POSITIVENESS OF NONNEGATIVE SUPERSOLUTIONS OF ELLIPTIC EQUATIONS

LIANG XITING AND YU MINGQI

(Communicated by Barbara L. Keyfitz)

ABSTRACT. We prove the positiveness of the nonnegative supersolutions of equation (1), provided that the solutions are nontrivial whenever the structure condition (2) is fulfilled by A and B .

Let G be a bounded domain in the n -dimensional Euclidean space E^n . Let $p > 1$, $W_p^1(G)$, and $\mathring{W}_p^1(G)$ be the usual Sobolev spaces. Consider in G the equation

$$(1) \quad \int_G (\nabla v \cdot A(x, u, \nabla u) + v B(x, u, \nabla u)) dx = 0, \quad \forall v \in C_c^1(G)$$

where $A(x, u, \xi)$ and $B(x, u, \xi)$ are defined on $G \times E^1 \times E^n$, continuous in u, ξ for fixed x , and measurable in x for fixed (u, ξ) . J. Serrin [1] proved that, when A, B satisfies the following structure condition,

$$\begin{cases} \nabla u \cdot A(x, u, \nabla u) \geq \kappa_0 |\nabla u|^p - d(x) |u|^{p-1}, & \kappa_0 > 0, \\ |A(x, u, \nabla u)| \leq \kappa_1 |\nabla u|^{p-1} + b(x) |u|^{p-1}, & \kappa_1 \geq \kappa_0, \\ |B(x, u, \nabla u)| \leq C(x) |\nabla u|^{p-1} + d(x) |u|^{p-1}, \end{cases}$$

the Harnack inequality holds for the nonnegative solution $u \in W_p^1(G)$ of equation (1). N. S. Trudinger [2] and Yu Mingqi and Liang Xiting [3] had given other approaches to the above-mentioned result. At the same time, [2], [3] obtained a weak Harnack inequality for the nonnegative solutions of equation (1). As a corollary, we arrive at the following conclusion: The nonnegative solution u of equation (1) is strictly positive in G , provided it is not trivial. That is,

$$\text{ess inf}_{G'} u > 0, \quad \forall G' \Subset G.$$

Alternative approaches to this conclusion have been given by Liang Xiting [4] without using the weak Harnack inequality. Now, by an improvement of the approach used in [4], we prove the positiveness of the nonnegative supersolutions of equation (1) under a more general structure condition (2), which

Received by the editors March 21, 1989 and, in revised form, April 2, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35B99, 35J70, 35J99.

Key words and phrases. Elliptic equation, supersolution, positiveness.

comprises not only the case of controllable growth but also the case of natural growth.

The main result of this paper is as follows:

Theorem. Let $\mathbf{A}$, B satisfy the following structure condition:

$$(2) \quad \begin{cases} \nabla u \cdot \mathbf{A}(x, u, \nabla u) \geq \kappa_0 |\nabla u|^p - d(x)|u|^\alpha, \\ |\mathbf{A}(x, u, \nabla u)| \leq \kappa_1 |\nabla u|^{p-1} + b(x)|u|^{\alpha/p'}, \\ |B(x, u, \nabla u)| \leq C(x)|\nabla u|^\gamma + d(x)|u|^{\alpha-1}, \end{cases}$$

where $\kappa_1 \geq \kappa_0 > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $p-1 \leq \gamma \leq p$; $p \leq \alpha \leq p^* = \frac{np}{n-p}$ for $1 < p < n$; $p \leq \alpha < +\infty$ for $p \geq n$; $b(x)$, $c(x)$, and $d(x)$ are nonnegative in G and satisfy the following integrability conditions:

$$(3) \quad d(x) \in L_{r_0}(G), \quad b(x) \in L_{r_1}(G), \quad c(x) \in L_r(G);$$

$$(4) \quad \begin{cases} r_0 > \frac{n}{p} & \text{for } 1 < p < n, \alpha = p, \\ r_0 > \frac{p^*}{p^* - \alpha} \text{ (or } 1 - \frac{1}{r_0} - \frac{\alpha}{p^*} > 0) & \text{for } 1 < p < n, p < \alpha < p^*, \\ r_0 = +\infty & \text{for } 1 < p < n, \alpha = p^*, \\ r_0 > 1 & \text{for } p \geq n, p \leq \alpha < +\infty; \end{cases}$$

$$(5) \quad \begin{cases} r_1 = \frac{n}{p-1} & \text{for } 1 < p < n, \alpha = p, \\ r_1 = (\frac{p}{p-1})(\frac{p^*}{p^* - \alpha}) \text{ (or } \frac{1}{r_1} = (p-1)(\frac{1}{n} - \frac{\alpha-p}{p^*p}) & \text{for } 1 < p < n, p < \alpha < p^*, \\ r_1 = +\infty & \text{for } 1 < p < n, \alpha = p^*, \\ r_1 > \frac{p}{p-1} & \text{for } p \geq n, p \leq \alpha < +\infty; \end{cases}$$

and r satisfies one of the following three conditions:

$$(6) \quad \begin{cases} \frac{1}{r} = 1 - \frac{\gamma}{p} - \frac{1}{p^*} & \text{for } p-1 \leq \gamma < p-1 + \frac{p}{n}, 1 < p < n, \\ r = +\infty & \text{for } \gamma = p-1 + \frac{p}{n}, 1 < p < n; \end{cases}$$

$$(7) \quad \begin{cases} r > \frac{n}{p-\gamma} & \text{for } p-1 \leq \gamma < p, 1 < p < n, \\ r > \frac{p}{p-\gamma} & \text{for } p-1 \leq \gamma < p, p \geq n; \end{cases}$$

$$(8) \quad r = \infty \quad \text{for } \gamma = p, p > 1.$$

Suppose

$$(9) \quad \begin{cases} u \in W_p^1(G) & \text{when condition (6) is satisfied,} \\ u \in W_p^1(G) \cap L_t(G) & \text{when condition (7) is satisfied,} \\ u \in W_p^1(G) \cap L_\infty(G) & \text{when condition (8) is satisfied,} \end{cases}$$

and

$$(10) \quad t = \begin{cases} \frac{n(\gamma+1-p)}{p-\gamma} & \text{for } r = +\infty, \\ \frac{rn(\gamma+1-p)}{r(p-\gamma)-n} & \text{for } \frac{n}{p-\gamma} < r < +\infty, 1 < p < n, \\ \frac{rn(\gamma+1-p)}{r(p-\gamma)-p} & \text{for } \frac{p}{p-\gamma} < r < +\infty, p \geq n. \end{cases}$$

Suppose u is a nonnegative supersolution of equation (1); that is,

$$(1') \quad \int_G \{ \nabla \nu \cdot \mathbf{A}(x, u, \nabla u) + \nu B(x, u, \nabla u) \} dx \geq 0, \quad \forall \nu \in C_c^1(G), \quad \nu \geq 0.$$

If u is not identically zero in G , then we have

$$(11) \quad \text{ess inf}_{G'} u > 0, \quad \forall G' \Subset G.$$

Denote $B(x_0, \rho) = \{|x - x_0| < \rho\}$, $B(0, \rho) = B(\rho)$. To prove the theorem, we need the following lemmas:

Lemma 1. Let $1 < p < n$, $u \in W_p^1(B(\rho))$. Let $u = 0$ on a subset S of $B(\rho)$ with positive measure. Let $\eta(x) = \eta(|x|)$ be a nonincreasing, continuous function of $|x|$, satisfying $0 \leq \eta(x) \leq 1$, and let

$$\eta(x) = 1 \quad \text{for } x \in S.$$

Then

$$\left(\int_{B(\rho)} |\eta(x)u(x)|^{p^*} dx \right)^{1/p^*} \leq c \left(n, p, \frac{\rho^n}{\text{meas } S} \right) \left(\int_{B(\rho)} |\eta(x)\nabla u(x)|^p dx \right)^{1/p}.$$

Lemma 1 is a slight variation of Lemma 5.1 of O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva [6]. In fact, let u be the function in Lemma 1. We have (see [6, p. 90])

$$\eta(x)|u(x)| \leq c(n) \frac{\rho^n}{\text{meas } S} \int_{B(\rho)} \frac{\eta(y)|\nabla_y u(y)|}{|y - x|^{n-1}} dy,$$

and Lemma 1 follows from Sobolev's theorem on potential integrals.

Lemma 2. Let u be the function appearing in the theorem. Suppose u is bounded on G and there exist $\lambda > 0$ and $\theta \in (0, 1)$, such that for some fixed $x_0 \in G$ and $\rho > 0$, we have

$$(12) \quad \text{meas } B(x_0, \rho) \cap \{u > \lambda\} \geq \theta \text{meas } B(\rho), \quad B(x_0, 2\rho) \subset G.$$

Then there must be a constant $H > 0$ (appearing in (30)), which depends only on n, θ , the coefficients appearing in (2), $\text{meas}(G)$, and the bound of u on G , such that

$$(13) \quad \text{ess inf}_{B(x_0, \rho)} u \geq \lambda e^{-H}.$$

Proof of Lemma 2. According to the assumption, we have $0 \leq u \leq M$. Let $\varepsilon > 0$, and let

$$w = \ln \frac{\lambda}{u + \varepsilon}.$$

For the sake of simplicity, let x_0 be the origin in (12). By (12), for any $k \geq 0$, we have

$$(14) \quad \begin{aligned} \text{meas } B(\rho) \cap \{w < k\} &\geq \text{meas } B(\rho) \cap \{w < 0\} \\ &\geq \text{meas } B(\rho) \cap \{u > \lambda\} \geq \frac{\theta}{2^n} \text{meas } B(2\rho). \end{aligned}$$

Under the hypothesis of the theorem, by approximation, we can take $v \in \mathring{W}_p^1 \cap L_\infty$ as the test function instead of $v \in C_c^1(G)$. Suppose in the following that $\rho \leq \rho_1 < \rho_0 \leq 2\rho$ and $\zeta(x) = \zeta(|x|)$ is a piecewise linear continuous function of $|x|$ satisfying

$$\zeta(x) = \begin{cases} 1 & \text{for } |x| \leq \rho_1 < \rho_0 \\ 0 & \text{for } |x| \geq \rho_0 \end{cases} \quad \text{and} \quad |\nabla \zeta(x)| \leq \frac{1}{\rho_0 - \rho_1}.$$

Let $\sigma > 0$, to be determined later, $\tau = \frac{\rho^2}{p-1}$, $(w - k)^+ = \max(w - k, 0)$, and

$$v = e^{-\sigma u} \frac{\zeta^\tau (w - k)^+}{(u + \varepsilon)^{p-1}}.$$

Then v is nonnegative, bounded in G , and $v \in \mathring{W}_p^1(G)$; thus we may take it as a test function. Inserting v into (1') gives
(15)

$$\begin{aligned} 0 \leq \int_G e^{-\sigma u} \left\{ \left[\frac{\zeta^\tau \nabla(w - k)^+}{(u + \varepsilon)^{p-1}} + \frac{(p-1)\zeta^\tau (w - k)^+ (-\nabla u)}{(u + \varepsilon)^p} \right. \right. \\ \left. \left. + \frac{(-\sigma)\zeta^\tau (w - k)^+ \nabla u}{(u + \varepsilon)^{p-1}} + \frac{\tau \zeta^{\tau-1} (w - k)^+ \nabla \zeta}{(u + \varepsilon)^{p-1}} \right] \cdot \mathbf{A} \right. \\ \left. + \frac{\zeta^\tau (w - k)^+}{(u + \varepsilon)^{p-1}} B \right\} dx. \end{aligned}$$

Taking into account structure condition (2) and the nonnegativeness of u , we have

$$\begin{aligned} \int_G e^{-\sigma u} \frac{\zeta^\tau \nabla(w - k)^+}{(u + \varepsilon)^{p-1}} \cdot \mathbf{A} dx &= \int_{G \cap \{w > k\}} e^{-\sigma u} \frac{\zeta^\tau (-\nabla u) \cdot \mathbf{A}}{(u + \varepsilon)^p} dx \\ &\leq - \int_{G \cap \{w > k\}} e^{-\sigma u} \zeta^\tau (\kappa_0 |\nabla w|^p - d(x) u^{\alpha-p}) dx; \\ \int_G e^{-\sigma u} \frac{\zeta^\tau (w - k)^+ (-\nabla u) \cdot \mathbf{A}}{(u + \varepsilon)^p} dx \\ &\leq - \int_{G \cap \{w > k\}} e^{-\sigma u} \zeta^\tau (w - k) (\kappa_0 |\nabla w|^p - d(x) u^{\alpha-p}) dx; \\ \int_G e^{-\sigma u} \frac{(-\sigma) \zeta^\tau (w - k)^+ \nabla u \cdot \mathbf{A}}{(u + \varepsilon)^{p-1}} dx \\ &\leq -\sigma \int_{G \cap \{w > k\}} e^{-\sigma u} \zeta^\tau (w - k) \left\{ \frac{\kappa_0 |\nabla u|^p}{(u + \varepsilon)^{p-1}} - d(x) u^{\alpha-p+1} \right\} dx; \end{aligned}$$

$$\begin{aligned}
& \int_G e^{-\sigma u} \frac{\zeta^{\tau-1} (w-k)^+ \nabla \zeta \cdot \mathbf{A}}{(u+\varepsilon)^{p-1}} dx \\
& \leq \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^{\tau-1} (w-k) |\nabla \zeta| (\kappa_1 |\nabla w|^{p-1} + b(x) u^{\alpha/p'-(p-1)}) dx \\
& \leq \delta \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) |\nabla w|^p dx \\
& \quad + c(\delta) \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^{p/p-1} |\nabla \zeta|^p (w-k) dx \\
& \quad + \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^{\tau-1} (w-k) |\nabla \zeta| b(x) u^{\alpha/p'-(p-1)} dx \\
& \quad (\delta > 0 \text{ arbitrarily small});
\end{aligned}$$

$$\begin{aligned}
& \int_G e^{-\sigma u} \frac{\zeta^\tau (w-k)^+}{(u-\varepsilon)^{p-1}} B dx \\
& \leq \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) \left(\frac{c(x) |\nabla u|^\gamma}{(u+\varepsilon)^{p-1}} + d(x) u^{\alpha-p} \right) dx; \\
& \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) \frac{c(x) |\nabla u|^\gamma}{(u+\varepsilon)^{p-1}} dx \\
& \leq \begin{cases} \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) c(x) |\nabla w|^{p-1} dx & \text{for } \gamma = p-1, \\ \|c(x)\|_{L_r(G)} \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) \frac{|\nabla u|^p}{(u+\varepsilon)^{p-1}} dx & \text{for } \gamma = p, \\ \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) c(x)^{1/(p-\gamma)} |\nabla w|^{p-1} dx \\ \quad + \int_{G \cap \{w>k\}} e^{-\sigma u} \zeta^\tau (w-k) \frac{|\nabla u|^p}{(u+\varepsilon)^{p-1}} dx & \text{for } p-1 < \gamma < p. \end{cases}
\end{aligned}$$

Combining the above results and taking $\sigma > 0$ large enough, $\delta > 0$ small enough (σ, δ depend only on n, p, κ_0, κ_1 and $\|c(x)\|_{L_r(G)}$), from (15) we get

$$\begin{aligned}
& e^{-\sigma M} \int_{G \cap \{w>k\}} \zeta^\tau |\nabla w|^p dx \\
& \leq \int_{G \cap \{w>k\}} [e^{-\sigma u} \zeta^\tau |\nabla w|^p + e^{-\sigma u} \zeta^\tau (w-k) |\nabla w|^p] dx \\
(16) \quad & \leq C \int_{G \cap \{w>k\}} \{ \zeta^\tau d(x) [u^{\alpha-p} + (w-k) u^{\alpha-p} + (w-k) u^{\alpha-p+1}] \\
& \quad + \zeta^{p/(p-1)} |\nabla \zeta|^p (w-k) + \zeta^{\tau-1} (w-k) |\nabla \zeta| b(x) u^{\alpha/p'-(p-1)} \\
& \quad + K(r) \zeta^\tau (w-k) C(x)^{1/(p-\gamma)} |\nabla w|^{p-1} \} dx.
\end{aligned}$$

$K(\gamma) = 0$ when $\gamma = p$; $K(\gamma) = 1$ when $p-1 \leq \gamma < p$; the constant $c > 0$ depends only on n, p, κ_0, κ_1 and $\|c(x)\|_{L_r(G)}$. From the definition of ζ , the

region of the integrals in (16) is

$$(17) \quad A(k, \rho_0) = B(\rho_0) \cap \{w > k\}.$$

Take $S = B(\rho_1) \cap \{w \leq k\} \subset B(\rho_0)$. From (14) we get

$$\begin{aligned} \text{meas } S &> \text{meas } B(\rho) \cap \{w \leq k\} \geq \left(1 - \frac{\theta}{2^n}\right) \text{meas } B(2\rho) \\ &\geq \left(1 - \frac{\theta}{2^n}\right) \text{meas } B(\rho_0). \end{aligned}$$

If $1 < p < n$, we take $\eta = \zeta^{p/(p-1)}$ and $q = p^*$, and we apply Lemma 1 to $(w - k)^+$ in $B(\rho_0)$, getting

$$(18) \quad \left(\int_{A(k, \rho_0)} [\zeta^{p/(p-1)}(w - k)]^q dx \right)^{1/q} \leq C(n, p, \theta) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p}.$$

If $p \geq n$, for any $q > p$ we take $p' \in (1, n)$ such that

$$\frac{1}{q} = \frac{1}{p'} - \frac{1}{n}.$$

By Lemma 1, we get

$$\begin{aligned} (19) \quad & \left(\int_{A(k, \rho_0)} [\zeta^{p/(p-1)}(w - k)]^q dx \right)^{1/q} \\ & \leq C(n, p', \theta) \left(\int_{A(k, \rho_0)} \zeta^{pp'/(p-1)} |\nabla w|^{p'} dx \right)^{1/p'} \\ & \leq C(n, q, \theta) \text{meas}^{1/p' - 1/p} A(k, \rho_0) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p} \\ & \leq C(n, q, \theta) \text{meas}^{1/q - 1/p + 1/n} A(k, \rho_0) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p}. \end{aligned}$$

Through the boundedness of u , taking $q = p^*$ when $1 < p < n$ and $q > p$ large enough when $p \geq n$, so that we can use Hölder's inequality, we get

$$(20) \quad \int_{A(k, \rho_0)} \zeta^\tau d(x) u^{\alpha-p} dx \leq M^{\alpha-p} \|d\|_{L_{r_0}(G)} \text{meas}^{1-1/r_0} A(k, \rho_0)$$

and

$$\begin{aligned}
 & \int_{A(k, \rho_0)} \zeta^\tau d(x)(w-k)(u^{\alpha-p} + u^{\alpha-p+1}) dx \\
 (21) \quad & \leq (M^{\alpha-p} + M^{\alpha-p+1}) \|d\|_{L_{r_0}(G)} \left(\int_{A(k, \rho_0)} [\zeta^{p/(p-1)}(w-k)]^q dx \right)^{1/q} \\
 & \quad \times \text{meas}^{1-1/r_0-1/q} A(k, \rho_0) \\
 & \leq C(M) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p} \text{meas}^{1-1/r_0-1/p+1/n} A(k, \rho_0),
 \end{aligned}$$

where the constant $c(M) > 0$ is independent of w , k , ρ_0 , and ρ_1 .

$$\begin{aligned}
 & \int_{A(k, \rho_0)} \zeta^{p/(p-1)} |\nabla \zeta|^p (w-k) dx \\
 (22) \quad & \leq \frac{1}{(\rho_0 - \rho_1)^p} \int_{A(k, \rho_0)} \zeta^{p/(p-1)} (w-k) dx \\
 & \leq \frac{C}{(\rho_0 - \rho_1)^p} \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p} \text{meas}^{1-1/p+1/n} A(k, \rho_0)
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{A(k, \rho_0)} \zeta^{\tau-1} (w-k) |\nabla \zeta| b(x) |u^{\alpha/p'-(p-1)}| dx \\
 (23) \quad & \leq \frac{M^{\alpha/p'-(p-1)}}{\rho_0 - \rho_1} \|b\|_{L_{r_1}(G)} \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p} \\
 & \quad \times \text{meas}^{1-1/r_1-1/p+1/n} A(k, \rho_0),
 \end{aligned}$$

where we have used (18) and (19).

If $p-1 \leq r < p$, we have

$$\begin{aligned}
 & \int_{A(k, \rho_0)} \zeta^\tau (w-k) C(x)^{1/(p-\gamma)} |\nabla w|^{p-1} dx \\
 (24) \quad & = \int_{A(k, \rho_0) \cap \{C(x) > N\}} + \int_{A(k, \rho_0) \cap \{C(x) \leq N\}} = I_1 + I_2.
 \end{aligned}$$

Because $c^r(x)$ is in $L_1(\dot{G})$,

$$\text{meas } G \cap \{c(x) > N\} \leq \frac{1}{N^r} \int_G |C(x)|^r dx \rightarrow 0, \quad \text{when } N \rightarrow \infty.$$

Thus, by the absolute convergence of the Lebesgue integral, we have

$$\int_{G \cap \{C(x) > N\}} C(x)^r dx \leq \varepsilon(N) \rightarrow 0, \quad \text{when } N \rightarrow \infty.$$

By (6), (7), and taking $q = p^*$ when $1 < p < n$, $q > 0$ large enough for $p \geq n$, we may use Hölder's inequality to get

$$\begin{aligned}
 (25) \quad I_1 &\leq \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1-1/p} \left(\int_{A(k, \rho_0)} [\zeta^{p/(p-1)} (w-k)]^q dx \right)^{1/q} \\
 &\quad \times \left(\int_{G \cap \{C(x) > N\}} C(x)^r dx \right)^{1/(p-\gamma)} \text{meas}^{1-(p-1)/p-1/q-1/r(p-r)} A(k, \rho_0) \\
 &\leq \varepsilon(N) \text{meas}^{1/n-1/(r(p-\gamma))} A(k, \rho_0) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right),
 \end{aligned}$$

$$\begin{aligned}
 (26) \quad I_2 &\leq N^{1/(p-r)} \int_{A(k, \rho_0)} \zeta^\tau (w-k) |\nabla w|^{p-1} dx \\
 &\leq \tilde{\delta} \int_{A(k, \rho_0)} \zeta^\tau (w-k) |\nabla w|^p dx + C(\tilde{\delta}, N) \int_{A(k, \rho_0)} \zeta^\tau (w-k) dx \\
 &\leq \tilde{\delta} \int_{A(k, \rho_0)} \zeta^\tau (w-k) |\nabla w|^p dx + C(\tilde{\delta}, N) \left(\int_{A(k, \rho_0)} \zeta^\tau |\nabla w|^p dx \right)^{1/p} \\
 &\quad \times \text{meas}^{1-1/p+1/n} A(k, \rho_0).
 \end{aligned}$$

Combining (16) and (20)–(26) and taking $\tilde{\delta} > 0$ sufficiently small, $N > 0$ sufficiently large so that $\varepsilon(N)$ can be small enough, we arrive at

$$\begin{aligned}
 &\int_{A(k, \rho_0)} (\zeta^\tau |\nabla w|^p + \zeta^\tau (w-k) |\nabla w|^p) dx \\
 &\leq C(M) \left\{ \text{meas}^{1-1/r_0} A(k, \rho_0) + \text{meas}^{(p/(p-1))(1-1/r_0-1/p+1/n)} A(k, \rho_0) \right. \\
 &\quad + \left(\frac{1}{\rho - \rho_1} \right)^\tau \text{meas}^{(p/(p-1))(1-1/p+1/n)} A(k, \rho_0) \\
 &\quad + \left(\frac{1}{\rho_0 - \rho_1} \right)^{p/(p-1)} \text{meas}^{(p/(p-1))(1-1/r_1-1/p+1/n)} A(k, \rho_0) \\
 &\quad \left. + \text{meas}^{(p/(p-1))(1-1/p+1/n)} A(k, \rho_0) \right\},
 \end{aligned}$$

where the constant $c(M) > 0$ is independent of w , k , ρ_0 , and ρ_1 . Combining

this with (18) or (19), we get

$$\begin{aligned}
 \int_{A(k, \rho_0)} \zeta^{p/(p-1)} (w - k) dx &\leq C(M) \text{meas}^{1-1/p+1/n} A(k, \rho_0) \\
 &\times \left\{ \text{meas}^{(1/p)(1-1/r_0)} A(k, \rho_0) + \text{meas}^{(1/(p-1))(1-1/r_0-1/p+1/n)} A(k, \rho_0) \right. \\
 (27) \quad &+ \left(\frac{1}{\rho_0 - \rho_1} \right)^{p/(p-1)} \text{meas}^{(1/(p-1))(1-1/p+1/n)} A(k, \rho_0) \\
 &+ \left[\frac{1}{\rho_0 - \rho_1} \text{meas}^{1-1/r_1-1/p+1/n} A(k, \rho_0) \right]^{1/(p-1)} \\
 &\left. + \text{meas}^{(1/(p-1))(1-1/p+1/n)} A(k, \rho_0) \right\}.
 \end{aligned}$$

By (4), we can take $\beta > 0$ in such a way that

$$\begin{aligned}
 1 + \beta &\leq 1 - \frac{1}{p} + \frac{1}{n} + \frac{1}{p} \left(1 - \frac{1}{r_0} \right) \\
 &= \begin{cases} 1 + \frac{1}{p} \left(\frac{p}{n} - \frac{1}{r_0} \right) & \text{if } \alpha = p, \ 1 < p < n, \\ 1 + \frac{\alpha-p}{p^*p} + \left(1 - \frac{1}{r_0} - \frac{1}{p^*} \right) & \text{if } p < \alpha < p^*, \ 1 < p < n, \\ 1 + \frac{1}{n} & \text{if } \alpha = p^*, \ 1 < p < n, \\ 1 + \frac{1}{p} \left(\frac{p}{n} - \frac{1}{r_0} \right) & \text{if } p \leq \alpha < +\infty, \ p \geq n; \text{ and} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 1 + \beta &\leq 1 - \frac{1}{p} + \frac{1}{n} + \frac{1}{p-1} \left(1 - \frac{1}{r_0} - \frac{1}{p} + \frac{1}{n} \right) \\
 &= \begin{cases} 1 + \left(\frac{p}{n} - \frac{1}{r_0} \right) & \text{if } \alpha = p, \ 1 < p < n, \\ 1 + \frac{1}{p^*} \left(\frac{\alpha-p}{p-1} \right) & \text{if } p < \alpha < p^*, \ 1 < p < n, \\ 1 + \frac{p}{p-1} \frac{1}{n} & \text{if } \alpha = p^*, \ 1 < p < n, \\ 1 + \frac{p}{p-1} \left(\frac{1}{n} - \frac{1}{p} \right) + \frac{1}{p-1} \left(1 - \frac{1}{r_0} \right) & \text{if } p \leq \alpha < +\infty, \ p \geq n. \end{cases}
 \end{aligned}$$

Also, according to (5),

$$\begin{aligned}
 1 + \frac{1}{(p-1)n} &\leq 1 - \frac{1}{p} + \frac{1}{n} + \frac{1}{p-1} \left(1 - \frac{1}{r_1} - \frac{1}{p} + \frac{1}{n} \right) \\
 &= \begin{cases} 1 + \frac{1}{(p-1)n} + \frac{\alpha-p}{p^*p} & \text{if } p \leq \alpha < p^*, \ 1 < p < n, \\ 1 + \frac{1}{(p-1)n} + \frac{1}{n} & \text{if } \alpha = p^*, \ 1 < p < n, \\ 1 + \frac{1}{(p-1)n} + \frac{1}{p-1} \left(\frac{p-1}{n} - \frac{1}{r_1} \right) & \text{if } p \leq \alpha < +\infty, \ p \geq n. \end{cases}
 \end{aligned}$$

Because G is bounded, from (27) we have

$$\begin{aligned}
 (28) \quad & (h-k) \text{meas } A(h, \rho_1) \\
 & \leq \int_{A(k, \rho_0)} \zeta^{p/(p-1)} (w-k) dx \\
 & \leq C(M) \left\{ \text{meas}^{1+\beta} A(k, \rho_0) + \left(\frac{1}{\rho_0 - \rho_1} \right)^{p/(p-1)} \text{meas}^{1+p/((p-1)n)} A(k, \rho_0) \right. \\
 & \quad \left. + \left(\frac{1}{\rho_0 - \rho_1} \right)^{1/(p-1)} \text{meas}^{1+1/((p-1)n)} A(k, \rho_0) \right\}, \\
 & \quad \forall h > k \geq 0, \quad \rho \leq \rho_1 < \rho_0 \leq 2\rho.
 \end{aligned}$$

For $m = 0, 1, 2, \dots$, let

$$\begin{aligned}
 \rho_m &= \rho + \frac{\rho}{2^m}, \\
 k_m &= H - \frac{H}{2^m} \quad (H > 0 \text{ to be determined later}), \\
 A_m &= \text{meas } A(k_m, \rho_m).
 \end{aligned}$$

Substitute k_{m+1}, k_m for h, k and ρ_m, ρ_{m+1} for ρ_0, ρ_1 , respectively. (Note that the constant $c(M) > 0$ is independent of w, h, k, ρ_0 , and ρ_1 , in (28).) From (28), we get

$$\begin{aligned}
 (29) \quad & \frac{H}{2^{m+1}} A_{m+1} \leq C(M) \left\{ A_m^{1+\beta} + \left(\frac{2^{m+1}}{\rho} \right)^{p/(p-1)} A_m^{1+p/((p-1)n)} \right. \\
 & \quad \left. + \left(\frac{2^{m+1}}{\rho} \right)^{1/(p-1)} A_m^{1+1/((p-1)n)} \right\},
 \end{aligned}$$

for $m = 0, 1, 2, \dots$. If we take $H > 0$ satisfying

$$(30) \quad \frac{2C(M)}{H} \max\{\text{meas}^\beta G, (2 \text{meas}^{1/n} B(2))^{p/(p-1)}, (2 \text{meas}^{1/n} B(2))^{1/m}\} \leq \frac{\delta}{3},$$

where $\delta > 0$ satisfies

$$2\delta^\beta \leq 1, \quad 2^{1+p/(p-1)} \delta^{p/((p-1)n)} \leq 1, \quad 2^{1+1/(p-1)} \delta^{1/((p-1)n)} \leq 1,$$

then, deducing from the dependence of $c(M)$, $H > 0$ depends only on the constants $n, p, r_0, r_1, r, \kappa_0, \kappa_1, \alpha, r, \beta$, and M , $\text{meas } G$, the norms $\|d(x)\|_{L_{r_0}(G)}$, $\|b(x)\|_{L_{r_1}(G)}$, and $\|c(x)\|_{L_r(G)}$, and the function $\varepsilon(N)$. From (29) and (30), it can be shown that

$$(31) \quad A_m \leq \delta^m \rho^n \text{meas } B(2).$$

In fact, (31) holds obviously for $m = 0$. Suppose it is true for m ; then it

follows from (29) and (30) that

$$\begin{aligned} A_{m+1} &\leq A_m \frac{2^{m+1} C(M)}{H} \{ (\delta^m \text{meas } G)^\beta + (2^{m+1})^{p/(p-1)} (\delta^m \text{meas } B(2))^{p/((p-1)n)} \\ &\quad + (2^{m+1})^{1/(p-1)} (\delta^m \text{meas } B(2))^{1/((p-1)n)} \} \\ &\leq \delta A_m = \delta^{m+1} \rho^n \text{meas } B(2). \end{aligned}$$

This implies by induction that (31) holds for any integer, which in turn implies that

$$\text{meas } A(H, \rho) = \lim_{m \rightarrow \infty} A_m = 0;$$

i.e.,

$$\text{ess inf}_{B(\rho)} w \leq H.$$

From the definition of w and the above inequality, we get

$$\text{ess inf}_{B(\rho)} u + \varepsilon \geq \lambda e^{-H}.$$

Letting $\varepsilon \rightarrow 0$, we arrive immediately at (13). This concludes the proof of Lemma 2.

Proof of the theorem. First, let us suppose that u is bounded. By the assumptions, if u is not identically zero, then we can find an $x_0 \in G$ such that

$$(32) \quad \text{meas } B(x_0, \rho) \cap \{u > 0\} > 0, \quad B(x_0, 2\rho) \subset G,$$

in which $\rho > 0$ can be arbitrarily small. From (32), we have some $\lambda > 0$, $\theta \in (0, 1)$, depending on u, ρ , such that

$$(33) \quad \text{meas } B(x_0, \rho) \cap \{u > \lambda\} \geq \theta \text{meas } B(\rho).$$

Otherwise, there exists $\{\lambda_n\}$

$$\begin{aligned} \lambda_n &\geq \lambda_{n+1} \geq \dots, \quad \lambda_n \rightarrow 0, \quad \text{when } n \rightarrow \infty, \\ (34) \quad \text{meas } B(x_0, \rho) \cap \{u \geq \lambda_n\} &\leq \frac{1}{n} \text{meas } B(\rho). \end{aligned}$$

Taking into account that

$$B(x_0, \rho) \cap \{u > 0\} = \bigcup_n (B(x_0, \rho) \cap \{u > \lambda_n\})$$

and

$$B(x_0, \rho) \cap \{u > \lambda_n\} \subset B(x_0, \rho) \cap \{u > \lambda_{n+1}\}$$

and using (34), we get

$$\text{meas } B(x_0, \rho) \cap \{u > 0\} = \lim_{n \rightarrow \infty} \text{meas } B(x_0, \rho) \cap \{u > \lambda_n\} = 0,$$

which contradicts (32)! Hence (33) must be true for some $\lambda > 0$ and $\theta \in (0, 1)$. Using Lemma 2, we have $\text{ess inf}_{B(x_0, \rho)} u > 0$. By extension, we have for any $G' \Subset G$, $\text{ess inf}_{G'} u > 0$. This concludes the proof of the theorem in the case of bounded u .

If u is not bounded, let $G' \in G$. For fixed $M > 0$, we write G' as $G' = G'_1 + G'_2$

$$G'_1 = G' \cap \left\{ u < \frac{M}{2} \right\} \in G \cap \{u < M\} = G, \quad \text{and} \quad G'_2 = G' \cap \left\{ u \geq \frac{M}{2} \right\}.$$

Then we have

$$(35) \quad \text{ess inf}_{G'_2} u \geq \frac{M}{2} > 0.$$

Substituting G_1 for G , G'_1 for G' , from the previous results, we have

$$(36) \quad \text{ess inf}_{G'_1} u > 0,$$

and (35) together with (36) implies (11). The theorem has been proved.

The authors are grateful to the referee for many good suggestions.

REFERENCES

1. J. Serrin, *Local behavior of solutions of quasilinear elliptic equations*, Acta Math. **111** (1964), 247–303.
2. N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.
3. Yu Mingqi and Liang Xiting, *The Harnack inequality for nonnegative generalized solutions of quasilinear elliptic equations*, J. Shanxi Univ. **12** (1989), 1–8.
4. Liang Xiting, *Positiveness of nonnegative generalized solutions of quasilinear elliptic equations*, J. of Qufu Norm. Univ. **13** (1987), 161–169.
5. ———, *The maximum principle for generalized solutions of quasilinear elliptic equations*, Acta Sci. Nat. Univ. Sunyatsen **27** (1988), 107–112.
6. O. A. Ladyzenskaja, V. A. Solonnikov, and H. H. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs **23**, American Mathematical Society, Providence, R.I., 1968.

DEPARTMENT OF MATHEMATICS, ZHONGSHAN UNIVERSITY, GUANGZHOU, PEOPLE'S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, SHANXI UNIVERSITY, TAIYUAN, PEOPLE'S REPUBLIC OF CHINA