PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 3, March 1991

A NOTE ON COBORDISM OF SURFACE LINKS IN S^4

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(Communicated by Frederick R. Cohen)

ABSTRACT. Sato's idea of the asymmetric linking number is used in cyclic branched coverings to give an invariant of the cobordism of embedded surfaces in the 4-sphere.

In this article we consider the link cobordism of surface links in S^4. We work in the smooth category.

Let $L = J \cup K$ be a link in S^4, where J, K are embedded, oriented connected surfaces. L is called *semi-boundary* [S] if each component bounds an embedded, orientable 3-manifold in S^4 which misses the other component. Sato [S] defined the *asymmetric linking number*, denoted by $\operatorname{alk}(J, K)$, to be the nonnegative generator of the image of $H_1(K : \mathbf{Z}) \to H_1(S^4 \backslash J : \mathbf{Z}) \cong \mathbf{Z}$. He proved that a link is semi-boundary iff

$$\operatorname{alk}(J, K) = 0 = \operatorname{alk}(K, J),$$

and being semi-boundary is preserved under link cobordism. We call two surface links $L_0 = J_0 \cup K_0$ and $L_1 = J_1 \cup K_1$ cobordant if there are disjointly embedded, orientable 3-manifolds C and E in $S^4 \times I$ such that $\partial C = J_0 \cup (-J_1)$, $\partial E = K_0 \cup (-K_1)$, and C, E are homeomorphic to $J_0 \times I$, $K_0 \times I$, respectively, where we regard L_i as lying in $S^4 \times \{i\}$. A link is called null-cobordant if it is cobordant to the standardly embedded surfaces (which bound disjoint handlebodies) in S^4. Thus alk can be regarded as the first obstruction to links being null-cobordant, and we focus on semi-boundary links from now on. The Sato-Levine invariant was defined [S] for semi-boundary links, and Cochran [C] defined the derived series of this invariant. In this paper we observe that the *covering asymmetric linking number* can be used as a link cobordism invariant and give examples of links with vanishing Sato-Levine invariant and trivial derivatives which belong to different cobordism classes.

Let $L = J \cup K$ be a 2-component, oriented semi-boundary link. Consider the n-fold cyclic branched covering M of S^4 along J, where $n = p^r$ is a prime power. Then there are n lifts $k_0, \dots, k_{n-1}$ of K to M since L is

Received by the editors December 5, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57Q45; Secondary 57Q60.
Key words and phrases. Surface links, cobordism, covering asymmetric linking number.

semi-boundary. We can assume that $k_{j+1} = \tau k_j$ where τ is the generator of covering translations. We regard n as 0 and $n+1$ as 1, so that the indices of lifts are regarded as lying in $\mathbf{Z}_n$.

Lemma 1. *$H_1(M \backslash k_j : \mathbf{Q})$ is isomorphic to either 0 or $\mathbf{Q}$. Furthermore, this is a link cobordism invariant. More precisely, let $L_i = J_i \cup K_i$ $(i = 0, 1)$ be cobordant links and M_i, k_j^i be their cyclic branched coverings and the lifts of K_i respectively $(i = 0, 1, j = 0, \dots, n-1)$. Then*

$$H_1(M_0 \backslash k_j^0 : \mathbf{Q}) \cong H_1(M_1 \backslash k_j^1 : \mathbf{Q}).$$

In particular, this is $\mathbf{Q}$ if L is null-cobordant.

The proof is given later. If $H_1(M \backslash k_j : \mathbf{Q}) = 0$, then L is not null-cobordant. Thus we focus on links with $H_1(M \backslash k_j : \mathbf{Q}) = \mathbf{Q}$. Consider the following homomorphism

$$H_1(k_j : \mathbf{Q}) \to H_1(M \backslash k_0 : \mathbf{Q}) \cong \mathbf{Q}.$$

Definition 2. Define $\xi_j^n = 0$ if this homomorphism is zero, $\xi_j^n = 1$ otherwise $(j = 1, \dots, n-1)$.

Theorem 3. *$\xi_j^n \in \mathbf{Z}_2$, $j \in \mathbf{Z}_n \backslash \{0\}$, are link cobordism invariants.*

Proof of Lemma 1. Let X_n be the n-fold cyclic (unbranched) covering (X_∞ denotes the infinite cyclic covering) of $X = \overline{S^4 \backslash N(J)}$. Then we have an exact sequence ([S-S]) with integral coefficient

$$\cdots \to H_q(X_\infty) \xrightarrow{t^n - 1} H_q(X_\infty) \to H_q(X_n) \to H_{q-1}(X_\infty) \to \cdots,$$

where t is the homomorphism induced from the generator of covering transformations. Thus we have

$$\begin{array}{ccccccc} H_1(X) & \to & H_0(X_\infty) & & \xrightarrow{t-1} & & H_0(X_\infty) \\ \| & \cong & \| & \searrow & & \nearrow & \\ \mathbf{Z} & & \mathbf{Z} & & 0 & & \end{array}.$$

Hence $(t-1)\colon H_1(X_\infty : \mathbf{Z}) \to H_1(X_\infty : \mathbf{Z})$ is surjective. Therefore $(t^n - 1) = (t-1)^n\colon H_1(X_\infty : \mathbf{Z}_p) \to H_1(X_\infty : \mathbf{Z}_p)$ is also surjective ($n = p^r$ is a prime power). Again using the exact sequence, we have $H_1(X_n : \mathbf{Z}_p) = \mathbf{Z}_p$. But the lift of the meridian of J represents a nontrivial element of infinite order in $H_1(X_n : \mathbf{Z})$. Hence $H_1(X_n : \mathbf{Q}) = \mathbf{Q}$. Thus we have $H_1(M : \mathbf{Q}) = 0$. A Mayer-Vietoris sequence with $\mathbf{Q}$-coefficients gives

$$\begin{array}{ccccccc} H_1(\partial N(k_j)) & \to & H_1(N(k_j)) & \oplus & H_1(\overline{M \backslash N(k_j)}) & \to & H_1(M) \\ \| & & \| & & & & \| \\ \mathbf{Q}^{2g} \oplus \mathbf{Q} & & \mathbf{Q}^{2g} & & & & 0 \end{array},$$

where g is the genus of K. Hence $H_1(M \backslash k_j : \mathbf{Q}) = 0$ or $\mathbf{Q}$.

Let $L_i = J_i \cup K_i$ $(i = 0, 1)$ be cobordant links via C, E, and let W be the n-fold cyclic branched covering of $S^4 \times I$ along C, E_j $(j = 0, \dots, n-1)$ be the lifts of E to W. Then $\partial E_j = k_j^0 \cup (-k_j^1)$, where k_j^i are the lifts of K_i to M_i, the n-fold branched covering of S^3 along J_i. (Note that we have exactly n lifts of E because L_i's are semi-boundary and E is homeomorphic to the product $K_0 \times I$.) The same argument shows that $H_1(W : \mathbf{Q}) = 0$ and $H_1(W \backslash E_0 : \mathbf{Q}) = 0$ or $\mathbf{Q}$.

We need to know the homomorphism $H_2(M_i) \to H_2(W)$. Let $Y = S^4 \times I \backslash N(C)$, Y_n (resp. Y_∞) be the n-fold (resp. infinite) cyclic covering of Y. Let $X^i = S^4 \backslash J_i$, X_n^i (resp. X_∞^i) be the n-fold (resp. infinite) cyclic covering of X^i. Since $(t-1)\colon H_1(X_\infty^i : \mathbf{Z}) \to H_1(X_\infty^i : \mathbf{Z})$ is surjective, it is an isomorphism (because $H_1(X_\infty^i : \mathbf{Z})$ is a finitely generated module over a Noetherian ring $\Lambda = \mathbf{Z}[t, t^{-1}]$, see [S-S]). Also we have $H_2(Y, X^i) = 0$ since the inclusion induces an isomorphism $H_*(X^i) \cong H_*(Y)$. Therefore we have the following commutative diagram with $\mathbf{Z}$-coefficients:

$$\begin{array}{ccccccccc}
0 & \to & H_2(X_\infty^i) & \xrightarrow{t-1} & H_2(X_\infty^i) & \to & H_2(X^i) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_2(Y_\infty) & \xrightarrow{t-1} & H_2(Y_\infty) & \to & H_2(Y) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_2(Y_\infty, X_\infty^i) & \xrightarrow{t-1} & H_2(Y_\infty, X_\infty^i) & \to & 0 & &
\end{array}$$

Note that $i_*\colon H_2(X^i) \to H_2(Y)$ (the homomorphism induced from the inclusion map) is an isomorphism, and $H_2(X^i) \cong \mathbf{Z}^{2h}$, where h is the genus of J_i.

Consider the following splitting

$$H_2(X_\infty^i : \mathbf{Q}) \cong F(X_\infty^i : \mathbf{Q}) \oplus T(X_\infty^i : \mathbf{Q})$$

where $F(\ )$, $T(\ )$ denote the free and torsion part of $H_2(\ )$ as a $\Gamma = \Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$-module respectively ($\Gamma$ is a PID). Furthermore, $T(X_\infty^i : \mathbf{Q}) = T^0(X_\infty^i : \mathbf{Q}) \oplus T^1(X_\infty^i : \mathbf{Q})$, where

$$T^0(X_\infty^i : \mathbf{Q}) \cong \Gamma/(t-1)^{p_1} \oplus \cdots \oplus \Gamma/(t-1)^{p_e}$$

is the $(t-1)$-summand and $T^1(\ )$ is the $(t-1)$-free summand.

Comparing the above sequence to the following sequence

$$0 \to \Gamma/(t-1) \to \Gamma/(t-1)^{p_k} \xrightarrow{t-1} \Gamma/(t-1)^{p_k} \to \Gamma/(t-1) \to 0,$$

where $\Gamma/(t-1) \cong \mathbf{Q}$, we conclude that $T^0(X_\infty^i) = 0$. (The same is true for $H_2(Y_\infty)$ and $H_2(Y_\infty, X_\infty^i)$.)

Hence $T(X_\infty^i : \mathbf{Q}) \cong \Gamma/\lambda_1 \oplus \cdots \oplus \Gamma/\lambda_m$ where λ_j is normalized so that $\lambda_j \in \Lambda$ and coefficients are relatively prime $(j = 1, \dots, m)$.

On the other hand, $\operatorname{Cok}(t-1\colon H_2(X^i_\infty : \mathbf{Z}) \to H_2(X^i_\infty : \mathbf{Z}))$ is isomorphic to $H_2(X^i_\infty) \otimes_\Lambda \mathbf{Z}$, where $\mathbf{Z}$ is regarded as Λ-module via the augmentation map $\Lambda \to \mathbf{Z}$, $t \to 1$ [S-S]. Since $H_2(X)$ is torsion free, we have $\lambda_j(1) = \pm 1$, $j = 1, \dots, m$ [S-S]. Then the same argument as Theorem 3 in [Sum] shows that

$$\operatorname{Cok}(t^n - 1\colon T(X^i_\infty : \mathbf{Q}) \to T(X^i_\infty : \mathbf{Q})) = 0 \qquad (n = p^r).$$

The same is true for $H_2(Y_\infty, X^i_\infty)$ and we have $H_2(Y_n, X^i_n : \mathbf{Q}) = 0$ (since $H_2(Y_\infty, X^i_\infty)$ is Γ-torsion), and hence $i_*\colon H_2(X^i_n : \mathbf{Q}) \to H_2(Y_n : \mathbf{Q})$ is an epimorphism.

Since $t-1$ is an isomorphism on $T(X^i_\infty)$, the sequence

$$0 \to F(X^i_\infty) \xrightarrow{t-1} F(X^i_\infty) \to H_2(X^i) \to 0$$

shows that $\operatorname{rank}_\Gamma F(X^i_\infty) = 2g$ where g is the genus of J_i. The same is true for Y_∞ and the similar exact sequences for $t^n - 1$ show that $\dim_{\mathbf{Q}} H_2(X^i_n) = 2gn = \dim_{\mathbf{Q}} H_2(Y_n)$. Since $i_*\colon H_2(X^i_n) \to H_2(Y_n)$ is an epimorphism of vector spaces of the same dimension, it is an isomorphism.

A Mayer-Vietoris sequence shows that $i_*\colon H_2(M_i; \mathbf{Q}) \to H_2(W : \mathbf{Q})$ is an isomorphism. Also we have the following Mayer-Vietoris sequences with $\mathbf{Q}$-coefficients:

$$\begin{array}{ccccccc}
 & H_2(M_i) & \to & H_1(\partial N(k^i_j)) & & & \\
 & \downarrow\cong & & \downarrow\cong & & & \\
 & H_2(W) & \to & H_1(\partial N(E_j)) & & & \\
\to & H_1(N(k^i_j)) & \oplus & H_1(M_i\backslash \operatorname{Int} N(k^i_j)) & \to & 0 \\
 & \cong\downarrow & & \downarrow & & \\
\to & H_1(N(E_j)) & \oplus & H_1(W\backslash \operatorname{Int} N(E_j)) & \to & 0
\end{array}$$

It follows that

$$i_*\colon H_1(M_i\backslash k^i_j : \mathbf{Q}) \to H_1(W\backslash E_j : \mathbf{Q})$$

is an isomorphism for $i = 0, 1$ and the lemma follows. Q.E.D.

Proof of Theorem 3. We use the same notation as in the proof of Lemma 1. Consider the following diagram with $\mathbf{Q}$-coefficients:

$$\begin{array}{ccccc}
H_1(k^0_j) & \to & H_1(M_0\backslash k^0_0) & \cong & \mathbf{Q} \\
\downarrow & & \downarrow & & \\
H_1(E_j) & \to & H_1(W\backslash E_0) & \cong & \mathbf{Q} \\
\uparrow & & \uparrow & & \\
H_1(k^1_j) & \to & H_1(M_1\backslash k^1_0) & \cong & \mathbf{Q}
\end{array}$$

Each vertical homomorphism is induced from an inclusion map and an isomorphism in $\mathbf{Q}$-coefficients. Hence the top homomorphism is zero iff so is the bottom homomorphism. Therefore ξ^n_j is well-defined. Q.E.D.

Example. Let $L_m = K_{m,0} \cup K_{m,1}$ be an "untwisted spun link" indicated in Figure 1, where m is a positive integer. (Regard S^4 as $B^3 \times S^1 \cup S^2 \times B^2$.

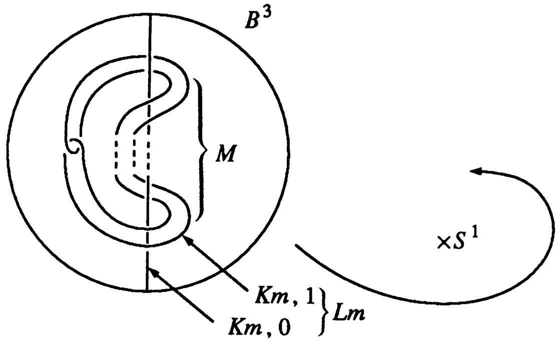

FIGURE 1

The circle in the 3-ball becomes a torus in S^4 after spinning, and the spun arc together with two disks in $S^2 \times B^2$ forms an S^2 in S^4.)· Thus $K_{m,0}$ is homeomorphic to S^2, and $K_{m,1}$ to a torus. Furthermore, $K_{m,0}$ and $K_{m,1}$ are unknotted. One can calculate

$$\xi_j^n(L_m) = \begin{cases} 1 & \text{if } j \equiv \pm m \pmod{n}, \\ 0 & \text{otherwise}, \end{cases}$$

for any prime power n. Hence L_m and $L_{m'}$ are not cobordant to each other unless $m = m'$. Note that the Sato-Levine invariant vanishes and Cochran's derivative is trivial for any m.

ACKNOWLEDGMENT

The author is grateful to Professor C. McA. Gordon for his encouragement and valuable conversations. The author is also thankful to the referee for pointing out mistakes in an earlier version.

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