{Irr}(\phi) \subset \text{Irr}(G)$, then $\text{Irr}(\phi) \subset \text{Irr}(\chi\phi)$.

Proof. Trivially, $\text{Irr}(\phi) \subseteq \text{Irr}(\chi\phi)$. If these two sets are equal, then $\text{Irr}(\phi) = \text{Irr}(\chi^k\phi)$ for every natural number k . Choose k such that $\text{Irr}(\chi^k) = \text{Irr}(G)$.

The existence of such k is proved in [ACH]. Using Lemma 6, we get $\text{Irr}(\phi) = \text{Irr}(G)$, in contradiction to our assumptions. $\square$

Theorem 3. *Let $1_{A_n} \neq \chi \in \text{Irr}(A_n)$, $n \geq 5$. Then $|\text{Irr}(\chi^4)| \geq 4$. Furthermore, $1_{A_n} \in \text{Irr}(\chi^4)$.*

Proof. First, notice that we may assume that $\text{Irr}(\chi^2) \neq \text{Irr}(G)$. For, otherwise, Lemma 6 would imply $|\text{Irr}(\chi^4)| = |\text{Irr}(A_n)| \geq 4$. Suppose that χ is a real character. In this case $1_{A_n} \in \text{Irr}(\chi^2)$. It follows that $1_{A_n} \in \text{Irr}(\chi^4)$. Using Lemma 7 we get $|\text{Irr}(\chi^4)| > |\text{Irr}(\chi^2)| \geq 3$.

We may assume now that χ is not real-valued. In this case, by Corollary 5, there is an algebraic conjugate χ^σ of χ such that χ^σ is a constituent of χ^2 . Therefore all of the constituents of $(\chi^\sigma)^2$ are in $\text{Irr}(\chi^4)$. It follows that χ^4 has at least 3 nonprincipal irreducible constituents.

Our proof will be complete once we show that $1_{A_n} \in \text{Irr}(\chi^4)$. Let $\phi \in \text{Irr}(\chi^2)$. We may choose a ϕ which is not conjugate to χ . If ϕ is real-valued, then $1_{A_n} \in \text{Irr}(\phi^2) \subseteq \text{Irr}(\chi^4)$. If ϕ is not real-valued but $\bar{\phi} \in \text{Irr}(\chi^2)$, then $1_{A_n} \in \text{Irr}(\phi\bar{\phi}) \subseteq \text{Irr}(\chi^4)$. We show that one of the above two cases must occur. This can be seen as follows: Suppose, to the contrary, that ϕ is not real-valued and that $\bar{\phi} \notin \text{Irr}(\chi^2)$. Take $g \in A_n$ such that $\phi(g)$ is not real. Since $\bar{\phi} \notin \text{Irr}(\chi^2)$, then all of the irreducible constituents of χ^2 except ϕ are rational at g . This is impossible, since χ is not conjugate to ϕ and therefore is rational at g . $\square$

Lemma 8. *Let $1_{A_n} \neq \chi \in \text{Irr}(A_n)$. Then $|\text{Irr}(\chi^5)| \geq 4$.*

Proof. If χ is real-valued then $1_{A_n} \in \text{Irr}(\chi^2)$. By Theorem 1 $|\text{Irr}(\chi^3)| \geq 3$. Since $\chi^5 = \chi^2\chi^3$, Lemma 7 implies that $|\text{Irr}(\chi^5)| \geq 4$.

Suppose that χ is not real-valued. In this case, one of χ or $\bar{\chi}$ must be a constituent of χ^2 . If $\bar{\chi} \in \text{Irr}(\chi^2)$, then $\chi\bar{\chi}$ is a constituent of χ^3 . Therefore $1_{A_n} \in \text{Irr}(\chi^3)$. Our result follows in this case from Lemma 7. Assume now that $\chi \in \text{Irr}(\chi^2)$. In this case, $\chi \in \text{Irr}(\chi^3)$, and χ^2 is a constituent of χ^5 . By Theorem 3, $\bar{\chi} \in \text{Irr}(\chi^3)$. Hence, $\chi\bar{\chi} \in \text{Irr}(\chi^5)$. It follows that $1_{A_n} \in \text{Irr}(\chi^5)$ in addition to the 3 nonprincipal constituents of χ^2 , thus proving our claim. $\square$

Lemma 9. *Let $1_{A_n} \neq \chi \in \text{Irr}(A_n)$ and $k \geq 6$. Then $|\text{Irr}(\chi^k)| \geq 4$.*

Proof. Write $\chi^k = \chi^{k-4}\chi^4$, and apply Lemma 7, Theorem 3, and Theorem 1. $\square$

Combining the above results we get

Theorem 4. *If $n \geq 5$ and $1_{A_n} \neq \chi \in \text{Irr}(A_n)$ then $|\text{Irr}(\chi^k)| \geq 4$ for every integer $k \geq 4$. Furthermore, 1_{A_n} is a constituent of χ^4 .*

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