INTEGRAL OPERATORS ON SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

PAULETTE SAAB

(Communicated by William J. Davis)

ABSTRACT. Let X be a compact Hausdorff space, let E be a Banach space, and let $C(X, E)$ stand for the Banach space of E -valued continuous functions on X under the uniform norm. In this paper we characterize integral operators (in the sense of Grothendieck) on $C(X, E)$ spaces in terms of their representing vector measures. This is then used to give some applications to nuclear operators on $C(X, E)$ spaces.

INTRODUCTION

Let X be a compact Hausdorff space; let E and F be Banach spaces. Denote by $C(X, E)$ the space of all continuous E -valued functions defined on X under the uniform norm. C. Swartz showed [9] that a bounded linear operator $T : C(X, E) \rightarrow F$ with representing measure G is absolutely summing if and only if each of the values of G is an absolutely summing operator from E to F and G is of bounded variation as a measure taking its values in the space of absolutely summing operators from E to F equipped with the absolutely summing norm. In this paper we shall extend Swartz's result to the class of (Grothendieck) integral operators on $C(X, E)$ spaces. More precisely, we shall show that a bounded linear operator $T : C(X, E) \rightarrow F$ with representing measure G is an integral operator if and only if each of the values of G is an integral operator from E to F and G is of bounded variation as a vector measure taking its values in the space of integral operators from E to F equipped with the integral norm. This result is then used to give some applications to nuclear operators on $C(X, E)$ spaces.

1. PRELIMINARIES

If X is a compact Hausdorff space and E is a Banach space, then $C(X, E)$ will denote the Banach space of all continuous E -valued functions equipped with the uniform norm. It is well known [4, p. 182] that the dual of $C(X, E)$

Received by the editors August 8, 1989 and, in revised form, January 12, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46E40, 46G10; Secondary 28B05, 28B20.

Supported in part by NSF Grant DMS-87500750.

is isometrically isomorphic to the space $M(X, E^*)$ of all regular E^* -valued measures on X that are of bounded variation. When E is the scalar field, we will simply write $C(X)$ and $M(X)$ for $C(X, E)$ and $M(X, E^*)$. If $\mu \in M(X, E^*)$ and $e \in E$, we will denote by $|\mu|$ the variation of μ and by $\langle e, \mu \rangle$ the element of $M(X)$ defined on each Borel subset B of X by

$$\langle e, \mu \rangle(B) = \mu(B)(e).$$

The duality between $M(X, E^*)$ and $C(X, E)$ is then defined as follows: for each $f \in C(X)$ and $e \in E$,

$$\mu(f \otimes e) = \int_X f d\langle e, \mu \rangle,$$

where $f \otimes e$ is the element of $C(X, E)$ defined by

$$f \otimes e(x) = f(x)e, \text{ for all } x \in X.$$

If B is a Borel subset of X , then 1_B will denote the characteristic function of B , and if $e \in E$ we let $1_B \otimes e$ denote the element of $C(X, E)^{**}$ defined by

$$1_B \otimes e(\mu) = \langle e, \mu \rangle(B) = \mu(B)(e),$$

for each $\mu \in M(X, E^*)$.

If X is a compact Hausdorff space, E and F are Banach spaces, every bounded linear operator $T : C(X, E) \rightarrow F$ has a representing measure G . The measure G is defined on the σ -field Σ of Borel subsets of X and takes its values in $\mathcal{L}(E, F^{**})$, the space of all bounded linear operators from E to F^{**} . The measure G is such that, for each Borel subset B of X and for each $e \in E$,

$$G(B)e = T^{**}(1_B \otimes e).$$

For $y^* \in F^*$, if we denote by G_{y^*} the E^* -valued measure on X such that, for each Borel subset B of X and each $e \in E$,

$$\langle e, G_{y^*} \rangle(B) = \langle y^*, G(B)e \rangle,$$

then G_{y^*} is the unique element of $M(X, E^*)$ that represents T^*y^* in the sense that, for each $f \in C(X, E)$,

$$\langle y^*, Tf \rangle = \int_X f(x) dG_{y^*}(x).$$

If E and F are Banach spaces, we denote by $E \otimes_\epsilon F$ the algebraic tensor product of E and F endowed with the norm $\|\cdot\|_\epsilon$:

$$\left\| \sum_{i=1}^m x_i \otimes y_i \right\|_\epsilon = \sup \left\{ \left| \sum_{i=1}^m x^*(x_i) y^*(y_i) \right| : \|x^*\|, \|y^*\| \leq 1 \right\}.$$

The completion $E \widehat{\otimes}_\epsilon F$ of $E \otimes_\epsilon F$ is called the injective tensor product of E and F .

A bounded linear operator $T : E \rightarrow F$ from a Banach space E into a Banach space F is said to be an *integral operator* if the bilinear form τ on $E \times F^*$ defined by $\tau(e, y^*) = y^*(Te)$ for $e \in E$ and $y^* \in F^*$ determines an element of $(E \widehat{\otimes}_e F^*)^*$, the dual of the injective tensor product $E \widehat{\otimes}_e F^*$ of the Banach spaces E and F^* . The integral norm of T , which we will denote by $\|T\|_{\text{int}}$, is just the norm of the bounded linear functional induced by τ as an element of the dual space $(E \widehat{\otimes}_e F^*)^*$. Hence, if $T : E \rightarrow F$ is an integral operator, then

$$\|T\|_{\text{int}} = \sup \left\{ \left\| \sum_{i=1}^n y_i^*(Te_i) \right\| : \left\| \sum_{i=1}^n e_i \otimes y_i^* \right\|_e \leq 1 \right\}.$$

Finally, $I(E, F)$ will stand for the Banach space of all integral operators from E to F equipped with the integral norm. For all undefined notions and notations we refer the reader to [3], [4], or [5].

One of the most useful tools in the study of an integral operator T between two Banach spaces E and F is its representation by a finite regular Borel scalar measure on the compact $B(E^*) \times B(F^{**})$, the product of the closed unit balls of E^* and F^{**} equipped with their weak*-topologies. This, of course, goes back to Grothendieck [6] and can be used to characterize integral operators by the following, which we will state and prove before proving the main result:

Proposition 1. *A bounded linear operator $T : E \rightarrow F$ between two Banach spaces is an integral operator if and only if there exists a regular F^{**} -valued vector measure m of bounded variation defined on the σ -field of Borel subsets of the closed unit ball $B(E^*)$, such that, for each $e \in E$,*

$$Te = \int_{B(E^*)} e^*(e) dm(e^*).$$

In this case m can be chosen so that $\|T\|_{\text{int}} = |m|(B(E^))$.*

Proof. Assume that there exists a regular F^{**} -valued vector measure m defined on the σ -field of Borel subsets of $B(E^*)$ with $|m|(B(E^*)) < \infty$ such that, for each $e \in E$,

$$Te = \int_{B(E^*)} e^*(e) dm(e^*).$$

This in particular shows that the operator $J \circ T : E \rightarrow F^{**}$, where J denotes the natural embedding of F into F^{**} , has an extension $\widehat{T}$ to an integral operator from $C(B(E^*))$ to F^{**} . Hence T is an integral operator [4, p. 233], and

$$\|T\|_{\text{int}} \leq \|\widehat{T}\|_{\text{int}} = |m|(B(E^*)).$$

Conversely, suppose that $T : E \rightarrow F$ is an integral operator; then it follows from [4, page 231] that there exists a regular Borel measure μ on $B(E^*) \times B(F^{**})$ such that, for each $e \in E$ and $y^* \in F^*$,

$$\langle y^*, Te \rangle = \int_{B(E^*) \times B(F^{**})} e^*(e) y^{**}(y^*) d\mu(e^*, y^{**})$$

and

$$\|T\|_{\text{int}} = |\mu|(B(E^*) \times B(F^{**})).$$

Following [4, p. 234], define

$$S : E \rightarrow L_\infty(\mu) \quad \text{by} \quad Se(e^*, y^{**}) = e^*(e)$$

and

$$R : F^* \rightarrow L_\infty(\mu) \quad \text{by} \quad Ry^*(e^*, y^{**}) = y^{**}(y^*).$$

Then S and R are bounded linear operators with $\|S\|, \|R\| \leq 1$. Let Q be the restriction of R^* to $L_1(\mu)$; then Q is a bounded linear operator from $L_1(\mu)$ into F^{**} . It is immediate that $J \circ T = Q \circ I \circ S$ where $I : L_\infty(\mu) \rightarrow L_1(\mu)$ is the natural inclusion and $J : F \rightarrow F^{**}$ is the natural embedding. This in particular shows that the operator $Q \circ I$ restricted to $C(B(E^*) \times B(F^{**}))$ is an integral operator whose representing F^{**} -valued measure $\hat{\mu}$ is such that

$$\hat{\mu}(C) = Q(1_C)$$

for each Borel subset C of $B(E^*) \times B(F^{**})$. Finally, note that since the natural projection of $B(E^*) \times B(F^{**})$ onto $B(E^*)$ is a continuous mapping, it induces a bounded linear operator from $C(B(E^*))$ into $C(B(E^*) \times B(F^{**}))$ as follows: for $\phi \in C(B(E^*))$, let $\hat{\phi} \in C(B(E^*) \times B(F^{**}))$ be such that

$$\hat{\phi}(e^*, y^{**}) = \phi(e^*).$$

This, of course, shows that the operator $J \circ T$ extends to an integral operator

$$\hat{T} : C(B(E^*)) \rightarrow F^{**}$$

such that, for each $\phi \in C(B(E^*))$,

$$\hat{T}\phi = Q \circ I(\hat{\phi}).$$

It is immediate that the F^{**} -valued measure m representing $\hat{T}$ is such that

$$m(B) = Q(1_{B \times B(F^{**})})$$

and

$$|m|(B(E^*)) \leq |\mu|(B(E^*) \times B(F^{**})) = \|T\|_{\text{int}}.$$

Since $\hat{T}$ extends T , it follows that

$$\|T\|_{\text{int}} = |m|(B(E^*)).$$

MAIN RESULT

Throughout this section, X is a compact Hausdorff space, E and F are Banach spaces, and $T : C(X, E) \rightarrow F$ is a bounded linear operator with representing measure G . The main result of this paper gives a characterization of integral operators T in terms of some properties of G . The first step to achieve such a characterization is to show that, if T is an integral operator on $C(X, E)$, then one can do a little better than Proposition 1 by representing the

operator T by a regular F^{**} -valued measure of bounded variation defined on the σ -field of Borel subsets of $X \times B(E^*)$ rather than on the whole unit ball of $C(X, E)^*$. The proof we present here is different from our earlier proof, which relied on our result [7]. We would like to thank the referee for suggesting the following approach, which relies more on basic knowledge and classical results of vector measures that can be found in [4] or [5].

Lemma 2. *If $T : C(X, E) \rightarrow F$ is an integral operator, then there exists a regular F^{**} -valued measure of bounded variation θ on the σ -field of Borel subsets of $X \times B(E^*)$, such that for each $f \in C(X, E)$ and $y^* \in F^*$,*

$$\langle y^*, Tf \rangle = \int_{X \times B(E^*)} e^*(f(x)) d\theta(x, e^*)$$

and

$$\|T\|_{\text{int}} = |\theta|(X \times B(E^*)).$$

Proof. Suppose that $T : C(X, E) \rightarrow F$ is an integral operator, then the bilinear map $\tau(f, y^*) = y^*(T(f))$ for $f \in C(X, E)$ and $y^* \in F^*$ defines an element of $(C(X, E) \otimes_e F^*)^*$. It is easy to check that $C(X, E) \otimes_e F^*$ embeds isometrically in $C(X \times B(E^*) \times B(F^{**}))$ such that, if $f \in C(X, E)$ and $y^* \in F^*$,

$$f \otimes y^*(x, e^*, y^{**}) = e^*(f(x))y^{**}(y^*)$$

for each $(x, e^*, y^{**}) \in \Omega = X \times B(E^*) \times B(F^{**})$. Hence, by the Hahn-Banach theorem, there exists a regular Borel measure μ on Ω so that, for each $f \in C(X, E)$ and $y^* \in F^*$,

$$\langle y^*, Tf \rangle = \int_{\Omega} e^*(f(x))y^{**}(y^*) d\mu(x, e^*, y^{**})$$

and

$$\|T\|_{\text{int}} = |\mu|(\Omega).$$

The proof now follows the steps of Proposition 1. Indeed, since the natural projection of Ω onto $X \times B(E^*)$ is continuous, it induces a bounded linear operator

$$S : C(X \times B(E^*)) \rightarrow L_{\infty}(\mu)$$

such that, if $\phi \in C(X \times B(E^*))$, then

$$S(\phi)(x, e^*, y^{**}) = \phi(x, e^*)$$

for all $(x, e^*, y^{**}) \in \Omega$.

Let $R : F^* \rightarrow L_{\infty}(\mu)$ be defined by $R(y^*)(x, e^*, y^{**}) = y^{**}(y^*)$, and let Q be the restriction of R^* to $L_1(\mu)$. It is straightforward to check that the operator $Q \circ I \circ S$ is an integral operator whose representing measure θ is such that

$$\theta(C) = Q(1_{C \times B(F^{**})})$$

for each Borel subset C of $X \times B(E^*)$, and

$$|\theta|(X \times B(E^*)) \leq |\mu|(\Omega) = \|T\|_{\text{int}}.$$

Moreover, note that $C(X, E)$ embeds isometrically in $C(X \times B(E^*))$ by

$$f(x, e^*) = e^*(f(x))$$

for each $f \in C(X, E)$ and $(x, e^*) \in X \times B(E^*)$. It is easy to see that, for each $f \in C(X, E)$,

$$J \circ T(f) = Q \circ I \circ S(f);$$

hence

$$T(f) = \int_{X \times B(E^*)} e^*(f(x)) d\theta(x, e^*)$$

and

$$\|T\|_{\text{int}} \leq \|Q \circ I \circ S\|_{\text{int}} \leq |\theta|(X \times B(E^*)).$$

This complete the proof of Lemma 2.

We are now ready to state and prove the main result of this paper.

Theorem 3. *Let X be a compact Hausdorff space, let E and F be Banach spaces, and let $T : C(X, E) \rightarrow F$ be a bounded linear operator with representing measure G . Then T is an integral operator if and only if, for each Borel subset B of X , the operator $G(B) : E \rightarrow F$ is integral and the measure G is of finite variation as a vector measure taking its values in $I(E, F)$, which is equipped with the integral norm.*

Proof. Assume that $T : C(X, E) \rightarrow F$ is an integral operator. Let G denote the vector measure representing the operator T ; hence, for each Borel subset B of X and each $e \in E$,

$$G(B)e = T^{**}(1_B \otimes e).$$

It is clear at this stage that, since T^{**} is also integral [4, p. 236] then, for each Borel subset B of X , the operator $G(B) : E \rightarrow F$ is an integral operator since it is the composition of T^{**} and the bounded linear operator $E \rightarrow C(X, E)^{**}$ which to each e in E associates the element $1_B \otimes e$. In what follows we shall concentrate on estimating the value of $\|G(B)\|_{\text{int}}$. For this, note that by Lemma 2 there exists a regular F^{**} -valued measure θ defined on the σ -field of Borel subsets of $X \times B(E^*)$ such that, for each $f \in C(X, E)$,

$$T(f) = \int_{X \times B(E^*)} e^*(f(x)) d\theta(x, e^*)$$

and

$$\|T\|_{\text{int}} = |\theta|(X \times B(E^*)).$$

It follows that, for each $y^* \in F^*$ and $f \in C(X, E)$,

$$\langle T^* y^*, f \rangle = \int_{X \times B(E^*)} e^*(f(x)) d\theta_{y^*}(x, e^*),$$

where θ_{y^*} is the scalar measure on $X \times B(E^*)$ defined by

$$\theta_{y^*}(C) = \theta(C)(y^*)$$

for each Borel subset C of $X \times B(E^*)$.

We claim that, for each Borel subset B of X and each $e \in E$,

$$(*) \quad G(B)e = \int_{B \times B(E^*)} e^*(e) d\theta(x, e^*).$$

For this, suppose that $y^* \in F^*$ and K is a compact G_δ subset of X . Let $(u_n)_{n \geq 1}$ be a sequence of continuous real-valued functions so that $0 \leq u_n \leq 1$ for all $n \geq 1$ and u_n converges to 1_K pointwise. If $e \in E$, then

$$\begin{aligned} \langle G(K)e, y^* \rangle &= \langle e, G_{y^*} \rangle(K) \\ &= \lim_{n \rightarrow \infty} \langle y^*, T(u_n \otimes e) \rangle \\ &= \lim_{n \rightarrow \infty} \int_{X \times B(E^*)} u_n(x) e^*(e) d\theta_{y^*}(x, e^*) \\ &= \int_{X \times B(E^*)} 1_K(x) e^*(e) d\theta_{y^*}(x, e^*) \\ &= \int_{K \times B(E^*)} e^*(e) d\theta_{y^*}(x, e^*). \end{aligned}$$

Therefore,

$$G(K)e = \int_{K \times B(E^*)} e^*(e) d\theta(x, e^*).$$

Moreover, since for each $y^* \in F^*$ the E^* -valued vector measure G_{y^*} is regular, it follows that

$$G(B)e = \int_{B \times B(E^*)} e^*(e) d\theta(x, e^*)$$

for all Borel subsets B of X . This proves our claim. If we denote by $p : X \times B(E^*) \rightarrow E^*$ the projection mapping which to each (x, e^*) in $X \times B(E^*)$ associates e^* in $B(E^*)$, then for each Borel subset B of X , let λ_B denote the regular F^{**} -valued measure defined on the σ -field of Borel subsets of $B(E^*)$ as follows: for each Borel subset V of $B(E^*)$,

$$\lambda_B(V) = \theta(B \times V).$$

In other words, $\lambda_B = \theta|_{B \times B(E^*)} \circ p^{-1}$, which is the image measure of the restriction of θ to $B \times B(E^*)$ by p . This implies that, for each $e \in E$,

$$(**) \quad \int_{B(E^*)} e^*(e) d\lambda_B(e^*) = \int_{B \times B(E^*)} e^*(e) d\theta(x, e^*).$$

Equations $(*)$ and $(**)$ show that the measure λ_B is a regular, F^{**} -valued measure that represents the operator $G(B) : E \rightarrow F$. Hence, by Proposition 1,

$$\|G(B)\|_{\text{int}} \leq |\lambda_B|(B(E^*)) = |\theta|(B \times B(E^*)),$$

and therefore

$$(\dagger) \quad \|G(B)\|_{\text{int}} \leq |\theta|(B \times B(E^*)).$$

Hence the vector measure G representing the operator T takes its values in $I(E, F)$, and it follows from (†) that

$$|G|_{\text{int}}(X) \leq |\theta|(X \times B(E^*)).$$

Here, of course, $|G|_{\text{int}}(X) = \sup \sum_{B_i \in \pi} \|G(B_i)\|_{\text{int}}$, where the sup is taken over all the finite partitions of X into Borel subsets of X .

Conversely, suppose $T : C(X, E) \rightarrow F$ is such that

$$G : \Sigma \rightarrow I(E, F), \text{ and } |G|_{\text{int}}(X) < \infty;$$

we need to show that T is an integral operator. For this, note that if J denotes the natural embedding of F into F^{**} , then by [4, p. 233] it is enough to show that $J \circ T : C(X, E) \rightarrow F^{**}$ is an integral operator. The mapping J induces a mapping $\hat{J} : I(E, F) \rightarrow I(E, F^{**})$ defined as follows, for each $U \in I(E, F)$:

$$\hat{J}(U) = J \circ U;$$

this in turn induces a vector measure $\hat{G} : \Sigma \rightarrow I(E, F^{**})$ such that

$$\hat{G}(B) = \hat{J}(G(B)),$$

for all $B \in \Sigma$. It is immediate that $\hat{G}$ is the measure representing the operator $J \circ T$. Since $|G|_{\text{int}}(X) < \infty$, it follows easily that $|\hat{G}|_{\text{int}}(X) < \infty$. Moreover, since $I(E, F^{**})$ is isometric to the dual space of the injective tensor product $E \hat{\otimes}_e F^*$ [4, p. 237], it follows that $\hat{G}$ is a vector measure of bounded variation taking its values in the dual space $(E \hat{\otimes}_e F^*)^*$. Since $|\hat{G}|_{\text{int}}(X) < \infty$, it follows from [4, p. 7] that $\hat{G}$ is strongly additive. Recall that, for each $e \in E$ and $y^* \in F^*$, the scalar measure $\langle e, G_{y^*} \rangle$ is in $M(X)$. Hence, for $u = \sum_{i=1}^n e_i \otimes y_i^* \in E \hat{\otimes}_e F^*$, let $\langle u, \hat{G} \rangle$ be the scalar measure defined on X as follows: for each Borel subset B of X ,

$$\langle u, \hat{G} \rangle(B) = \hat{G}(B)(u) = \sum_{i=1}^n \langle G(B)e_i, y_i^* \rangle = \sum_{i=1}^n \langle e_i, G_{y_i^*} \rangle(B).$$

This implies that for each $u \in E \hat{\otimes}_e F^*$, the measure $\langle u, \hat{G} \rangle \in M(X)$. This in particular shows that $\hat{G} : \Sigma \rightarrow (E \hat{\otimes}_e F^*)^*$ is countably additive with respect to the weak* topology on $(E \hat{\otimes}_e F^*)^*$. Since $\hat{G}$ is strongly additive, it follows easily that G is countably additive. Finally, since $\hat{G}$ is weak* regular, a glance at [4, p. 157] reveals that it is also regular. It follows that $\hat{G}$ defines an element of $C(X, (E \hat{\otimes}_e F^*)^*)$; hence, by [5, p. 269], there exists a weak* $|\hat{G}|_{\text{int}}$ -integrable function

$$h : X \rightarrow I(E, F^{**}) \simeq (E \hat{\otimes}_e F^*)^*$$

such that $\|h(x)\|_{\text{int}} = 1 \cdot |\hat{G}|_{\text{int}}$ a.e., $\hat{G} = h|\hat{G}|_{\text{int}}$ and, for each $f \in C(X, E \hat{\otimes}_e F^*)$,

$$\langle f, \hat{G} \rangle = \int_X \langle f(x), h(x) \rangle d|\hat{G}|_{\text{int}}(x).$$

In particular, if $\varphi \in C(X, E)$ and $y^* \in F^*$, let $f = \varphi \otimes y^*$ be the element of $C(X, E \widehat{\otimes}_e F^*)$ defined by

$$\varphi \otimes y^*(x) = \varphi(x) \otimes y^* \quad \text{for all } x \in X,$$

then

$$\langle \varphi \otimes y^*, \widehat{G} \rangle = \int_X y^* \langle h(x), \varphi(x) \rangle d|\widehat{G}|_{\text{int}}(x).$$

But we also have

$$(***) \quad \langle J \circ T\varphi, y^* \rangle = \langle \varphi, \widehat{G}_{y^*} \rangle,$$

where $\widehat{G}_{y^*}$ is the element of $M(X, E^*)$ such that, for each Borel subset B of X ,

$$\widehat{G}_{y^*}(B) = y^*(\widehat{G}(B)) = y^*(G(B)).$$

Hence since $\widehat{G} = h \cdot |\widehat{G}|_{\text{int}}$, it follows that for each Borel subset B of X ,

$$\widehat{G}_{y^*}(B) = \int_B y^* h(x) d|\widehat{G}|_{\text{int}}(x).$$

This implies that, for each $\varphi \in C(X, E)$,

$$(***) \quad \langle \varphi, \widehat{G}_{y^*} \rangle = \int_X y^* \langle h(x), \varphi(x) \rangle d|\widehat{G}|_{\text{int}}(x).$$

It follows that if $\sum_{i=1}^n \varphi_i \otimes y_i^*$ is in $C(X, E) \otimes_e F^*$ with $\|\sum_{i=1}^n \varphi_i \otimes y_i^*\|_e \leq 1$, then, for each x in X ,

$$\left| \langle h(x), \sum_{i=1}^n \varphi_i(x) \otimes y_i^* \rangle \right| \leq \|h(x)\|_{\text{int}},$$

since

$$\left\| \sum_{i=1}^n \varphi_i(x) \otimes y_i^* \right\|_e \leq \left\| \sum_{i=1}^n \varphi_i \otimes y_i^* \right\|_e \leq 1.$$

It follows then, from (***) and (****), that

$$\left\| \sum_{i=1}^n \langle T\varphi_i, y_i^* \rangle \right\| \leq |\widehat{G}|_{\text{int}}(X) < \infty.$$

Hence $J \circ T$ is an element of $I(C(X, E), F^{**})$, which implies that $T \in I(C(X, E), F)$. This completes the proof.

APPLICATIONS

The study of integral operators on $C(X, E)$ spaces was motivated by some problems that arose in [8] concerning nuclear operators on $C(X, E)$ spaces. Recall that an operator T between two Banach spaces Y and Z is said to be a *nuclear operator* if there exist sequences (y_n^*) and (z_n) in Y^* and Z , respectively, such that $\sum_n \|y_n^*\| \|z_n\| < \infty$ and, for each $y \in Y$,

$$Ty = \sum_n y_n^*(y) z_n.$$

The nuclear norm is defined by

$$\|T\|_{\text{nuc}} = \inf \left\{ \sum_n \|y_n^*\| \|z_n\| \right\},$$

where the infimum is taken over all sequences (y_n^*) and (z_n) such that $Ty = \sum_n y_n^*(y)z_n$ for all $y \in Y$. We shall denote by $N(Y, Z)$ the space of nuclear operators from Y to Z under the nuclear norm. The study of nuclear operators on $C(X, E)$ spaces was initiated in [1], where some of the known results in the scalar case were extended. Nuclear operators on $C(X, E)$ spaces were also considered in [2], where it was shown that if $T : C(X, E) \rightarrow F$ is a nuclear operator, then for each Borel subset B of X , the operator $G(B) : E \rightarrow F$ is nuclear. In [8] it was shown that the representing measure G of a nuclear operator is in fact of bounded variation as a vector measure taking its values in $N(E, F)$. Hence all the above known results can be summarized as follows:

Proposition 4. *Let X be a compact Hausdorff space, and let E and F be two Banach spaces. If T is a nuclear operator from $C(X, E)$ into F with representing measure G , then*

- (i) *For any Borel subset B of X , the operator $G(B) : E \rightarrow F$ is nuclear, and*
- (ii) *G is of bounded variation as a vector measure taking its values in $N(E, F)$ under the nuclear norm.*

Easy examples show that conditions (i) and (ii) above do not characterize nuclear operators on $C(X, E)$ spaces. As a matter of fact, it was shown in [8] that counterexamples can be given as soon as the range space F fails to have the so-called Radon–Nikodym property (RNP) [4]. The interesting question that arises is then the following:

Question 5. Assuming that F has the (RNP) and that $T : C(X, E) \rightarrow F$ is an operator whose representing measure G satisfies conditions (i) and (ii) of Proposition 4, is T a nuclear operator?

In what follows, we shall show how Theorem 3 can be used to give a positive answer to Question 5 in case F is assumed to be complemented in its bidual.

Theorem 6. *Let F be a Banach space with the Radon–Nikodym property, and assume that F is complemented in its bidual. A bounded linear operator $T : C(X, E) \rightarrow F$ is nuclear whenever its representing vector measure G satisfies the following conditions:*

- (i) *For each Borel subset B of X , the operator $G(B) : E \rightarrow F$ is nuclear, and*
- (ii) *G is of bounded variation as a vector measure taking its values in $N(E, F)$.*

Proof. First note that, for an arbitrary Banach space F , any element U in $N(E, F)$ is in $I(E, F)$ with $\|U\|_{\text{int}} \leq \|U\|_{\text{nuc}}$. Hence, by Theorem 3, an

operator $T : C(X, E) \rightarrow F$ that satisfies conditions (i) and (ii) above is indeed integral. Therefore, when in addition we assume that F is complemented in its bidual F^{**} , the operator T becomes Pietsch integral (also called strictly integral) [4, p. 235]. Under these conditions, and if F is assumed to have the Radon–Nikodym property, it is well known [4, p. 175] that T becomes a nuclear operator. This completes the proof.

Remark. As we have just seen, Theorem 6 is a direct application of the main result of this paper, it should be noted that Theorem 6 gives a positive answer to Question 5 when F is a dual space and when F is a Banach lattice.

BIBLIOGRAPHY

1. G. Alexander, *Linear operators on the space of vector-valued continuous functions*, Ph.D. thesis, New Mexico State University, Las Cruces, New Mexico, 1972.
2. R. Bilyeu and P. Lewis, *Some mapping properties of representing measures*, Ann. Math. Pure Appl. **109** (1976), 273–287.
3. G. Choquet, *Lecture on analysis*, Vol. II, Benjamin, New York, 1969.
4. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys **15** Amer. Math. Soc., Providence, RI, 1977.
5. N. Dinculeanu, *Vector measures*, Pergamon, Press, 1967.
6. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc. **16** (1955), 1–140.
7. P. Saab, *Integral representation by boundary vector measures*, Canad. Math. Bull. **25** (1982), 164–168.
8. B. Smith, *Some bounded linear operators on the spaces $C(\Omega, E)$ and $A(K, E)$* , Ph.D. thesis, University of Missouri, Columbia, MO, 1989.
9. C. Swartz, *Absolutely summing and dominated operators on spaces of vector valued continuous functions*, Trans. Amer. Math. Soc. **179** (1973), 123–132.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211