NORM OF SCHUR MULTIPLICATION FOR SCHATTEN NORM

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ABSTRACT. Let M_n denote the algebra of all $n \times n$ complex matrices and $\|\cdot\|_p$ be the Schatten's p -norm on M_n . For each $A \in M_n$, a linear operator S_A on M_n is defined by $S_A(X) := A \circ X$ for all $X \in M_n$, where $\circ$ denotes the Schur product and $\|S_A\|_{p,q}$ is defined as the operator norm from $(M_n, \|\cdot\|_q)$ to $(M_n, \|\cdot\|_p)$ for $p, q \geq 1$.

Given an $A \in M_n$, suppose $0 < \lambda < 1$, and p, p_1, p_2, q, q_1 , and q_2 are not smaller than 1, and

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}$$

are satisfied. Then we will show that $\|S_A\|_{p,q} \leq \|S_A\|_{p_1,q_1}^\lambda \cdot \|S_A\|_{p_2,q_2}^{1-\lambda}$.

1. INTRODUCTION AND NOTATIONS

Let M_n be the algebra of all $n \times n$ complex matrices. The *Schur product* (also called *Hadamard product*) $A \circ B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in M_n is defined by

$$(1) \quad A \circ B := (a_{ij} \cdot b_{ij}),$$

and for given $A \in M_n$, the *Schur multiplication operator* S_A on M_n is defined by

$$(2) \quad S_A(X) := A \circ X \quad \text{for all } X \in M_n.$$

We denote the singular values of $A \in M_n$, arranged in nonincreasing order, by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$; that is, $s_i(A)$ is the i th largest (with multiplicities counted) eigenvalue of $(A^*A)^{1/2}$. For $A \in M_n$ and $p \geq 1$, the *Schatten's p -norm* $\|A\|_p$ of A is defined by

$$(3) \quad \|A\|_p := \left(\sum_{i=1}^n s_i^p(A) \right)^{1/p}.$$

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In particular, when $p = 1, 2$ and ∞ , norm $\|\cdot\|_p$ is called *trace norm*, *Frobenius norm*, and *spectral norm*, respectively.

We denote the operator norm of S_A from $(\mathbb{M}_n, \|\cdot\|_p)$ to $(\mathbb{M}_n, \|\cdot\|_q)$ by $\|S_A\|_{p,q}$, i.e.,

$$(4) \quad \|S_A\|_{p,q} = \sup_{B \in \mathbb{M}_n} \|S_A(B)\|_q / \|B\|_p,$$

and we write $\|S_A\|_p \equiv \|S_A\|_{p,p}$.

A norm $\|\cdot\|$ is called *unitary invariant* if $\|A\| = \|UAV\|$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. For arbitrary $p \geq 1$, $\|\cdot\|_p$ is an example of a unitarily invariant norm.

Given a norm $\|\cdot\|$ on $\mathbb{M}_n$, under the coupling $\langle X, Y \rangle \equiv \text{tr}(XY^*)$ for $X, Y \in \mathbb{M}_n$ the dual norm $\|\cdot\|^D$ (on $\mathbb{M}_n$) of $\|\cdot\|$ is defined by

$$(5) \quad \|A\|^D := \sup_{B \in \mathbb{M}_n} |\text{tr}(AB^*)| / \|B\|.$$

Then it is known (see [2, pp. 130–135]) that $\|\cdot\|_q$ becomes the dual norm of $\|\cdot\|_p$ for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Since the dual map of S_A on the Banach space $(\mathbb{M}_n, \|\cdot\|_q)$ is given as $S_{\bar{A}}$ on $(\mathbb{M}_n, \|\cdot\|_q)$ where $\bar{A}$ is the complex conjugation of A , it is easy to show $\|S_A\|_p = \|S_{\bar{A}}\|_q$ if $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We shall obtain some convex property of the function $p \rightarrow \|S_A\|_p$ ($p \geq 1$) for given $A \in \mathbb{M}_n$.

In the sequel, for a row vector $\mathbf{x} = (\xi_1, \xi_2, \dots, \xi_n)$ of n -tuple of complex numbers, ${}^t\mathbf{x}$ denotes the transpose of $\mathbf{x}$, that is, ${}^t\mathbf{x}$ is a column vector such that

$${}^t\mathbf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}.$$

2. RESULTS AND REMARKS

First of all, we may mention the known results of norm S_A .

$$(6) \quad \|A \circ B\|_\infty \leq \|A\|_\infty \cdot \|B\|_\infty \quad \text{for all } A, B \in \mathbb{M}_n. \quad (\text{Schur [7]})$$

This Schur's result implies

$$(7) \quad \|S_A\|_\infty \leq \|A\|_\infty \quad (A \in \mathbb{M}_n).$$

We denote the Euclidean norms of the columns of A , arranged in nonincreasing order, by $c_1(A) \geq c_2(A) \geq \dots \geq c_n(A)$ and the Euclidean norms of the rows of A by $r_1(A) \geq r_2(A) \geq \dots \geq r_n(A)$. For $A \in \mathbb{M}_n$ and $\alpha \in (0, 1)$, we write $t_i(A, \alpha) = \{p_i(A, \alpha) \cdot q_i(A, 1-\alpha)\}^{1/2}$ where $p_i(A, \alpha)$ is the i th largest main diagonal entry of $(AA^*)^\alpha$ and $q_i(A, 1-\alpha)$ is the i th largest main diagonal entry of $(A^*A)^{1-\alpha}$.

$$(8) \quad \|S_A\|_\infty \leq \min\{r_1(A), c_1(A)\} \quad \text{for all } A \in \mathbb{M}_n. \quad (\text{Ong [5]})$$

$$(9) \quad \|S_A\|_\infty \leq t_1(A, \alpha) \quad \text{for all } A \in \mathbb{M}_n \text{ and } \alpha \in (0, 1). \quad (\text{Walter [8]})$$

Next, in (10) and (11) we consider a unitarily invariant norm $\|\cdot\|_\mu$ and operator norm of S_A with respect to $\|\cdot\|_\mu$; that is,

$$(10) \quad \|S_A\|_\mu := \sup_{B \in \mathbb{M}_n} \|S_A(B)\|_\mu / \|B\|_\mu.$$

Let $\|\cdot\|_\mu$ be a unitarily invariant norm on $\mathbb{M}_n$. Then

$$(11) \quad \|A \circ B\|_\mu \leq \|A\|_\infty \cdot \|B\|_\mu \quad \text{for all } A, B \in \mathbb{M}_n. \quad (\text{Okubo [4]})$$

We obtain from this result

$$(12) \quad \|S_A\|_\mu \leq \|A\|_\infty \quad \text{for all } A \in \mathbb{M}_n.$$

Let $\|\cdot\|_\mu$ be a unitarily invariant norm on $\mathbb{M}_n$. Then

$$(13) \quad \|S_A\|_\mu \leq \inf\{c_1(X) \cdot c_1(Y) \mid X, Y \in \mathbb{M}_{r,n}; X^*Y = A, r \geq 1\} \quad \text{for all } A \in \mathbb{M}_n. \\ (\text{Ando, Hern, and Johnson [1]})$$

Haagerup (cf. [6]) showed that the value of the right-hand side of (13) is equal to $\|S_A\|_\infty$; hence we have

$$(14) \quad \|S_A\|_\mu \leq \|S_A\|_\infty \quad \text{for all } A \in \mathbb{M}_n.$$

Proposition 1. *If $p \geq 1$, then*

$$(15) \quad \|S_A\|_2 \leq \|S_A\|_p \quad \text{for all } A \in \mathbb{M}_n.$$

Proof. Let $A = (a_{ij})$. It is easy to show that

$$(16) \quad \|S_A\|_2 = \max_{1 \leq i, j \leq n} |a_{ij}|.$$

(This is mentioned by Ando, Horn, and Johnson in [1].) Let

$$e_i = {}^t(0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0) \quad \text{for } i = 1, 2, \dots, n.$$

Since $S_A(E_{ij}) = a_{ij}E_{ij}$ where $E_{ij} = e_i \otimes e_j^*$, a_{ij} is an eigenvalue of S_A for all $1 \leq i, j \leq n$. It follows from this fact that

$$(17) \quad \|S_A\| \geq \|S_A(E_{ij})\| / \|E_{ij}\| = |a_{ij}|$$

for any norm $\|\cdot\|$ on $\mathbb{M}_n$ and all i, j of $1 \leq i, j \leq n$; hence we have, from (17), $\|S_A\|_2 \leq \|S_A\|_p$ for $p \geq 1$. $\square$

Remark. It is known that if $A \geq 0$ (nonnegative definite), then

$$(18) \quad \|S_A\|_\infty = \max_{1 \leq i \leq n} |a_{ii}| \quad \text{where } A = (a_{ij}).$$

Therefore, as the proof of Proposition 1, we obtain that if $A \geq 0$ then $\|S_A\|_\mu = \max_{1 \leq i \leq n} |a_{ii}|$.

As mentioned before we have $\|S_A\|_p = \|S_A\|_q$ for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $\|S_A\|_1 = \|S_A\|_\infty$, and from Proposition 1 it follows that

$$(19) \quad \min_{1 \leq p \leq \infty} \|S_A\|_p = \|S_A\|_2.$$

We have interest in the behavior of the function $[1, \infty] \ni p \rightarrow \|S_A\|_p$ for given $A \in \mathbb{M}_n$.

To show the main result in this section we need the following lemma obtained by M. Riesz (cf. [3, pp. 214–215]).

Let $A \in \mathbb{M}_n$ and $1 \leq s, t \leq \infty$ be given. Then we define $\|A\|_{s,t}$ by

$$(20) \quad \|A\|_{s,t} := \sup_{\xi \in C^n} \|A\xi\|_t / \|\xi\|_s,$$

where $\|\xi\|_s = (\sum_{i=1}^n |\xi_i|^s)^{1/s}$ for $\xi = {}^t(\xi_1, \xi_2, \dots, \xi_n)$.

Lemma 2 [M. Riesz]. Let $A \in \mathbb{M}_n$ and $\lambda \in (0, 1)$ be given. If $s, s_1, s_2, t, t_1, t_2 \geq 1$ and

$$\frac{1}{s} = \frac{\lambda}{s_1} + \frac{1-\lambda}{s_2}, \quad \frac{1}{t} = \frac{\lambda}{t_1} + \frac{1-\lambda}{t_2},$$

then

$$(21) \quad \|A\|_{s,t} \leq \|A\|_{s_1,t_1}^\lambda \cdot \|A\|_{s_2,t_2}^{1-\lambda}.$$

Theorem 3. Let $A \in \mathbb{M}_n$ and $\lambda \in (0, 1)$ be given. If $p, p_1, p_2, q, q_1, q_2 \geq 1$ and

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}, \quad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2},$$

then

$$(22) \quad \|S_A\|_{p,q} \leq \|S_A\|_{p_1,q_1}^\lambda \cdot \|S_A\|_{p_2,q_2}^{1-\lambda}.$$

Proof. By the singular value decomposition of a matrix, $\|B\|_s \leq 1$ if and only if B is written by

$$(23) \quad B = U \Sigma V$$

where U, V are unitary and $\Sigma = \text{diag}(s_1(A), s_2(A), \dots, s_n(A))$ such that $\sum_{i=1}^n s_i^s(A) \leq 1$. We may assume $t, t' \geq 1$ and $\frac{1}{t} + \frac{1}{t'} = 1$. By the definition of dual norm,

$$(24) \quad \|S_A(B)\|_t = \sup_{\|C\|_{t'} \leq 1} |\text{tr}(A \circ B) \cdot C^*|.$$

From $\|C\|_{t'} \leq 1$, C is written by $C = U' \Sigma' V'$ where U', V' are unitary and $\Sigma' = \text{diag}(s_1(C), s_2(C), \dots, s_n(C))$ such that $\sum_{i=1}^n s_i^{t'}(C) \leq 1$.

Therefore

$$(25) \quad \begin{aligned} \|S_A\|_{s,t} &= \sup_{\|B\|_s \leq 1} \|S_A(B)\|_t \\ &= \sup_{\|B\|_s \leq 1} \sup_{\|C\|_{t'} \leq 1} |\text{tr}(A \circ B) \cdot C^*| \\ &= \sup_{\|B\|_s \leq 1} \sup_{\|C\|_{t'} \leq 1} |\text{tr}(A \circ (U \cdot \text{diag}(s(B)) \cdot V)) \cdot V'^* \cdot \text{diag}(s(C)) \cdot U'^*|, \end{aligned}$$

where

$$s(B) = {}^t(s_1(B), s_2(B), \dots, s_n(B))$$

and

$$s(C) = {}^t(s_1(C), s_2(C), \dots, s_n(C)).$$

Since there exists the matrix $T = T_{U, V, U', V'}$ dependent on unitary $U, V, U',$ and V' such that

$$\langle T(s(B)), s(C) \rangle = \text{tr}((A \circ U \cdot \text{diag}(s(B)) \cdot V) \cdot V'^* \cdot \text{diag}(s(C)) \cdot U'^*),$$

$$(26) \quad \begin{aligned} \|S_A\|_{s,t} &= \sup_{U, V, U', V'} \sup_{\|\xi\|_s \leq 1, \|\eta\|_{t'} \leq 1} |\langle T_{U, V, U', V'} \xi, \eta \rangle| \\ &= \sup_{U, V, U', V'} \|T_{U, V, U', V'}\|_{s,t}. \end{aligned}$$

Hence, according to Lemma 2 we have

$$\begin{aligned} \|S_A\|_{s,t} &\leq \sup_{U, V, U', V'} \|T_{U, V, U', V'}\|_{s_1, t_1}^\lambda \cdot \|T_{U, V, U', V'}\|_{s_2, t_2}^{1-\lambda} \\ &\leq \|S_A\|_{s_1, t_1}^\lambda \cdot \|S_A\|_{s_2, t_2}^{1-\lambda}. \quad \square \end{aligned}$$

Corollary 4. Let $A \in \mathbb{M}_n$ be given. Then $\|S_A\|_p \leq \|S_A\|_q$ ($1 \leq q \leq p \leq 2$) and $\|S_A\|_p \leq \|S_A\|_q$ ($2 \leq p \leq q < \infty$).

Proof. From remark (19), $\min\{\|S_A\|_p \mid p \geq 1\} = \|S_A\|_2$, and when $p = q$, $p_1 = q_1$, and $p_2 = q_2$, it follows from Theorem 3 that

$$(27) \quad \|S_A\|_p \leq \|S_A\|_{p_1}^\lambda \cdot \|S_A\|_{p_2}^{1-\lambda} \quad (A \in \mathbb{M}_n)$$

for $p, p_1, p_2 \geq 1$ and $\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}$, that is $p \rightarrow \|S_A\|_p$ is a log-convex function of $\frac{1}{p}$ ($p \geq 1$).

Hence we can easily show Corollary 4. $\square$

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