A FUNCTIONAL CENTRAL LIMIT THEOREM IN DIOPHANTINE APPROXIMATION

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(Communicated by William D. Sudderth)

ABSTRACT. A functional central limit theorem is proved for the number of solutions (p, q) of the inequality $|q\omega - p| < f(q)q^{-1}$, $q \leq n$ (respectively $0 < q\omega - p < f(q)q^{-1}$, $q \leq n$) for some functions f having a positive limit.

1. INTRODUCTION AND STATEMENT

Let Ω be the set of irrational numbers in $(0, 1)$ and let $\mathcal{B}$ be the class of its Borel subsets.

Given a function $f(q)$ of the positive integer q with $0 < f(q) \leq 1/2$, define

$$\begin{aligned}\Lambda_n(\omega) &= \Lambda_{f,n}(\omega) = \# \left\{ (p, q) \in \mathbf{N} \times \mathbf{N}^* : |q\omega - p| < \frac{f(q)}{q}, 1 \leq q \leq n \right\} \\ &= \# \left\{ q \in \mathbf{N}^* : \langle q\omega \rangle < \frac{f(q)}{q}, 1 \leq q \leq n \right\},\end{aligned}$$

where $\langle \alpha \rangle$ is the distance between α and the nearest integer, and

$$\begin{aligned}\Lambda_n^+(\omega) &= \Lambda_{f,n}^+(\omega) = \# \left\{ (p, q) \in \mathbf{N} \times \mathbf{N}^* : 0 < q\omega - p < \frac{f(q)}{q}, 1 \leq q \leq n \right\} \\ &= \# \left\{ q \in \mathbf{N}^* : q\omega - [q\omega] < \frac{f(q)}{q}, 1 \leq q \leq n \right\}\end{aligned}$$

($n \geq 1$, $\omega \in \Omega$; throughout the paper $[\cdot]$ denotes the integer part of a real number).

Assume that $\sum f(q)q^{-1}$ diverges. For fairly general f , the theorems of LeVeque, Erdős, and Schmidt ([5], [3], [9], [4]) show that $\Lambda_{f,n} \sim 2 \sum_{q \leq n} f(q)q^{-1}$ ($= b(n)$, say) a.e., thus improving a classical theorem of Khintchine. Philipp [6, §3] proved, in accordance with a conjecture of LeVeque [5b], the asymptotic normality of $\Lambda_{f,n}$ (explicitly, that $b(n)^{-1/2}(\Lambda_{f,n} - b(n))$ tends to the standard

Received by the editors August 25, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 11K60; Secondary 60F05, 60F17.

Key words and phrases. Diophantine approximation, functional central limit theorem, invariance principle, continued fraction expansion, mixing random variables.

normal law) and that of other related quantities, for f satisfying, besides other conditions, $\lim_q f(q) = 0$; also, [6] contains results of law of iterated logarithm type in this case (in addition, the theorems in [7]—see the introduction—would yield almost sure invariance principles).

Moreover, in [6, p. 43] it is asserted that, with appropriate norming constants, similar results would continue to hold for nonincreasing functions f for which $f(q) = c + O((\log q)^{-1-\varepsilon})$ for some constants $\varepsilon > 0$ and $0 < c < 10^{-3}$.

In this paper, we prove a functional central limit theorem (invariance principle) for $\Lambda_n(\Lambda_n^+)$ under a slightly weaker assumption on f .

We fix some notation. We will denote by λ the Lebesgue measure on $(\Omega, \mathcal{B})$. If ρ is any probability measure on this space we shall write E_ρ for the corresponding expectation operator and $\mathcal{L}_\rho(\xi)$ for the law of a random element ξ defined on $(\Omega, \mathcal{B}, \rho)$; moreover, if ρ is absolutely continuous with respect to λ , we shall write $\rho \ll \lambda$. For the concepts of Skorohod space, Wiener measure, and weak convergence of probability measures, which shall be denoted by $\rightarrow_w$, we refer to [2].

The following result has a version in $C([0, 1])$.

Theorem. *Let $f(q)$ be a nonincreasing function of the positive integer q such that $0 < f(q) \leq 1/2$ and*

$$f(q) = c + o\left(\frac{\log(\log q)}{\log q}\right) \quad \text{for some constant } 0 < c \leq 1/2.$$

Consider the random functions

$$\begin{aligned} \xi_n(t) &= \left(\frac{\pi^2/6}{2c \log n \log(\log n)} \right)^{1/2} \{ \Lambda_{f, [n^t]} - 2c \log[n^t] \}, \\ \xi_n^+(t) &= \left(\frac{\pi^2/6}{c \log n \log(\log n)} \right)^{1/2} \{ \Lambda_{f, [n^t]}^+ - c \log[n^t] \}, \end{aligned}$$

$t \in [0, 1]$, $n \geq 3$. Then for any $\rho \ll \lambda$, $\mathcal{L}_\rho(\xi_n) \rightarrow_w W$ and $\mathcal{L}_\rho(\xi_n^+) \rightarrow_w W$ in the Skorohod space $D([0, 1])$, W being standard Wiener measure.

The proof involves the continued fraction expansion $[0, a_1, a_2, \dots]$ of a number in Ω , and a limit theorem for mixing random variables with infinite variances. As is suggested by an estimation made in [4, p. 35], we compare Λ_n with (random) partial sums of $\{a_j^{1/2}\}$ and we apply the functional limit theorem proved for these sums in [8, Example 2.14.1] (see 2.15 there for references). The comparison is carried out starting from relation (5) below (note that when $f \equiv c$, we have $r_j = [(ca_j)^{1/2}]$ a.e.) and using Lemma 1 (to show that $\tilde{r}_j \approx (ca_j)^{1/2}$, where $\tilde{r}_j$ is a suitable approximation of r_j), which follows from arguments of Ibragimov and Billingsley.

2. PROOF

Given $\omega \in \Omega$, let $\omega = [0, a_1(\omega), a_2(\omega), \dots]$ be its simple continued fraction expansion. We will deal also with the following quantities:

$$\begin{aligned} x_n(\omega) &= [a_n(\omega), a_{n+1}(\omega), \dots], \\ p_0(\omega) &= 0, p_1(\omega) = 1, p_n(\omega) = a_n(\omega)p_{n-1}(\omega) + p_{n-2}(\omega) \quad \text{if } n \geq 2, \\ q_0(\omega) &= 1, q_1(\omega) = a_1(\omega), q_n(\omega) = a_n(\omega)q_{n-1}(\omega) + q_{n-2}(\omega) \quad \text{if } n \geq 2, \\ y_n(\omega) &= \frac{q_n(\omega)}{q_{n-1}(\omega)} = [a_n(\omega), a_{n-1}(\omega), \dots, a_1(\omega)], \\ \frac{1}{u_n(\omega)} &= \left| \omega - \frac{p_{n-1}(\omega)}{q_{n-1}(\omega)} \right| q_{n-1}^2(\omega). \end{aligned}$$

Then $p_n(\omega)/q_n(\omega) = [0, a_1(\omega), \dots, a_n(\omega)]$ ($n \geq 0$), $u_1(\omega) = x_1(\omega)$ and $u_n(\omega) = x_n(\omega) + (y_{n-1}(\omega))^{-1}$ if $n \geq 2$ (see [1, §4]).

It is well known that endowing $(\Omega, \mathcal{B})$ with Gauss's measure

$$P(B) = \frac{1}{\log 2} \int_B \frac{d\omega}{1+\omega} \quad (B \in \mathcal{B}),$$

$\{a_j: j \geq 1\}$ is a stationary sequence of random variables which has good properties of weak dependence. Note that $P(a_j = k) = (\log 2)^{-1} \log(1 + (k(k+2))^{-1})$ ($k \geq 1$).

We shall consider the σ -algebras

$$\mathcal{M}_{jl} = \begin{cases} \sigma(a_{j-l}, \dots, a_{j+l}) & \text{if } j-l \geq 1, \\ \sigma(a_1, \dots, a_{j+l}) & \text{if } j-l < 1. \end{cases}$$

The following lemma is Corollary 3.4 of [8]. Lemma 2 is due to Lévy ([1, (4.12)]).

Lemma 1. Let $\{\eta_j: j \geq 1\}$ be a sequence of measurable real functions on $(\Omega, \mathcal{B})$, and $\{b_n: n \geq 1\} \subset (0, \infty)$. Assume

- (a) $\sup_j E_P \eta_j^2 < \infty$ and $E_P \eta_j = 0$ for every $j \geq 1$.
- (b) $\lim_n n b_n^{-2} = 0$.
- (c) $\sum_{l=1}^{\infty} \sup_j E_P^{1/2}(\eta_j - E_P(\eta_j | \mathcal{M}_{jl}))^2 < \infty$.

Then $\max_{1 \leq i \leq n} |b_n^{-1} \sum_{j=1}^i \eta_j| \rightarrow 0$ in measure.

Lemma 2. If $n \geq 2$, $y = [k_{n-1}, \dots, k_1]$ with $k_1, \dots, k_{n-1} \in \mathbb{N}^*$ and $1 \leq a < b$, then

$$\lambda(a < x_n \leq b | y_{n-1} = y) = \lambda((a, b]) \frac{y(y+1)}{(ay+1)(by+1)}.$$

After these preliminaries, we proceed to prove the theorem. From now on fix a function f as in the statement.

(I) Convergence of ξ_n

Step 1. Let $n \geq 1$ and $\omega \in \Omega$. It is well known that

$$(1) \quad |q'\omega - p'| < (2q')^{-1}, \text{ g.c.d. } (p', q') = 1, \text{ implies } p' = p_k(\omega), \\ q' = q_k(\omega) \text{ for some } k \geq 0.$$

Then, if we define

$$(2) \quad r_j(\omega) = \#\{r \in \mathbf{N}^* : r < (f(rq_{j-1}(\omega))u_j(\omega))^{1/2}\} \quad (j \geq 1),$$

we have

$$\Lambda_{q_n(\omega)-1}(\omega) = \#\left\{(k, r) \in \mathbf{N} \times \mathbf{N}^* : |rq_k(\omega)\omega - rp_k(\omega)| < \frac{f(rq_k(\omega))}{rq_k(\omega)}, \right. \\ \left. 1 \leq rq_k(\omega) < q_n(\omega)\right\} \\ = \sum_{k=0}^{n-1} r_{k+1}(\omega);$$

for the proof of the second equality, use the fact that

$$|q_k(\omega)\omega - p_k(\omega)| = (x_{k+1}(\omega)q_k(\omega) + q_{k-1}(\omega))^{-1} \quad \text{if } k \geq 1,$$

and

$$(3) \quad r < \left(\frac{1}{2}u_{k+1}(\omega)\right)^{1/2}, \quad k \geq 0, \text{ implies } rq_k(\omega) < q_{k+1}(\omega)$$

(observe that if $k \geq 1$ and $r^2 < (1/2)u_{k+1}$ then $(rq_k)^2 < (1/2)(x_{k+1}q_k^2 + q_kq_{k-1}) \leq a_{k+1}q_k^2 + (1/2)q_kq_{k-1} < (a_{k+1}q_k + q_{k-1})^2 = q_{k+1}^2$).

Define the r.v. $\tau_n: \Omega \rightarrow \mathbf{N}^*$ by

$$(4) \quad \tau_n(\omega) = k \quad \text{if } q_{k-1}(\omega) \leq n < q_k(\omega) \quad (k = 1, 2, \dots; n \geq 1).$$

Therefore we have, for every $\omega \in \Omega$,

$$(5) \quad \Lambda_n(\omega) = \sum_{j=1}^{\tau_n(\omega)} r_j(\omega) - \beta_n(\omega),$$

where

$$(6) \quad \beta_n(\omega) := \#\left\{(p, q) \in \mathbf{N} \times \mathbf{N}^* : |q\omega - p| < \frac{f(q)}{q}, \quad n < q < q_{\tau_n}(\omega)\right\} \\ \leq \left(\frac{3}{2}a_{\tau_n}(\omega)\right)^{1/2}$$

(we write $q_{\tau_n(\omega)}(\omega) = q_{\tau_n}(\omega)$, etc.). In order to prove the inequality in (6), note that if (p, q) belongs to the set defining β_n then, by (1) and (3), $p = rp_{\tau_n-1}(\omega)$, $q = rq_{\tau_n-1}(\omega)$ with a positive integer $r < ((1/2)u_{\tau_n}(\omega))^{1/2}$.

Step 2. From now on fix $\rho \ll \lambda$. By [8, Example 2.14.1] we know that, writing

$$(7) \quad \xi_n^{(1)}(t) = \left(\frac{\log 2}{cn \log n} \right)^{1/2} \sum_{j \leq [nt]} \{(ca_j)^{1/2} - E_p(ca_1)^{1/2}\} \quad (t \in [0, 1]),$$

$$\mathcal{L}_\rho(\xi_n^{(1)}) \rightarrow_w W.$$

In place of r_j we will deal first with $\tilde{r}_j$ which is defined by

$$(8) \quad \tilde{r}_j(\omega) = \#\{r \in \mathbf{N}^* : r < (f(r\gamma_j)u_j(\omega))^{1/2}\} \quad (j \geq 1),$$

where $\gamma_1 = 1$, $\gamma_j = [(2^{1/2})^{j-2}]$ if $j \geq 2$. We deduce from (7) that if

$$\xi_n^{(2)}(t) = \left(\frac{\log 2}{cn \log n} \right)^{1/2} \sum_{j \leq [nt]} \{\tilde{r}_j - E_p \tilde{r}_j\}$$

($E_p \tilde{r}_j \leq E_p((3/2)a_j)^{1/2} < \infty$), we have $\mathcal{L}_\rho(\xi_n^{(2)}) \rightarrow_w W$; for this purpose, it is sufficient to show that if

$$g_j = \tilde{r}_j - (ca_j)^{1/2}, \quad \eta_j = g_j - E_p g_j, \quad j \geq 1$$

(we shall see below that $E_p |g_j| < \infty$), then $\{\eta_j\}$ satisfies (a) and (c) of Lemma 1.

In order to verify (a) it is enough to prove that $\sup_{j \geq 1} E_p g_j^2 < \infty$. First, we observe that for every $j \geq 1$ and $\omega \in \Omega$, if $\tilde{r}_j(\omega) \geq 1$ we have

$$\tilde{r}_j(\omega) = \max\{r \in \mathbf{N}^* : r < (f(r\gamma_j)u_j(\omega))^{1/2}\};$$

in fact, since f is nonincreasing, if r, r' are positive integers, $r' < r$, and r belongs to the set defining $\tilde{r}_j(\omega)$, then r' also belongs to that set. Thus we have, for each $j \geq 1$,

$$(9') \quad \tilde{r}_j(\omega) = 0 \Leftrightarrow u_j(\omega) \leq \frac{1}{f(\gamma_j)},$$

and

$$(9'') \quad \text{for every } r \in \mathbf{N}^*, \quad \tilde{r}_j(\omega) = r \Leftrightarrow \frac{r^2}{f(r\gamma_j)} < u_j(\omega) \leq \frac{(r+1)^2}{f((r+1)\gamma_j)}.$$

These relations show that $|g_j| \leq 1$ if $\tilde{r}_j = 0$ and that

$$-1 \leq g_j \leq (f([(ca_j)^{1/2}] - 1)u_j)^{1/2} - (ca_j)^{1/2} \quad \text{if } \tilde{r}_j \geq 1.$$

Then

$$|g_j| \leq 3(2c^{1/2})^{-1} a_j^{1/2} \{f([(ca_j)^{1/2}] - 1) - c\} + 1$$

and (a) of Lemma 1 holds if

$$\sum_{k=1}^{\infty} \{f([(ck)^{1/2}] - 1) - c\}^2 \frac{1}{k} < \infty$$

($P(a_1 = k) \sim (\log 2)^{-1} k^{-2}$); but this follows from the hypothesis on f .

In what follows we will write $\tilde{r}_{jl} = E_P(\tilde{r}_j | \mathcal{M}_{jl})$; note that $\eta_j - E_P(\eta_j | \mathcal{M}_{jl}) = \tilde{r}_j - \tilde{r}_{jl}$. Define

$$A_{jl} = \left\{ u_j \leq \frac{1}{f(\gamma_j)} - \frac{6}{2^l} \right\} \cup \bigcup_{r=1}^{\infty} \left\{ \frac{r^2}{f(r\gamma_j)} + \frac{6}{2^l} < u_j \leq \frac{(r+1)^2}{f((r+1)\gamma_j)} - \frac{6}{2^l} \right\}.$$

It holds that $\tilde{r}_j = \tilde{r}_{jl}$ a.s. on A_{jl} . In order to see this, take positive integers $k_{j-l}, \dots, k_{j+l}$ and $\omega \in A_{jl} \cap \Delta$, $\omega' \in \Delta$, where

$$\Delta = \Delta_{jl}(k_{j-l}, \dots, k_{j+l}) = \begin{cases} \{a_{j-l} = k_{j-l}, \dots, a_{j+l} = k_{j+l}\} & \text{if } j-l \geq 1, \\ \{a_1 = k_1, \dots, a_{j+l} = k_{j+l}\} & \text{if } j-l < 1; \end{cases}$$

we have $\tilde{r}_j(\omega) = \tilde{r}_j(\omega')$ by the definition of A_{jl} , (9), and the fact that $|u_j(\omega) - u_j(\omega')| < 62^{-l}$ (see, for example, [8, (4.4)]).

Therefore, since $\tilde{r}_j \leq ((3/2)a_j)^{1/2}$,

$$(10) \quad E_P(\eta_j - E_P(\eta_j | \mathcal{M}_{jl}))^2 = E_P(\tilde{r}_j - \tilde{r}_{jl})^2 \leq 6 \int_{A_{jl}^c} a_j dP.$$

Now observe that

$$(11) \quad A_{jl}^c = \bigcup_{r=1}^{\infty} J_{jlr}$$

where

$$J_{jlr} = \left\{ \frac{r^2}{f(r\gamma_j)} - \frac{6}{2^l} < u_j \leq \frac{r^2}{f(r\gamma_j)} + \frac{6}{2^l} \right\} \quad (r \in \mathbf{N}^*).$$

Assume $l \geq 2$. We will show that for some constant K ,

$$(12) \quad \lambda(J_{jlr}) \leq K 2^{-l} r^{-4}.$$

Suppose $j \geq 2$ and take $r \in \mathbf{N}^*$. Writing $\alpha(r) = r^2(f(r\gamma_j))^{-1}$, we have

$$\begin{aligned} \lambda(J_{jlr}) &= \sum \lambda(1 < x_j \leq \alpha(r) + 62^{-l} - y^{-1} | y_{j-1} = y) \lambda(y_{j-1} = y) \\ &\quad + \sum \lambda(\alpha(r) - 62^{-l} - y^{-1} < x_j \leq \alpha(r) + 62^{-l} - y^{-1} | y_{j-1} = y) \\ &\quad \times \lambda(y_{j-1} = y) \\ &= A + B \quad (\text{say}), \end{aligned}$$

where the first sum is taken over all possible values y of y_{j-1} which satisfy $\alpha(r) - 62^{-l} - y^{-1} < 1$ (this implies that $\alpha(r) \geq r^4$), and in the second one $\alpha(r) - 62^{-l} - y^{-1} \geq 1$ (in this case $\alpha(r) - 62^{-l} \geq (1/2)r^2$ because $l \geq 2$). By Lemma 2,

$$\begin{aligned} A &= \sum (\alpha(r) - y^{-1} - 1 + 62^{-l})(\alpha(r) + 62^{-l})^{-1} \lambda(y_{j-1} = y) \\ &\leq 12 \cdot 2^{-l} r^{-4}, \\ B &= 12 \cdot 2^{-l} \sum (1 + y^{-1})((\alpha(r) - 62^{-l})(\alpha(r) + 62^{-l}))^{-1} \lambda(y_{j-1} = y) \\ &\leq 24 \cdot 2^{-l} r^{-4}. \end{aligned}$$

Hence $\lambda(J_{jlr}) \leq 36 \cdot 2^{-l} r^{-4}$ when $j \geq 2$ and we obtain a similar bound for $j = 1$ using the fact that $\mathcal{L}_P(x_1)(dt) = I_{(1, \infty)}(t)(t(t+1) \log 2)^{-1} dt$. This proves (12).

On the other hand, since $a_j \leq u_j$, there exists a constant K' such that

$$(13) \quad a_j \leq K' r^2 \quad \text{on } J_{jlr}.$$

From (10)–(13), which are valid for $j \geq 1$ and $l \geq 2$, we conclude that $\{\eta_j\}$ satisfies (c) of Lemma 1 (for $j \geq 1$, $l = 1$, $E_P(\eta_j - E_P(\eta_j | \mathcal{M}_{j1}))^2 \leq 4 \sup_j E_P \eta_j^2 < \infty$). Therefore $\mathcal{L}_P(\xi_n^{(2)}) \rightarrow_w W$.

Step 3. We want to replace $E_P \tilde{r}_j$ in $\xi_n^{(2)}$ by a constant. First we prove that

$$(14) \quad |E_P \tilde{r}_j - E_\lambda \tilde{r}_j| = O(s_1^j) \quad \text{for some } s_1 \in (0, 1).$$

The preceding step shows that for some constant K , $E_P^{1/2}(\tilde{r}_j - \tilde{r}_{jl})^2 \leq K(2^{1/2})^{-l}$. Moreover, arguing as in the proof of [8, (4.15)], we have $|E_P \tilde{r}_{jl} - E_\lambda \tilde{r}_{jl}| \leq K'(s')^{j-l}$ if $j > l$, for some constants K' and $s' \in (0, 1)$. Applying these two inequalities (with $l_j = [j/2]$) we deduce (14).

Using (9) and Lemma 2 we have, for $j \geq 2$ and any positive integer r (with the notation $\alpha(r)$ of the preceding step),

$$\begin{aligned} \lambda(\tilde{r}_j \geq r) &= \sum_y \lambda(x_j > \alpha(r) - y^{-1} | y_{j-1} = y) \lambda(y_{j-1} = y) \\ &= r^{-2} f(r\gamma_j) E_\lambda(1 + y_{j-1}^{-1}); \end{aligned}$$

then

$$E_\lambda \tilde{r}_j = \left\{ \sum_{r=1}^{\infty} r^{-2} f(r\gamma_j) \right\} E_\lambda(1 + y_{j-1}^{-1}) \quad (j \geq 2).$$

Denoting by m_j this quantity for $f \equiv c$, we can assert that

$$(15) \quad m_j = c \frac{\pi^2}{6} E_\lambda(1 + y_{j-1}^{-1}) = c \frac{\pi^2}{6} \left(\frac{1}{\log 2} + O(s_2^j) \right) \quad \text{for some } s_2 \in (0, 1).$$

The last relation follows from a theorem of Lévy which is used in the proof of [8, Lemma 4.5(b)] (or apply directly that result).

Now if we show that

$$(16) \quad (n \log n)^{-1/2} \sum_{j=1}^n (f(\gamma_j) - c) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

then (14)–(16) will yield, by noting that $|E_\lambda \tilde{r}_j - m_j| \leq (\pi^2/3)(f(\gamma_j) - c)$ for $j \geq 2$ (f is nonincreasing),

$$(n \log n)^{-1/2} \sum_{j=1}^n |E_P \tilde{r}_j - (6 \log 2)^{-1} c \pi^2| \rightarrow 0$$

as $n \rightarrow \infty$. This will imply that, denoting

$$\xi_n^{(3)}(t) = \left(\frac{\log 2}{cn \log n} \right)^{1/2} \sum_{j \leq [nt]} \left\{ \tilde{r}_j - \frac{c\pi^2}{6 \log 2} \right\},$$

$$\mathcal{L}_\rho(\xi_n^{(3)}) \rightarrow_w W.$$

In order to prove (16), first observe that, given a sequence of positive integers $\{j(n)\}$ (to be defined later), the quantity there is

$$(17) \quad O(j(n)(n \log n)^{-1/2}) + O((n/\log n)^{1/2}(f(\gamma_{j(n)}) - c)).$$

Write $f(q) = c + \varepsilon(q) \log(\log q)(\log q)^{-1}$ ($q \in \mathbf{N}^*$) where $\lim_q \varepsilon(q) = 0$. There exist $\delta(n) \rightarrow 0$ such that, if we define $j(n) = [\delta(n)(n \log n)^{1/2}]$ then $j(n) \rightarrow \infty$ and $\delta(n)^{-1} \varepsilon(\gamma_{j(n)}) \rightarrow 0$ as $n \rightarrow \infty$ (first choose $n_1 < n_2 < \dots$ such that for all $n \geq n_i$ it holds that $i \varepsilon(\gamma_{[i^{-1}(n \log n)^{1/2}]}) \leq i^{-1}$ and $i^{-1}(n \log n)^{1/2} \geq i$; then define $\delta(n) = i^{-1}$ if $n_i \leq n < n_{i+1}$). This implies (16) since the second term in (17) is $O(\delta(n)^{-1} \varepsilon(\gamma_{j(n)}))$ for this $j(n)$.

As indicated above, we deduce the convergence of $\mathcal{L}_\rho(\xi_n^{(3)})$.

Step 4. The r_j satisfy similar relations to (9):

$$r_j(\omega) = 0 \Leftrightarrow u_j(\omega) \leq \frac{1}{f(q_{j-1}(\omega))}$$

and, for every $r \in \mathbf{N}^*$,

$$r_j(\omega) = r \Leftrightarrow \frac{r^2}{f(rq_{j-1}(\omega))} < u_j(\omega) \leq \frac{(r+1)^2}{f((r+1)q_{j-1}(\omega))}.$$

Suppose $j \geq 2$. From elementary properties of continued fractions, $q_{j-1} = \tilde{q}_{j-1}(\gamma_{j-1})$ where $\tilde{q}_{j-1}$ is a function defined on the set of possible values of γ_{j-1} . Given $r \in \mathbf{N}^*$, write $\alpha(r, y) = r^2(f(r\tilde{q}_{j-1}(y)))^{-1}$ if y is a value of γ_{j-1} ; the preceding relations and Lemma 2 show that

$$\begin{aligned} \lambda(r_j \geq r) &= \sum_y \lambda(x_j > \alpha(r, y) - y^{-1} | \gamma_{j-1} = y) \lambda(\gamma_{j-1} = y) \\ &= r^{-2} E_\lambda(f(rq_{j-1})(1 + y_{j-1}^{-1})). \end{aligned}$$

Therefore, since $r_j \leq \tilde{r}_j$, which is due to the monotonicity of f and the fact that $q_{j-1} \geq \gamma_j$ (see [1, (4.8)]), we have

$$\begin{aligned} E_\lambda |\tilde{r}_j - r_j| &= \sum_{r=1}^{\infty} r^{-2} E_\lambda (\{f(r\gamma_j) - f(rq_{j-1})\} (1 + y_{j-1}^{-1})) \\ &\leq (\pi^2/3)(f(\gamma_j) - c). \end{aligned}$$

Hence, by using (16) we conclude that $(n \log n)^{-1/2} \sum_{j=1}^n |\tilde{r}_j - r_j| \rightarrow 0$ in measure, which in turn yields that $\mathcal{L}_\rho(\xi_n^{(4)}) \rightarrow_w W$, where

$$\xi_n^{(4)}(t) = \left(\frac{\log 2}{cn \log n} \right)^{1/2} \sum_{j \leq [nt]} \left\{ r_j - \frac{c\pi^2}{6 \log 2} \right\}.$$

Step 5. By a well-known theorem of Khintchine, $\lim_k k^{-1} \log q_k = B := \pi^2 / (12 \log 2)$ a.e. This and (4) give

$$(18) \quad \lim_n \frac{\tau_n}{\log n} = B^{-1} \quad \text{a.e.}$$

Now we claim

$$(19) \quad \eta_n := (\log n \log(\log n))^{-1/2} \max_{1 \leq k \leq n} a_{\tau_k}^{1/2} \rightarrow 0 \quad \text{in measure.}$$

For every $\varepsilon > 0$, since $\tau_k \leq \tau_n$ if $k \leq n$,

$$\begin{aligned} P(\eta_n > \varepsilon) &\leq P(\tau_n > (1 + \varepsilon)B^{-1} \log n) \\ &\quad + P\left((\log n \log(\log n))^{-1/2} \max_{j \leq (1+\varepsilon)B^{-1} \log n} a_j^{1/2} > \varepsilon\right) \end{aligned}$$

and the second term on the right is bounded by

$$(1 + \varepsilon)B^{-1}(\log n)P(a_1 > \varepsilon^2 \log n \log(\log n));$$

then (18) implies (19) because $P(a_1 > x) \sim (x \log 2)^{-1}$ as $x \rightarrow \infty$.

We also need

$$(20) \quad \zeta_n := (\log n \log(\log n))^{-1/2} \max_{1 \leq k \leq n} |B\tau_k - \log k| \rightarrow 0 \quad \text{in measure.}$$

First observe that, since $q_{\tau_k-1} \leq k < q_{\tau_k}$, $0 \leq \log q_{\tau_k} - \log k \leq \log y_{\tau_k} \leq \log(a_{\tau_k} + 1) \leq a_{\tau_k}^{1/2}$, and we get $|B\tau_k - \log k| \leq |B\tau_k - \log q_{\tau_k}| + a_{\tau_k}^{1/2}$. Moreover, if $Y_n(t) = \tau_n^{-1/2}(\log q_{[t\tau_n]} - [t\tau_n]B)$ ($t \in [0, 1]$), we have

$$\max_{k \leq n} |\log q_{\tau_k} - B\tau_k| = \tau_n^{1/2} \sup_{t \in [0, 1]} |Y_n(t)|$$

and (20) follows from (18), (19), and a theorem of Billingsley [2, p. 194 and Theorem 17.1].

We claim that if

$$\xi_n^{(5)}(t) = \left(\frac{\pi^2/12}{c \log n \log(\log n)} \right)^{1/2} \sum_{j=1}^{\tau_{[n']}} \left\{ r_j - \frac{c\pi^2}{6 \log 2} \right\},$$

then $\mathcal{L}_\rho(\xi_n^{(5)}) \rightarrow_w W$. Here we will use the notation and results of [2, §17].

Define

$$\Phi_n(t, \omega) = \begin{cases} [\log n]^{-1} \tau_{[n']}(\omega), & \text{if } [\log n]^{-1} \tau_n(\omega) \leq 1, \\ tB^{-1}, & \text{otherwise,} \end{cases}$$

and $g(t) = tB^{-1}(t \in [0, 1], \omega \in \Omega)$. We have $\Phi_n(\cdot, \omega) \in D_0$ for each $\omega, g \in D_0$, and Φ_n converges in ρ -measure to g in D_0 ; this follows from (20) since

$$\sup_{t \in [0, 1]} |\Phi_n(t, \cdot) - g(t)| \leq [\log n]^{-1} \left\{ \max_{1 \leq k \leq n} |\tau_k - B^{-1} \log k| + B^{-1} \right\}.$$

Starting from the weak convergence of $\mathcal{L}_\rho(\xi_{[\log n]}^{(4)}, \Phi_n)$ (use [2, Theorem 4.4]), noting that if ω satisfies $[\log n]^{-1} \tau_n(\omega) \leq 1$ and $t \in [0, 1]$,

$$(\xi_{[\log n]}^{(4)} \circ \Phi_n)(t, \omega) = \left(\frac{\log 2}{c[\log n] \log([\log n])} \right)^{1/2} \sum_{j=1}^{\tau_{[n^t]}(\omega)} \left\{ r_j(\omega) - \frac{c\pi^2}{6 \log 2} \right\},$$

and using (18), we can argue as in [2, proof of Theorem 17.1] to conclude that $\mathcal{L}_\rho(\xi_n^{(5)}) \rightarrow_w W$.

In order to complete the proof, observe that by (5) and (6)

$$\sup_{t \in [0, 1]} |\xi_n(t) - \xi_n^{(5)}(t)| \leq (\pi^2/(12c))^{1/2} ((3/2)^{1/2} \eta_n + 2c\zeta_n) \quad \text{a.e.},$$

and use (19) and (20).

(II) **Convergence of ξ_n^+ .** Minor modifications in (I) are needed. We merely remark that since a convergent $p_k(\omega)/q_k(\omega)$ is less than ω if and only if k is even, (5) is true for Λ_n^+ if in the sum of the right member j runs through odd values and β_n is suitably defined; then the inequality in (6) also holds.

ACKNOWLEDGMENT

Part of this work was done while visiting the Département de Mathématique at Université Louis Pasteur, Strasbourg. I am grateful to that institution for its hospitality.

REFERENCES

1. P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965.
2. —, *Convergence of probability measures*, Wiley, New York, 1968.
3. P. Erdős, *Some results on diophantine approximation*, Acta Arith. **5** (1959), 359–369.
4. S. Lang, *Introduction to diophantine approximations*, Addison-Wesley, Reading, MA, 1966.
- 5a. W. J. LeVeque, *On the frequency of small fractional parts in certain real sequences*, Trans. Amer. Math. Soc. **87** (1958), 237–260.
- 5b. —, *On the frequency of small fractional parts in certain real sequences. II*, Trans. Amer. Math. Soc. **94** (1960), 130–149.
6. W. Philipp, *Mixing sequences of random variables and probabilistic number theory*, Mem. Amer. Math. Soc. **114** (1971).
7. W. Philipp and W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*, Mem. Amer. Math. Soc. **161** (1975).

8. J. D. Samur, *On some limit theorems for continued fractions*, Trans. Amer. Math. Soc. **316** (1989), 53–79.
9. W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Math., vol. 785, Springer-Verlag, Berlin, 1980.

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