VANISHING OF COEFFICIENTS IN OVERLAPPING GERM EXPANSIONS FOR p -ADIC $GL(n)$

FIONA MURNAGHAN AND JOE REPKA

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ABSTRACT. Shalika germs at the identity in p -adic $GL(n)$ will blow up at other singular points. Waldspurger has developed a technique for describing this behaviour at intermediate singular points, giving a “germ expansion for germs”. In this paper we discuss which germs occur in such an expansion and find that the answer is more complicated than expected.

1. INTRODUCTION

Waldspurger [3] has shown that no germ on $G = GL(n)$ is identically zero on the regular elliptic set. This is consistent with the intuition that every unipotent class should contribute at the regular elliptic level to some orbital integrals.

Given a unipotent class $\mathcal{O}$ in G , Waldspurger gives an expansion of the corresponding germ near a singular elliptic point γ_0 , expressing it in terms of germs on the centralizer $Z(\gamma_0)$. The authors originally expected nontrivial contributions from all germs corresponding to classes “near” the class $\mathcal{O}$ (as expressed in [2; Theorem 6.11 and Proposition 6.8]).

However in studying Waldspurger’s work we discovered more subtle behaviour, in that some classes which are near $\mathcal{O}$ do not contribute. For example, even for unipotent classes of small dimension there may not be a contribution from the germ associated to the identity class in $Z(\gamma_0)$.

In this paper we give partial results about the existence and nonexistence of nonzero contributions in various circumstances, but we are unable to discern an overall pattern or explain the phenomenon in terms of orbital integrals. We present these results in the hope that somebody may find an explanation.

The machinery established by Waldspurger is summarized in §2. In §3 we define the kind of anomalous behaviour to be considered, show that in certain cases it does not occur, and describe a large class of anomalies which can be constructed explicitly and systematically. In §4 we prove that there are no anomalies at the identity in the centralizer $Z(\gamma_0)$, provided that the

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intermediate field extension is of prime power order. Finally in §5 we list examples of anomalies.

2. SUMMARY OF RESULTS OF WALDSPURGER

Let F be a p -adic field of characteristic zero, with residue field of order q , and $G = \mathrm{GL}(n, F)$.

In order to state Waldspurger's results, some notation is needed; essentially we follow Waldspurger [3].

A unipotent conjugacy class in $\mathrm{GL}(m)$ corresponds to a partition $\mu = (m_1, \dots, m_r)$ of m , where the m_i 's are the sizes of the blocks in the Jordan canonical form of elements in the class.

Let R_m be the abelian group generated by the characters of irreducible representations of $\mathrm{GL}(m, \mathbb{F}_q)$ which occur as subquotients of the representation induced from the trivial representation of the standard Borel subgroup. Let $y_m \in R_m$ be the Steinberg representation of $\mathrm{GL}(m, \mathbb{F}_q)$.

The graded algebra $R = \bigoplus_{m \in \mathbb{N}} R_m$, has a multiplication defined as follows: if $x \in R_m$ and $y \in R_{m'}$, then $x \cdot y$ is the representation of $\mathrm{GL}(m+m')$ obtained by inducing $(x \otimes y) \otimes 1$ from the parabolic subgroup whose Levi component is $\mathrm{GL}(m) \times \mathrm{GL}(m')$.

If $\mu = (m_1, \dots, m_r)$ is a partition of m , we write $y_\mu = y_{m_1} \cdots y_{m_r}$, an element of R_m . Writing $\mathcal{P}(m)$ for the set of all partitions of m , we have that $\{y_\mu : \mu \in \mathcal{P}(m)\}$ is a basis of R_m .

For $\mu \in \mathcal{P}(m)$, with $\mu = (m_1, \dots, m_r)$, define

$$(2.1) \quad \zeta_\mu = \frac{(-1)^{r+m} m(r-1)!}{|\mathrm{St}(\mu)|},$$

where $\mathrm{St}(\mu)$ is the stabilizer of μ in the symmetric group S_r . In fact, $\zeta_\mu \in \mathbb{Z}$. We will use the notation $r(\mu)$ to denote the length of $\mu \in \mathcal{P}(m)$.

For $m \in \mathbb{N}$, define $z_m \in R_m \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$(2.2) \quad z_m = \sum_{\mu \in \mathcal{P}(m)} \zeta_\mu y_\mu.$$

For $\mu = (m_1, \dots, m_r) \in \mathcal{P}(m)$, define $z_\mu \in R_m \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$z_\mu = z_{m_1} \cdots z_{m_r}.$$

It turns out that $\{z_\mu : \mu \in \mathcal{P}(m)\}$ is a basis for $R_m \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let f be a positive integer. Define an endomorphism τ_f of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$\tau_f z_m = z_{fm}.$$

This restricts to an endomorphism of R (also denoted τ_f).

Using the formulas 3.17(c) [4, p. 46], it can easily be shown that

$$(2.3) \quad y_m = \sum_{\mu \in \mathcal{P}(m)} \eta_\mu z_\mu,$$

where, for $\mu = (m_1, \dots, m_r) \in \mathcal{P}(m)$,

$$\eta_\mu = \frac{(-1)^{r+m}}{m_1 \cdots m_r |\text{St}(\mu)|}.$$

The algebra $R_m \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the so-called Hall algebra $H(q) = \bigoplus_{m \in \mathbb{N}} H_m(q)$, where $H_m(q)$ is the space of class functions on the unipotent set of $\text{GL}(m, \mathbb{F}_q)$. For details of these constructions see [4].

For $m \in \mathbb{N}$, let $c_m(q) = \prod_{j=1}^m (q^j - 1)$, and for $\mu = (m_1, \dots, m_r) \in \mathcal{P}(m)$, let

$$c_\mu(q) = c_{m_1}(q) \cdots c_{m_r}(q).$$

If μ is a partition of m , we write μ^t for the “dual partition.”

Suppose K is a finite extension of F . Recall the theorem of Shalika (cf. [3]) which says that for a locally constant compactly supported function ϕ on $\text{GL}(m, K)$, the orbital integral of ϕ over the conjugacy class of γ can be written

$$\sum_{\mu \in \mathcal{P}(m)} \Lambda_{\mu^t}(\phi) \Gamma_{\mu^t}^K(\gamma),$$

for regular γ sufficiently near the identity. The functions $\Gamma_{\mu^t}^K$ are the Shalika germs and the coefficients $\Lambda_{\mu^t}(\phi)$ are the orbital integrals of ϕ over the unipotent classes $\mathcal{O}_{\mu^t}$.

For $\mu \in \mathcal{P}(m)$, let $\nu(\mu^t) = \frac{1}{2} \dim \mathcal{O}_{\mu^t}$.

It is now possible to state Waldspurger’s theorem. Let $T \subset G = \text{GL}(n, F)$ be an elliptic torus corresponding to the extension field E/F , let F' be an unramified intermediate field with $[F' : F] = f$, and let $n' = n/f$.

Suppose the element X in the ring of integers $\mathfrak{O}_{F'}$ is such that its reduction in $\mathbb{F}_{q^f}$ generates $\mathbb{F}_{q^f}$ over $\mathbb{F}_q$. If ϖ is a uniformizing element in F , then for $a \geq 0$, the element $\gamma_0 = 1 + \varpi^a X$ is a singular point in T . Waldspurger considers the behaviour near γ_0 of the Shalika germ expansion at the identity in G . He gives a formula relating it to germs at γ_0 , which are identified with germs at the identity in the centralizer $\text{GL}(n', F')$ of γ_0 [3].

Any regular point $\gamma \in T$ sufficiently close to γ_0 can be written in the form

$$\gamma = \gamma_0 + \varpi^b Y = 1 + \varpi^a X + \varpi^b Y,$$

with $b > a$, $Y \in \mathfrak{O}_E$, and $F'(Y) = E$.

With γ as above, define $\gamma' = 1 + \varpi^{b-a} Y$. Then the following identity holds in $R \otimes_{\mathbb{Z}} \mathbb{C}$:

$$(2.4) \quad \sum_{\mu \in \mathcal{P}(n)} \Gamma_{\mu^t}^F(\gamma) q^{-a\nu(\mu^t)} c_\mu(q) y_\mu = \tau_f \left[\sum_{\mu \in \mathcal{P}(n')} \Gamma_{\mu^t}^{F'}(\gamma') c_\mu(q^f) y_\mu \right].$$

It follows that it is possible to write

$$(2.5) \quad \Gamma_{\mu^t}^F(\gamma) = \sum_{\lambda \in \mathcal{P}(n')} a_{\mu, \lambda} \Gamma_{\lambda^t}^{F'}(\gamma'),$$

for some constants $a_{\mu,\lambda}$. In principle the $a_{\mu,\lambda}$'s can be evaluated by expressing y_μ on the right side of (2.4) in terms of the z_μ 's, using (2.3); applying τ_f ; expressing the resulting formula in terms of the y_μ 's, $\mu \in \mathcal{P}(n)$, again, using (2.2); and finally equating coefficients of the y_μ 's.

In practice it is difficult even to determine which of the $a_{\mu,\lambda}$'s are nonzero. In this paper we give some partial results which show that the answer is not as simple as one might expect.

3. DEFINITION OF ANOMALIES

For $\mu \in \mathcal{P}(n)$, only certain types of germs $\Gamma_{\lambda'}^{F'}$ can occur in the expression for $\Gamma_{\mu'}^F$. If the coefficient of y_μ in $\tau_f(y_\lambda)$ is zero, then $a_{\mu,\lambda} = 0$. The only way $a_{\mu,\lambda}$ could possibly be nonzero for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}(n')$ is if the unipotent class corresponding to μ' is "induced" (in the sense of Lusztig and Spaltenstein [1]) from some unipotent class in the Levi subgroup isomorphic to $\mathrm{GL}(f \cdot \lambda_1) \times \dots \times \mathrm{GL}(f \cdot \lambda_k)$. In terms of partitions, the following condition must be satisfied: for each i there exists a partition $(n_1^i, \dots, n_{r_i}^i)$ of $f \cdot \lambda_i$ with $r_i \leq f$ and such that

$$(3.1) \quad \mu = (n_1^1, \dots, n_{r_1}^1; n_1^2, \dots, n_{r_2}^2; \dots; n_1^k, \dots, n_{r_k}^k)$$

See ([2, Remark 8.1(1)]).

This condition on λ implies that if u is in the class $\mathcal{O}_{\lambda'} \in \mathrm{GL}(n', F') \subset \mathrm{GL}(n, F)$, then $\gamma_0 u$ has conjugates in G which are close to $\mathcal{O}_{\mu'}$. From this we might expect that $\Gamma_{\lambda'}^{F'}$ actually occurs in the expression of $\Gamma_{\mu'}^F$, i.e. that $a_{\mu,\lambda} \neq 0$.

The authors were surprised to find that this is not always the case, and this is the motivation for the following definition.

Definition 3.2. Suppose $\mu \in \mathcal{P}(n)$ and $\lambda \in \mathcal{P}(n')$. Then (μ, λ) is an anomaly if λ satisfies the condition (3.1) above but $a_{\mu,\lambda} = 0$.

Notice that this condition depends on $f = [F' : F]$ but does not depend on γ_0, γ' or on the elliptic torus T .

Proposition 3.3

- (1) $(\mu, (1^{n'}))$ is never an anomaly.
- (2) $(\tau_f(\lambda), \lambda)$ is never an anomaly, for any $\lambda \in \mathcal{P}(n')$, where we write $\tau_f(\lambda_1, \dots, \lambda_r) = (f \cdot \lambda_1, \dots, f \cdot \lambda_r)$.
- (3) (μ, λ) is not an anomaly if $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\lambda_1, \dots, \lambda_1; \lambda_2, \dots, \lambda_2; \dots; \lambda_r, \dots, \lambda_r)$, with each entry repeated f times.

Proof.

$$(i) \quad \tau_f(y_1^{n'}) = \tau_f(z_1^{n'}) = z_f^{n'} = \left(\sum_{\delta \in \mathcal{P}(f)} \zeta_\delta y_\delta \right)^{n'}.$$

If μ can be made up by concatenating partitions $\delta^1, \dots, \delta^{n'}$ of $\mathcal{P}(f)$, then the product $y_{\delta^1} \cdots y_{\delta^{n'}}$ occurs in the above power with a non-zero coefficient whose sign is the same as the sign of ζ_μ (the sign depends on the parity of $n + \text{length}(\delta^1) + \cdots + \text{length}(\delta^{n'}) = n + \text{length}(\mu) \cdots$).

This result says that the regular germ in the centralizer always occurs with a nonzero coefficient for those μ 's which satisfy (3.1).

(ii) We write $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (f\lambda_1, \dots, f\lambda_k)$, so

$$\begin{aligned} \tau_f(y_{\lambda_1} \cdots y_{\lambda_k}) &= \prod_i \left(\sum_{\delta \in \mathcal{P}(\lambda_i)} \eta_\delta \tau_f(z_\delta) \right) \\ &= \left(\prod_i \eta_{(\lambda_i)} \right) z_{f\lambda_1} \cdots z_{f\lambda_k} + \text{lower terms} \\ &= \left(\prod_i \eta_{(\lambda_i)} \right) \left(\prod_i \zeta_{(f\lambda_i)} \right) y_{f\lambda_1} \cdots y_{f\lambda_k} + \text{lower terms} \end{aligned}$$

We see that the coefficient of $y_\mu = y_{f\lambda_1} \cdots y_{f\lambda_k}$ is nonzero.

(iii) The proof of (iii) is almost identical, except that it considers the coefficient of the lowest order term. $\square$

The results of this proposition can also be proved using orbital integrals.

The smallest n so that anomalies exist for $\text{GL}(n)$ is $n = 12$. With $f = 6$ and $n' = 2$, there are six anomalies of the form $(\mu, (2))$, with

$$\begin{aligned} \mu &= (5421), \quad (5331), \quad (53211) \\ &\quad (44211), \quad (4332), \quad (43221). \end{aligned}$$

There are no other anomalies for $\text{GL}(12)$.

We now describe a technique for constructing a certain type of anomalies. With $n = fn'$ as before, we construct anomalies of the form $(\mu, (n'))$, in the case where f is not a power of a prime.

Consider μ of the form

$$(3.4) \quad \mu = ((n' - 1)f - 1, 1, m_3, \dots, m_r).$$

Proposition 3.5. Suppose f is not a prime power. With μ as in (3.4), suppose $r \leq f$ and $m_i \leq f - 2$ for $i = 3, \dots, r$. If

$$\frac{(r-1)(r-2)}{f}$$

equals the number of times 1 occurs in μ , then $(\mu, (n'))$ is an anomaly.

Conversely, given f not a prime power and n' , such a μ always exists.

Proof.

$$\begin{aligned} \tau_f(y_{n'}) &= \tau_f \left(\sum_{\lambda \in \mathcal{P}(n')} \eta_\lambda z_\lambda \right) = \sum_{\lambda \in \mathcal{P}(n')} \eta_\lambda \tau_f(z_\lambda) \\ &= \frac{(-1)^{n'-1}}{n'} z_n + \frac{(-1)^{n'}}{n'-1} z_{n-f} z_f + \cdots. \end{aligned}$$

With μ as above, y_μ can only occur in these first two terms. Its coefficient is

$$\frac{(-1)^{n'-1}}{n'} \zeta_\mu + \frac{(-1)^{n'}}{n'-1} \zeta_{(n-f-1, 1)} \zeta_{(m_3, \dots, m_r)}.$$

Using the fact that

$$|\text{St}(\mu)| = |\text{St}(\mu_3, \dots, \mu_r)| \cdot (\# \text{ of } 1\text{'s in } \mu),$$

we find that the coefficient of y_μ in $\tau_f(y_{n'})$ is

$$\frac{(-1)^{n+r+n'-1} f}{|\text{St}(m_3, \dots, m_r)|} \left(\frac{(r-1)(r-2)}{(\# \text{ of } 1\text{'s in } \mu)} - f \right).$$

The first statement of the proposition follows.

Using the Chinese Remainder Theorem it is easy to show that any non-prime-power f can be written in the form $f = (r-1)(r-2)/k$, for some r with $4 \leq r \leq f$ and $1 \leq k \leq r-1$. The last statement of the proposition follows from this. $\square$

Proposition 3.6. *If $\lambda = (a_1, \dots, a_s) \in \mathcal{P}(n')$, with $3 \leq a_1 \leq \dots \leq a_s$, consider*

$$\mu = ((a_1 - 1)f - 1, 1; m_3, \dots, m_r; a_2 f, \dots, a_s f).$$

Suppose $r \leq f$, f is not a prime power. Then (μ, λ) is an anomaly if $(r-1)(r-2)/f$ equals the number of times 1 occurs in μ .

Proof. The proof is essentially the same as that of the previous proposition. $\square$

It is not difficult to obtain a similar result if $2 = a_1 \leq a_2 \leq \dots \leq a_s$; in this case the condition is that $2 \cdot ((r-1)(r-2)/f)$ equals the number of times 1 occurs in μ .

4. NONEXISTENCE OF ANOMALIES AT THE IDENTITY WHEN f IS A PRIME POWER

The examples of the last section were constructed under the assumption that f is not a power of a prime number. In this section we show that if f is a prime power then there are no anomalies at the identity.

Theorem 4.1. *Suppose $f = \ell^k$, where ℓ is a prime. If $n = fn'$, then $(\mu, (n'))$ is never an anomaly for any $\mu = (m_1, \dots, m_r) \in \mathcal{P}(n)$.*

Proof. The idea of the proof is to write the coefficient of y_μ in $\tau_f(y_{(n')})$ as the sum of the coefficients of y_μ in $\eta_\lambda z_{\tau_f(\lambda)}$, as λ ranges over $\mathcal{P}(n')$. We will show that the coefficient of y_μ in $\eta_{(n')} z_{(n)}$, which equals ζ_μ/n' , has strictly lower ℓ -order than all the other coefficients; this shows that the sum is nonzero.

The coefficient of y_μ in $\eta_\mu z_{\tau_f(\lambda)}$ is a sum of terms which arise in the following way. If $\lambda = (\lambda_1, \dots, \lambda_s)$, for each $i = 1, \dots, s$ let $\sigma^i \in \mathcal{P}(\lambda_i f)$ in such a way that the partition of n obtained by “combining” all the σ^i ’s is μ . The coefficient of y_μ arising from such a family of partitions is, up to a sign,

$$(4.2) \quad \frac{1}{(\prod_i \lambda_i)} |\text{St}(\lambda)| \zeta_{\sigma^1} \cdot \zeta_{\sigma^2} \cdots \zeta_{\sigma^s},$$

where ζ_{σ^i} is the coefficient of y_{σ^i} in $z_{\lambda_i f}$ (cf. (2.2)).

It will be sufficient to show that, for $\lambda \neq (n')$, the ℓ -order of (4.2) is strictly greater than the ℓ -order of ζ_μ/n' .

Let r_i be the length of σ^i , $i = 1, \dots, s$. So, up to sign, (4.2) equals

$$(4.3) \quad \frac{1}{(\prod_i \lambda_i) |\text{St}(\lambda)|} \prod_i \frac{\lambda_i \ell^k (r_i - 1)!}{|\text{St}(\sigma^i)|} = \frac{\ell^{ks}}{|\text{St}(\lambda)|} \prod_i \frac{(r_i - 1)!}{|\text{St}(\sigma^i)|}.$$

We want to compare the ℓ -order of this with the ℓ -order of

$$(4.4) \quad \frac{\ell^k (r - 1)!}{|\text{St}(\mu)|}$$

Applying Lemma 4.6 to $r = r_1 + \dots + r_s$, we see that $\text{ord}_\ell(r - 1)! \leq \left(\sum_{i=1}^s \text{ord}_\ell(r_i - 1)! \right) + (s - 1)(k - 1)$, so the ℓ -order of (4.4) is less than or equal to

$$k + \left(\sum_{i=1}^s \text{ord}_\ell(r_i - 1)! \right) + (s - 1)(k - 1) - \text{ord}_\ell |\text{St}(\mu)|.$$

Since $\text{St}(\sigma^1) \times \dots \times \text{St}(\sigma^s)$ is a subgroup of $\text{St}(\mu)$, we see that $\prod_{i=1}^s |\text{St}(\sigma^i)|$ divides $|\text{St}(\mu)|$, so the ℓ -order of (4.4) is less than or equal to

$$\left(\sum_{i=1}^s \text{ord}_\ell(r_i - 1)! \right) + (s - 1)(k - 1) + k - \prod_{i=1}^s \text{ord}_\ell |\text{St}(\sigma^i)|.$$

This will be strictly less than the ℓ -order of the right-hand side of (4.3) provided we can show that $\text{ord}_\ell |\text{St}(\lambda)| < s - 1$.

Since $|\text{St}(\lambda)| \mid (s!)$ and $\text{ord}_\ell(s!) \leq (s - 1)/(\ell - 1)$ (cf. Lemma 4.5), the result follows if $\ell \geq 3$. If $\ell = 2$, the inequality is not necessarily strict; we only know $\text{ord}_\ell |\text{St}(\lambda)| \leq s - 1$.

Suppose $\ell = 2$. If s is not a power of 2, Lemma 4.5 tells us that $\text{ord}_\ell(s!) < s - 1$ as needed. Suppose $s = 2^\alpha$ and $\text{ord}_\ell |\text{St}(\lambda)| = s - 1$. Now $\text{ord}_2(2^\alpha)! = 2^\alpha - 1 = s - 1$.

Suppose the λ_i 's are not all equal, so $|\text{St}(\lambda)| < s! = (2^\alpha)!$. Since $|\text{St}(\lambda)|$ is a product of factorials it is easily seen that $\text{ord}_2 |\text{St}(\lambda)| < \text{ord}_2(s!)$, a contradiction.

Finally suppose $\lambda_1 = \lambda_2 = \dots = \lambda_s$. If the σ^i 's are not all the same, then y_μ occurs in $\eta_\lambda z_{\tau_f(\lambda)} = \eta_\lambda (z_{\lambda_1 f})^s$ as a sum of cross-terms, so the 2-order of its coefficient is strictly larger than the 2-order of ζ_μ/n' .

The only remaining case is the one in which $\sigma^1 = \sigma^2 = \dots = \sigma^s$. Here it is necessary to examine the orders of the stabilizers more closely. The 2-order of $|\text{St}(\mu)|$ is much greater than $\text{ord}_2\left(\prod_i |\text{St}(\sigma^i)|\right)$, from which the result follows. $\square$

Lemma 4.5. *If ℓ is a prime, then*

$$\text{ord}_\ell(s!) \leq \frac{s-1}{\ell-1},$$

and the inequality is strict unless s is a power of ℓ .

Proof. Observe that if $\ell^{k-1} < s \leq \ell^k$, then

$$\text{ord}_\ell(s!) = \left\lfloor \frac{s}{\ell} \right\rfloor + \left\lfloor \frac{s}{\ell^2} \right\rfloor + \dots + \left\lfloor \frac{s}{\ell^k} \right\rfloor \leq \frac{s}{\ell} + \dots + \frac{s}{\ell^k} = s \frac{1 - \ell^{-k}}{\ell - 1}.$$

If $s = \ell^k$ we have equality and the desired result. Otherwise

$$\text{ord}_\ell(s!) = \left\lfloor \frac{s}{\ell} \right\rfloor + \dots + \left\lfloor \frac{s}{\ell^{k-1}} \right\rfloor \leq \frac{s}{\ell} + \dots + \frac{s}{\ell^{k-1}} = s \frac{1 - \ell^{1-k}}{\ell - 1} < \frac{s-1}{\ell-1}. \quad \square$$

Lemma 4.6. *Suppose $r = \sum_{i=1}^s r_i$ with positive integers r_i . Let ℓ be a prime and suppose $r \leq \ell^k$. Then*

$$\text{ord}_\ell((r-1)!) \leq \left(\sum_{i=1}^s \text{ord}_\ell((r_i-1)!) \right) + (s-1)(k-1).$$

Proof. The proof follows by induction from the case where $s = 2$. So suppose $r = a + b$. Let $m = \text{ord}_\ell b$. Then b/ℓ^j is an integer for $0 \leq j \leq m$, and

$$\left\lfloor \frac{a-1}{\ell^j} + \frac{b}{\ell^j} \right\rfloor = \left\lfloor \frac{a-1}{\ell^j} \right\rfloor + \frac{b}{\ell^j} \quad \text{if } j \leq m.$$

So

$$\begin{aligned}
 \text{ord}_\ell((r-1)!) &= \left\lfloor \frac{a-1}{\ell} + \frac{b}{\ell} \right\rfloor + \cdots + \left\lfloor \frac{a-1}{\ell^{k-1}} + \frac{b}{\ell^{k-1}} \right\rfloor \\
 &= \left\lfloor \frac{a-1}{\ell} \right\rfloor + \frac{b}{\ell} + \cdots + \left\lfloor \frac{a-1}{\ell^m} \right\rfloor + \frac{b}{\ell^m} + \left\lfloor \frac{a-1}{\ell^{m+1}} + \frac{b}{\ell^{m+1}} \right\rfloor + \cdots \\
 &\quad + \left\lfloor \frac{a-1}{\ell^{k-1}} + \frac{b}{\ell^{k-1}} \right\rfloor \\
 &\leq \left\lfloor \frac{a-1}{\ell} \right\rfloor + \frac{b}{\ell} + \cdots + \left\lfloor \frac{a-1}{\ell^m} \right\rfloor + \frac{b}{\ell^m} + \left\lfloor \frac{a-1}{\ell^{m+1}} \right\rfloor + \left\lfloor \frac{b}{\ell^{m+1}} \right\rfloor + \cdots \\
 &\quad + \left\lfloor \frac{a-1}{\ell^{k-1}} \right\rfloor + \left\lfloor \frac{b}{\ell^{k-1}} \right\rfloor + (k-1-m) \\
 &= \text{ord}_\ell((a-1)!) + \text{ord}_\ell(b) + (k-1-m) \\
 &= \text{ord}_\ell((a-1)!) + \text{ord}_\ell((b-1)!) + \text{ord}_\ell(b) + (k-1-m) \\
 &= \text{ord}_\ell((a-1)!) + \text{ord}_\ell((b-1)!) + k-1. \quad \square
 \end{aligned}$$

5. EXAMPLES

In this last section we list some examples of anomalies. These include some which can be constructed by the technique of Proposition 3.5, but there are many others and they do not occur in any obvious pattern.

First we consider anomalies of the form $(\mu, (2))$; here $n = 2f$ and $n' = 2$. We list all such anomalies for $n \leq 24$.

For $f = 6$ there are 6 anomalies $(\mu, (2))$, with $\mu = (5421), (53^21), (5321^2), (4^221^2), (43^22), (432^21)$.

For $f = 10$ there are 48 anomalies:

$$\begin{aligned}
 \mu = & (43^32^31), (43^42^2), (4^232^31^3), (4^32^21^4), (4^332^21), (4^33^22), \\
 & (4^421^2), (53^22^31^3), (5432^21^4), (543^221^3), (543^22^21), \\
 & (543^32), (54^2321^2), (54^321), (5^23^221^2), (5^2421^4), (5^242^21^2) \\
 & (5^321^3), (63^22^21^4), (63^22^4), (6432^31), (64^221^4), \\
 & (652^41), (6532^3), (653^21^3), (6^22^31^2), (6^2321^3), \\
 & (6^2421^2), (732^41^2), (73^31^4), (7432^21^2), (752^31^2), \\
 & (75321^3), (753^21^2), (762^31), (7631^4), (7^231^3), \\
 & (832^31^3), (8432^21), (84^21^4), (8531^4), (8541^3), \\
 & (8631^3), (93^221^3), (93^31^2), (942^21^3), (94321^2), \\
 & (952^21^2)
 \end{aligned}$$

For $f = 12$ there are 110 anomalies $(\mu, (2))$, with $\mu =$

$(43^2 2^6 1^2), (4^2 3^2 2^4 1^2), (5432^5 1^2), (543^5), (54^2 32^2 1^4),$
 $(54^2 32^3 1^2), (5^2 4321^5), (5^2 43^2 21^2), (5^2 4^2 2^2 1^2), (5^2 4^2 321),$
 $(5^3 21^7), (5^3 2^2 1^5), (632^7 1), (6432^5 1), (64^2 321^5),$
 $(64^2 3^2 2^2), (65421^7), (6542^2 1^5), (65432^2 1^2), (6543^2 21),$
 $(654^2 2^2 1), (65^2 3^2 1^2), (6^2 321^7), (6^2 4^3), (6^2 543),$
 $(6^2 5^2 2), (73^4 2^2 1), (73^5 2), (74^2 21^7), (74^2 2^2 1^5),$
 $(75321^7), (753^3 21), (754^3), (763^2 2^2 1), (76531^3),$
 $(765321), (7653^2), (76542), (76^2 32), (76^2 41),$
 $(7^2 31^7), (7^2 321^5), (83^4 21^2), (83^5 1), (84321^7),$
 $(843^2 2^3), (843^3 21), (84^2 31^5), (8542^3 1), (854321^2),$
 $(85432^2), (85^2 42), (8631^7), (86321^5), (863^2 21^2),$
 $(8642^2 1^2), (86431^3), (864321), (8643^2), (86541),$
 $(86^2 2^2), (86^2 31), (8732^2 1^2), (87421^3), (8742^2 1),$
 $(87432), (8752^2), (87531), (93^2 21^7), (943^2 21^3),$
 $(94^3 3), (9531^7), (95321^5), (9532^3 1), (954321),$
 $(95^2 32), (963^2 1^3), (96432), (96531), (96^2 21),$
 $(97431), (97521), (9832^2), (98421), (10, 431^7),$
 $(10, 4^2 2^2 1^2), (10, 4^3 2), (10, 532^2 1^2), (10, 5421^3), (10, 5432),$
 $(10, 5^2 21^2), (10, 6321^3), (10, 63^2 2), (10, 6421^2), (10, 6521),$
 $(10, 6^2 1^2), (10, 7321^2), (10, 7421), (10, 751^2), (10, 8321),$
 $(10, 841^2), (10, 931^2), (11, 32^2 1^6), (11, 4^3 1), (11, 5431),$
 $(11, 5^2 21), (11, 63^2 1), (11, 6421), (11, 7321), (11, 82^2 1).$

For $f = 14$ there are 136 anomalies of type $(\mu, (2))$ and for $f = 15$ there are 329.

These and the ones listed above are all the anomalies of this form for $n < 36$, i.e. $f < 18$.

We also list anomalies of the form $(\mu, (21))$; here $n = 3f$ and $n' = 3$. What follows is a complete list of such anomalies for $f \leq 11$.

For $f = 6$, there are 5 anomalies, with

$$\mu = (32^7 1), (3^3 21^7), (43^2 2^3 1^2), (4^2 31^7), (532^2 1^6).$$

For $f = 8$, there are 2 anomalies, with $\mu = (753^3 21), (76531^3)$.

For $f = 9$, there are 18 anomalies, with

$$\begin{aligned}\mu = & (543^6), (5^2 4^2 3^3), (654^2 32^2 1), (6^2 4^2 3^2 1), \\ & (6^2 53^2 2^2), (73^6 2), (75^2 4321), (76532^2 1^2), \\ & (76^2 3^2 1^2), (7^2 3^3 2^2), (7^2 4^2 1^5), (7^2 5421^2), \\ & (83^6 1), (85^2 42^2 1), (865421^2), (87432^2 1), \\ & (874^2 21^2), (8^2 3^3 1^2).\end{aligned}$$

For $f = 10$, there are 39 anomalies, with

$$\begin{aligned}\mu = & (4^2 3^2 21^{14}), (4^2 3^3 1^{13}), (53^3 21^{14}), (54^3 1^{13}), \\ & (5^5 1^5), (643^2 2^5 1^4), (654^2 3^3 2), (65^2 4^2 2^3), \\ & (6^2 41^{14}), (6^2 54^2 32), (6^2 5^2 431), (6^3 432^2 1), \\ & (7541^{14}), (75^2 3^3 1^4), (764^2 32^3), (76^2 4^2 21), (7^2 543^2 1), \\ & (7^2 5^2 321), (7^2 63^2 2^2), (7^2 64321), (854^2 32^3), (85^3 21^5), \\ & (86^2 431^3), (86^2 4^2 1^2), (874^2 2^3 1), (875^2 2^2 1), (8^2 43^2 2^2), \\ & (8^2 5421^3), (8^2 542^2 1), (8^2 632^2 1), (8^2 7321^2), (932^3 1^{12}), \\ & (964^2 2^3 1), (964^2 3^2 1), (965^2 31^2), (98432^2 1^2), (986321^2), \\ & (9^2 4^2 21^2), (9^2 5321^2).\end{aligned}$$

For $f = 11$, there are 16 anomalies, with

$$\begin{aligned}\mu = & (54^2 3^6 2), (654^2 3^4 1^2), (75^2 3^4 21^2), (7^2 54^2 3^2), \\ & (865^2 431^2), (8^2 5^2 3^2 1), (95^2 4^2 21^4), (96^2 5^2 1^2), \\ & (974^2 3^2 21), (975432^2 1), (98532^4), (985^2 1^6), \\ & (9^2 4^2 32^2), (10, 83^4 21), (10, 8741^4), (10^2 72^2 1^2).\end{aligned}$$

We observe that in contrast to anomalies "at the identity", i.e. with $\lambda = (n')$, anomalies with $\lambda = (2, 1)$ can occur with f a prime power, or even a prime number.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 1A1, CANADA