ON YAMAMOTO'S RECIPROCITY LAW

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ABSTRACT. A simple proof of Yamamoto's reciprocity law is given.

Let p and q be distinct odd primes with $p \equiv q \equiv 1 \pmod{4}$. Define the symbol $[p, q] = \pm 1$ by

$$[p, q] \equiv p^{(q-1)/4} \left(\frac{p}{q}\right)^{h(pq)/2} \pmod{q},$$

where $h(pq)$ is the classnumber of the real quadratic field $Q(\sqrt{pq})$ and $\varepsilon(pq) = \frac{1}{2}(t + u\sqrt{pq}) > 1$ is its fundamental unit (t, u positive integers). Yamamoto's reciprocity law [5, Theorem 3] states that

$$[p, q] = [q, p].$$

We give a simple proof of a slightly stronger result:

Theorem. Let p and q be distinct primes with $p \equiv q \equiv 1 \pmod{4}$; let $A(p, q)$ be the number of pairs (x, y) of integers satisfying

$$0 < x < \frac{p}{2}, \quad 0 < y < \frac{q}{2}, \quad qx < py, \quad \left(\frac{x}{p}\right) = -\left(\frac{y}{q}\right),$$

where (x/p) is the familiar Legendre symbol. Then

$$[p, q] = [q, p] = (-1)^{A(p, q)} \left(\frac{p}{q}\right).$$

The following lemma is well known; notation is as above.

Lemma. $h(pq)$ is even; furthermore

- (a) $h(pq) \equiv 0 \pmod{4}$, if $\left(\frac{p}{q}\right) = 1$, $\left(\frac{p}{q}\right)_4 = \left(\frac{q}{p}\right)_4$ [1, Theorem 4; 4, p. 603],
- (b) $\left(\frac{p}{2}\right)^2 - pq\left(\frac{q}{2}\right)^2 = +1$, if $\left(\frac{p}{q}\right) = 1$, $\left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4$ [4, p. 603],
- (c) $h(pq) \equiv 2 \pmod{4}$, $\left(\frac{p}{2}\right)^2 - pq\left(\frac{q}{2}\right)^2 = -1$, if $\left(\frac{p}{q}\right) = -1$ [1, Theorem 4; 3, §3].

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Proof of the theorem. By the lemma, a straightforward calculation shows that

$$[p, q]^2 \equiv \left(p^{(q-1)/4} \left(\frac{t}{2} \right)^{h(pq)/2} \right)^2 \equiv 1 \pmod{q};$$

hence, the symbol $[p, q]$ is indeed $+1$ or -1 .

Let $\delta = \pm 1$, $\varepsilon = \pm 1$. An application of Dirichlet's classnumber formula applied to each of the real quadratic fields $Q(\sqrt{p})$, $Q(\sqrt{q})$, $Q(\sqrt{pq})$ gives

$$(1) \quad \prod_{\substack{0 < k < pq/2 \\ (k/p)=\delta, (k/q)=\varepsilon}} 2 \sin(k\pi/pq) \\ = \varepsilon(p)^{-\delta(1-(q/p))h(p)/4} \varepsilon(q)^{-\varepsilon(1-(p/q))h(p)/4} \varepsilon(pq)^{-\delta\varepsilon h(pq)/4}.$$

The trigonometric product on the left side of (1) is an integer of the cyclotomic field $Q(e^{2\pi i/pq})$. Next, if $\mathcal{P}$ (resp. $\mathcal{Q}$) is a prime ideal of $Q(e^{2\pi i/pq})$ dividing $1 - e^{2\pi i/p}$ (resp. $1 - e^{2\pi i/q}$), we can show using the method of Bucher [2]

$$(2) \quad \prod_{\substack{0 < k < pq/2 \\ (k/p)=\delta, (k/q)=\varepsilon}} 2 \sin(k\pi/pq) \\ \equiv \begin{cases} (-1)^{N(p, q, \delta, \varepsilon) + (p-1)(q-1)/16} q^{(p-1)/8} \varepsilon(q)^{-(p/q)\varepsilon h(q)(p-1)/4} \pmod{\mathcal{P}}, \\ (-1)^{N(p, q, \delta, \varepsilon)} p^{(q-1)/8} \varepsilon(q)^{-(q/p)\delta h(p)(q-1)/4} \pmod{\mathcal{Q}}, \end{cases}$$

where $N(p, q, \delta, \varepsilon)$ = number of pairs (x, y) of integers satisfying

$$(3) \quad 0 < x < p/2, \quad 0 < y < q/2, \quad qx < py, \\ \left(\frac{x}{p}\right) = \left(\frac{q}{p}\right)\delta, \quad \left(\frac{y}{q}\right) = \left(\frac{p}{q}\right)\varepsilon.$$

From (1) and (2), we have

$$(4) \quad q^{(p-1)/8} \equiv (-1)^{N(p, q, \delta, \varepsilon) + (p-1)(q-1)/16} \varepsilon(p)^{-\delta(1-(p/q))h(p)/4} \\ \times \varepsilon(q)^{\varepsilon((p/q)-1)h(q)/4} \varepsilon(pq)^{-\delta\varepsilon h(pq)/4} \pmod{\mathcal{P}}$$

and

$$(5) \quad p^{(q-1)/8} \equiv (-1)^{N(p, q, \delta, \varepsilon)} \varepsilon(p)^{\delta((q/p)q-1)h(p)/4} \\ \times \varepsilon(q)^{-\varepsilon(1-(p/q))h(q)/4} \varepsilon(pq)^{-\delta\varepsilon h(pq)/4} \pmod{\mathcal{Q}}.$$

Mapping $p \rightarrow q$, $q \rightarrow p$, $\delta \rightarrow \varepsilon$, $\varepsilon \rightarrow \delta$ in (4), so that $\mathcal{P} \rightarrow \mathcal{Q}$, we obtain

$$(6) \quad N(p, q, \delta, \varepsilon) \equiv N(q, p, \varepsilon, \delta) + (p-1)(q-1)/16 \pmod{2}.$$

Taking $(\delta, \varepsilon) = (1, -1)$ and $(-1, 1)$ in (5) and multiplying the resulting two congruences together, we deduce

$$(7) \quad p^{(q-1)/4} \equiv (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)} \varepsilon(pq)^{h(pq)/2} \pmod{\mathcal{Q}}.$$

As $\mathcal{Q}$ divides $(1 - e^{2\pi i/q})$ and $(1 - e^{2\pi i/q})$ divides $\sqrt{q}$, we deduce from (7)

$$(8) \quad p^{(q-1)/4} \equiv (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)} (t(pq)/2)^{h(pq)/2} \pmod{\mathcal{Q}}.$$

Further, as both sides of (8) are integers (mod q), we have

$$(9) \quad p^{(q-1)/4} \equiv (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)} (t(pq)/2)^{h(pq)/2} \pmod{q}.$$

Multiplying both sides of (9) by $p^{(q-1)/4} (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)}$, we obtain

$$(10) \quad [p, q] = (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)} \left(\frac{p}{q}\right).$$

Similarly, from (4), we obtain

$$(11) \quad [q, p] = (-1)^{N(p, q, 1, -1) + N(p, q, -1, 1)} \left(\frac{q}{p}\right).$$

Hence, by the law of quadratic reciprocity, we have Yamamoto's reciprocity law in the form

$$(12) \quad [p, q] = [q, p] = (-1)^{A(p, q)} \left(\frac{p}{q}\right),$$

where $A(p, q)$ = number of pairs (x, y) of integers satisfying

$$(13) \quad 0 < x < \frac{p}{2}, \quad 0 < y < \frac{q}{2}, \quad qx < py, \quad \left(\frac{x}{p}\right) = -\left(\frac{y}{q}\right).$$

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