AMENABILITY OF LOCALLY COMPACT GROUPS AND SUBSPACES OF $L^\infty(G)$

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ABSTRACT. If G is a locally compact group, let $\mathcal{A}$ be the set of all the functions which left average to a constant, i.e. the function $f \in L^\infty(G)$ such that there is a constant in the $\|\cdot\|_\infty$ -closed convex hull of $\{_x f : x \in G\}$. We prove in this paper that $\mathcal{A}$ is a subspace of $L^\infty(G)$ if and only if G is amenable as a discrete group. This answers a problem asked by Emerson, Rosenblatt and Yang, and Wong and Riazi. We also answer two other problems of Rosenblatt and Yang on whether the set $\mathcal{U}$ of functions in $L^\infty(G)$ admitting a unique left invariant mean value is a subspace of $L^\infty(G)$ when G is not amenable and whether there is a largest admissible subspace of $L^\infty(G)$ with a unique left invariant mean.

1. INTRODUCTION AND NOTATION

Let G be a locally compact group with a fixed Haar measure λ and let $L^p(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For each $f \in L^\infty(G)$ and $x \in G$, let $_x f \in L^\infty(G)$ be defined by $_x f(y) = f(xy)$ ($y \in G$). A subspace S of $L^\infty(G)$ is said to be *admissible* if it contains the constants and $_x f$ for each $f \in S$ and $x \in G$. Let $LIM(S)$ be the set of all the left invariant means on S , i.e. all $m \in S^*$ with $m \geq 0$, $m(1) = 1$, and $m(_x f) = m(f)$ ($x \in G$, $f \in S$). If $LIM(L^\infty(G)) \neq \phi$, we say that G is amenable. It is well known that, for any locally compact group G , G is amenable if G_d is amenable, where G_d is the group G with discrete topology. The converse is not always true. The purpose of this paper is to give a characterization of a locally compact group which is amenable as a discrete group and criteria for the amenability of a locally compact group and a discrete group in terms of the subspaces of $L^\infty(G)$. We also answer three of the problems raised by Rosenblatt and Yang in [7].

For $f \in L^\infty(G)$ and a constant c , we say that f left averages to c if $c \in \|\cdot\|_\infty$ -closed convex of $\{_x f : x \in G\}$. Let $\mathcal{A}$ denote the set of all functions

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which left average to some constant and $\mathcal{A}_0$ denote the set of functions which left average to 0. If $f \in L^\infty(G)$, let S_f be the smallest admissible subspace generated by f . $f \in L^\infty(G)$ is said to have a *unique left invariant mean value* if $LIM(S_f) \neq \emptyset$ and there is a constant c such that for each $m \in LIM(S_f)$, $m(f) = c$. See [7] for more details. The set of all functions with a unique left invariant mean value is denoted by $\mathcal{U}$ and let H be the linear span of $\{f - {}_x f : f \in L^\infty(G), x \in G\}$.

In §2, we use a theorem of Chou in [1] to prove that $\mathcal{A}_0$ is a subspace if and only if G_d is amenable. We also show that $\mathcal{A}_0$ is a subspace if and only if $\mathcal{A}$ is a subspace. This answers a problem asked by Emerson in [3, p. 187], Wong and Riazi in [10, p. 494] and Rosenblatt and Yang in [7].

In §3, we prove that $\mathcal{U}$ is not a subspace for any nonamenable group. This answers a problem asked by Rosenblatt and Yang in [7]. Consequently, we show that G is amenable if and only if $\mathcal{U}$ is a subspace for a discrete group. We also settle a problem asked by Rosenblatt and Yang in [7] on whether there is a largest admissible subspace with a unique left invariant mean. We show that there is such a subspace if and only if G is amenable.

2. LOCALLY COMPACT GROUPS WHICH ARE AMENABLE AS DISCRETE GROUPS

Let $\mathcal{D}$ be the maximal ideal space of $L^\infty(G)$ with the Gelfand topology. Then $\mathcal{D}$ is a compact Hausdorff space. The Gelfand transform $\wedge$ is an isometry of $L^\infty(G)$ onto $C(\mathcal{D})$, the algebra of real-valued continuous functions on $\mathcal{D}$ with the supremum norm. Note that if $\theta \in \mathcal{D}$, ${}_x \theta$ is defined by ${}_x \theta(f) = \theta({}_x f)$ for $f \in L^\infty(G)$ and $x \in G$.

For $\theta \in \mathcal{D}$, let $\rho_\theta : L^\infty(G) \rightarrow \ell^\infty(G)$ be defined by $(\rho_\theta f)(x) = \theta({}_x f)$, $x \in G$. Here $\ell^\infty(G) = L^\infty(G_d)$. Note that $\|\rho_\theta\| \leq 1$, $\rho_\theta \geq 0$, and $\rho_\theta({}_x f) = {}_x \rho_\theta(f)$ for $x \in G$ and $f \in L^\infty(G)$ (see Chou [1]).

In [1], Chou proved the following:

Theorem (Chou). *For a σ -compact locally compact group G , G_d is amenable if and only if for each $\theta \in \mathcal{D}$, the subspace $\{\rho_\theta(f) : f \in L^\infty(G)\}$ of $\ell^\infty(G)$ has a left invariant mean.*

Our proof of Theorem 2.3 depends on this theorem of Chou. We first establish some lemmas.

Lemma 2.1. *For any locally compact group G , the following statements are equivalent:*

- (i) $\mathcal{A}$ is a subspace;
- (ii) $\mathcal{A}_0$ is a subspace;
- (iii) $\mathcal{A}_0 = \overline{H}$.

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{A}$ is a subspace, let $f_1, f_2 \in \mathcal{A}_0 \subseteq \mathcal{A}$. There is a number c such that $f_1 + f_1$ left averages to c . By Theorem 1.5 of [7], G is amenable. Let $m \in LIM(L^\infty(G))$, then $m(f_1) = m(f_2) = 0$ since $f_1, f_2 \in \mathcal{A}_0$. Therefore $c = m(f_1 + f_2) = 0$, i.e. $f_1 + f_2 \in \mathcal{A}_0$.

(ii) $\Rightarrow$ (iii). For any $x \in G$ and $f \in L^\infty(G)$, since

$$\lim_n \frac{1}{n} \sum_{k=1}^n x^k (f - {}_x f) = 0$$

$f - {}_x f \in \mathcal{A}_0$. Hence if $\mathcal{A}_0$ is a subspace, then $H \subseteq \mathcal{A}_0$. It is easy to see that $\mathcal{A}_0$ is closed. So $\overline{H} \subseteq \mathcal{A}_0$. Let $f \notin \overline{H}$, then there is $m \in \text{LIM } (L^\infty(G))$ with $m(f) \neq 0$. So $f \notin \mathcal{A}_0$. (See [5, (B37) and p. 236]), i.e. $\mathcal{A}_0 \subseteq \overline{H}$.

(iii) $\Rightarrow$ (ii) is clear. To prove (ii) $\Rightarrow$ (i), let f_i left average to c_i . Then $f_i - c_i$ left averages to 0 ($i = 1, 2$). If $\mathcal{A}_0$ is a subspace, then $(f_1 - c_1) + (f_2 - c_2) \in \mathcal{A}_0$, i.e. $f_1 + f_2$ left averages to $c_1 + c_2$. $\square$

Theorem 2.2. For any locally compact group G and $\theta \in \mathcal{D}$, $S_\theta = \{\rho_\theta(f) : f \in L^\infty(G) \text{ is a simple function}\}$ is a subspace of $\ell^\infty(G)$ with the following properties:

- (i) $|F| \in S_\theta$ for any $F \in S_\theta$
- (ii) If $m \in S_\theta^*$ is left invariant, then there are nonnegative left invariant $m^+, m^- \in S_\theta^*$ such that $m = m^+ - m^-$ where $m^+ = \max(m, 0)$, $m^- = -\min(m, 0)$ (see [5] (B34)).

Proof. (i) If $f = \sum_{i=1}^n a_i 1_{E_i}$ is a measurable simple function, then $|\rho_\theta(f)| = \rho_\theta(|f|)$ i.e. $|\rho_\theta(f)| \in S_\theta$. Indeed, for any $x \in G$, since $\{x^{-1}E_i : i = 1, 2, \dots, n\}$ are pairwise disjoint, there is at most one i with $\theta({}_x 1_{E_i}) \neq 0$. Hence

$$\begin{aligned} |\rho_\theta(f)(x)| &= \left| \theta \left(\sum_{i=1}^n a_i {}_x 1_{E_i} \right) \right| = \left| \sum_{i=1}^n a_i \theta({}_x 1_{E_i}) \right| = \sum_{i=1}^n |a_i| \theta({}_x 1_{E_i}) \\ &= \theta \left(\sum_{i=1}^n |a_i| {}_x 1_{E_i} \right) = \theta(|f|)(x). \end{aligned}$$

(ii) Since $m \in S_\theta^*$ is bounded, there are nonnegative $m^+, m^- \in S_\theta^*$ such that $m = m^+ - m^-$, $m^+ = \max(m, 0)$ and $m^- = -\min(m, 0)$ by (i) and (B.37) of [5].

Let $F_0 \in S_\theta$ and $F_0 \geq 0$, since

$$m^+(F_0) = \sup\{m(F) : 0 \leq F \leq F_0, F \in S_\theta\}$$

if $x \in G$ and $0 \leq F \leq F_0$, then $0 \leq {}_x F \leq {}_x F_0$ and $m({}_x F) = m(F)$. Hence $m^+(F_0) = m^+({}_x F_0)$. Since for any $F \in S_\theta$, there are $F^+, F^- \in S_\theta$ such that $F^+ \geq 0$, $F^- \geq 0$ and $F = F^+ - F^-$, m^+ is left invariant. Similarly, we can prove that m^- is left invariant. $\square$

We are now ready to answer a question raised by Rosenblatt and Yang in [7] (Remark 3 after Theorem 1.5).

Theorem 2.3. For any locally compact group G , $\mathcal{A}$ is a subspace if and only if G_d is amenable.

Proof. If G_d is amenable, then $\mathcal{A}$ is a subspace (see [7, Theorem 1.1]). Let $\mathcal{A}$ be a subspace. By Lemma 2.1, $\mathcal{A}_0 = \overline{H}$. Since any countable subgroup of G is contained in a σ -compact, open and closed subgroup of G (see [6, Proposition 22.24]), it suffices to show that, for any σ -compact, open and closed subgroup G_0 of G , $(G_0)_d$ is amenable by (D) and (F) of [2, p. 516]. By Chou's theorem above, it suffices to show that the subspace $\{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}$ of $\ell^\infty(G_0)$ has a left invariant mean for each $\theta_0 \in \mathcal{D}_0$, where $\mathcal{D}_0$ is the maximal ideal space of $L^\infty(G_0)$.

Let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be the set of all the left cosets of G_0 in G . For each $f \in L^\infty(G_0)$, let $\overset{\circ}{f}$ be defined by $\overset{\circ}{f}(x_\alpha x) = f(x)$ for each $x \in G_0$, $\alpha \in \Lambda$. Since $G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0$, $\overset{\circ}{f}$ is a function on G .

Claim. $\overset{\circ}{f} \in L^\infty(G)$. Let $c \in \mathbb{C}$ and K be any compact subset of G . Since G_0 is open, there are only finite many $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $K \cap x_{\alpha_i} G_0 \neq \emptyset$ ($i = 1, 2, \dots, n$). Hence

$$\{x \in G : \overset{\circ}{f} > c\} \cap K = \bigcup_{i=1}^n K \cap x_{\alpha_i} G_0 \cap \{x : \overset{\circ}{f}(x) > c\}$$

is λ -measurable since f is measurable on G_0 . Therefore $\overset{\circ}{f} \in L^\infty(G)$ (see [5, (11.31) and (15.8)]).

Let $\theta_0 \in \mathcal{D}_0$ be given and $\theta \in \mathcal{D}$ be defined by $\theta(f) = \theta_0(f|G_0)$ for any $f \in L^\infty(G)$. We first show that there is a left invariant mean m_θ on S_θ where $S_\theta = \{\rho_\theta(f) : f \in L^\infty(G) \text{ is a simple function}\}$. Let

$$H_\theta = \text{span}\{x F - F : x \in G, F \in S_\theta\} \quad \text{and} \quad h_\theta \in H_\theta.$$

Then there is a $h \in H$ such that $h_\theta = \rho_\theta(h)$.

Claim. $\|1 - h_\theta\|_\infty \geq 1$. Suppose $\varepsilon > 0$ with $\|1 - h_\theta\|_\infty < 1 - \varepsilon$. Since $h \in H \subseteq \mathcal{A}_0$, there are $\lambda_i > 0$ and $x_i \in G$ ($i = 1, 2, \dots, n$) such that $\sum_{i=1}^n \lambda_i = 1$ and $\|\sum_{i=1}^n \lambda_{ix_i} h\|_\infty < \varepsilon/2$. Then

$$\left\| \sum_{i=1}^n \lambda_{ix_i} \rho_\theta(h) \right\|_\infty = \left\| \rho_\theta \left(\sum_{i=1}^n \lambda_{ix_i} h \right) \right\|_\infty \leq \|\rho_\theta\| \left\| \sum_{i=1}^n \lambda_{ix_i} h \right\|_\infty \leq \varepsilon/2.$$

It follows that

$$\|1 - h_\theta\|_\infty \geq \left\| \sum_{i=1}^n \lambda_{ix_i} (1 - h_\theta) \right\|_\infty = \left\| 1 - \sum_{i=1}^n \lambda_{ix_i} \rho_\theta(h) \right\|_\infty \geq 1 - \varepsilon/2$$

which is impossible. Hence $\|1 - h_\theta\|_\infty \geq 1$ for any $h_\theta \in H_\theta$. Now we can find $M \in S_\theta^*$ such that $M(1) = 1$ and $M(h_\theta) = 0$ for any $h_\theta \in H_\theta$, i.e. M is left

invariant. By Theorem 2.2, M^+ is a nonnegative left invariant functional on S_θ . Put $m_\theta = M^+/M^+(1)$, then m_θ is a left invariant mean on S_θ .

Let $S_{\theta_0} = \{\rho_{\theta_0}(f) : f \in L^\infty(G_0) \text{ is a simple function}\}$. By using m_θ , we can define a left invariant mean on S_{θ_0} as follows. For each $\rho_{\theta_0}(f) \in S_{\theta_0}$, define $m_{\theta_0}(\rho_{\theta_0}(f)) = m_\theta(\rho_\theta(\overset{\circ}{f}))$. Then m_{θ_0} is well defined. Indeed, for $f_1, f_2 \in L^\infty(G_0)$ and f_i is a simple function ($i = 1, 2$). Let $\rho_{\theta_0}(f_1) = \rho_{\theta_0}(f_2)$ i.e. $\rho_{\theta_0}(f) = 0$ where $f = f_1 - f_2$. Let $f = \sum_{i=1}^n a_i 1_{E_i}$. Note that for each $x \in G$, there is at most one i such that $\rho_{\theta_0}(1_{E_i})(x) \neq 0$. Hence we have $\rho_{\theta_0}(1_{E_i})(x) = 0$ for $i = 1, 2, \dots, n$ and $x \in G_0$. Let $1 \leq i \leq n$ be given,

$$\begin{aligned} \rho_\theta(\overset{\circ}{1}_{E_i})(x_{\alpha_0}x) &= \rho_\theta(x_{\alpha_0}x 1_{(\cup_{\alpha \in \Lambda} x_\alpha^{-1} x_{\alpha_0} E_i)}) = \theta(1_{(\cup_{\alpha \in \Lambda} x^{-1} x_{\alpha_0}^{-1} x_\alpha E_i)}) \\ &= \theta_0(1_{(\cup_{\alpha \in \Lambda} x^{-1} x_{\alpha_0}^{-1} x_\alpha E_i)} \mid G_0) = \theta_0(1_{x^{-1}E_i}) = 0 \end{aligned}$$

for any $x \in G_0$ and $\alpha_0 \in \Lambda$. Hence $\rho_\theta(\overset{\circ}{1}_{E_i}) = 0$ ($i = 1, 2, \dots, n$) i.e., $\rho_\theta(\overset{\circ}{f}) = 0$ and

$$\begin{aligned} m_{\theta_0}(\rho_{\theta_0}(f_1)) - m_{\theta_0}(\rho_{\theta_0}(f_2)) &= m_\theta(\rho_\theta(\overset{\circ}{f}_1)) - m_\theta(\rho_\theta(\overset{\circ}{f}_2)) \\ &= m_\theta(\rho_\theta(\overset{\circ}{f})) = 0 \end{aligned}$$

so $m_{\theta_0}(\rho_{\theta_0}(f_1)) = m_{\theta_0}(\rho_{\theta_0}(f_2))$. It is clear that m_{θ_0} is linear. To see that m_{θ_0} is left invariant, it suffices to show that for any $x \in G_0$ and measurable set E of G_0

$$(*) \quad {}_x\rho_\theta(\overset{\circ}{1}_E) = \rho_\theta(\overset{\circ}{({}_x1_E)}).$$

Indeed, (*) implies that

$$\begin{aligned} m_{\theta_0}({}_x\rho_{\theta_0}(1_E)) &= m_{\theta_0}(\rho_{\theta_0}({}_x1_E)) = m_\theta(\rho_\theta(\overset{\circ}{({}_x1_E)})) = m_\theta({}_x\rho_\theta(\overset{\circ}{1}_E)) \\ &= m_\theta(\rho_\theta(\overset{\circ}{1}_E)) = m_{\theta_0}(\rho_{\theta_0}(1_E)) \end{aligned}$$

for any $x \in G_0$ and measurable subset E of G_0 . Hence $m_{\theta_0}({}_x\rho_{\theta_0}(f)) = m_{\theta_0}(\rho_{\theta_0}(f))$ for any $x \in G_0$ and simple function $f \in L^\infty(G_0)$. To prove (*), note that

$$\begin{aligned} \rho_\theta(\overset{\circ}{({}_x1_E)}) &= \rho_\theta(1_{\cup_{\alpha \in \Lambda} x_\alpha x^{-1} E}) = \theta_0(1_{\cup_{\alpha \in \Lambda} x_\alpha x^{-1} E} \mid G_0) = \theta_0(1_{x^{-1}E}), \\ {}_x\rho_\theta(\overset{\circ}{1}_E) &= \rho_\theta({}_x1_{\cup_{\alpha \in \Lambda} x_\alpha E}) = \theta_0(1_{\cup_{\alpha \in \Lambda} x^{-1} x_\alpha E} \mid G_0) = \theta_0(1_{x^{-1}E}). \end{aligned}$$

Also m_{θ_0} is nonnegative. Indeed, let $f = \sum_{i=1}^n a_i 1_{E_i} \in L^\infty(G_0)$ be a simple function and $\rho_{\theta_0}(f) \geq 0$. Suppose that for each i_0 there exist $x \in G_0$ such

that $\theta_0(x1_{E_{i_0}}) \neq 0$ (otherwise take $a_{i_0} = 0$). Since $x^{-1}E_i \cap x^{-1}E_j = \emptyset$ ($i \neq j$), $\theta_0(x1_{E_j}) = 0$ if $j \neq i_0$. Hence

$$\rho_{\theta_0}(f)(x) = \sum_{i=1}^n \rho_{\theta_0}(a_{ix}1_{E_i}) = a_{i_0} \geq 0.$$

For each i , note that $\rho_{\theta_0}(\overset{\circ}{1}_{E_i}) = \rho_{\theta_0}(1_{\cup_{\alpha \in \Lambda} x_{\alpha} E_i}) \geq 0$, so $m_{\theta_0}(\rho_{\theta_0}(\overset{\circ}{1}_{E_i})) \geq 0$ ($i = 1, 2, \dots, n$). Therefore

$$m_{\theta_0}(\rho_{\theta_0}(f)) = \sum_{i=1}^n a_i m_{\theta_0}(\rho_{\theta_0}(1_{E_i})) = \sum_{i=1}^n a_i m_{\theta_0}(\rho_{\theta_0}(\overset{\circ}{1}_{E_i})) \geq 0.$$

It is clear that $m_{\theta_0}(1) = 1$. Hence m_{θ_0} is a left invariant mean on S_{θ_0} .

Since S_{θ_0} is dense in $\{\rho_{\theta_0}(f) : f \in L^{\infty}(G_0)\}$, we can extend m_{θ_0} to $\{\rho_{\theta_0}(f) : f \in L^{\infty}(G_0)\}$ such that m_{θ_0} is a bounded functional. It is easy to see that m_{θ_0} is nonnegative and left invariant, i.e. m_{θ_0} is a left invariant mean on $\{\rho_{\theta_0}(f) : f \in L^{\infty}(G_0)\}$. $\square$

Let X be a left invariant subspace of $L^{\infty}(G)$ and D denote the set of all finite convex combinations of Dirac measures. As in [10], we denote the set $\{f \in X : \inf\{\|\mu * f\|_{\infty} : \mu \in D\} = 0\}$ by $\mathcal{N}_1(X)$ (see [10, pp. 480, 491]). It is clear that $\mathcal{N}_1(X) = X \cap \mathcal{A}_0$. See [5, 20.9].

Corollary 2.4. $\mathcal{N}_1(L^{\infty}(G))$ is closed under addition if and only if G_d is amenable.

Proof. Since $\mathcal{N}_1(L^{\infty}(G)) = \mathcal{A}_0$, the Corollary is true by Lemma 2.1 and Theorem 2.3. $\square$

Remark 1. Wong and Riazi in [10, p. 493] proved that for any locally compact group G , G is amenable if and only if $\mathcal{N}_1(UCB(G))$ is closed under addition where $UCB(G)$ is the set of all the uniformly continuous functions on G . Since there are amenable locally compact groups which are not amenable as discrete groups, we have answered the problem raised by Wong and Raizi in [10, p. 494 (Remark 3)].

Remark 2. Let $P(G) = \{\varphi \in L^1(G) : \varphi \geq 0 \text{ and } \|\varphi\|_1 = 1\}$. Emerson proved in [3] that G is amenable if and only if $\{f \in L^{\infty}(G) : \inf\{\|\varphi * f\|_{\infty} : \varphi \in P\} = 0\}$ is closed under addition. This corollary provides an answer to his problem in replacing $P(G)$ by D [3, p. 187].

Corollary 2.5. (Granirer [4] and Rudin [9]). If G is not discrete and G_d is amenable, then

$$LIM(L^{\infty}(G)) \neq TLIM(L^{\infty}(G))$$

where $TLIM(L^{\infty}(G))$ is the set of all topological left invariant means on $L^{\infty}(G)$.

Proof. By Proposition 1.2 and the last proposition of [4], we can find an open dense subset V in G such that $m(1_V) < 1$ for all $m \in TLIM(L^{\infty}(G))$. By

Theorem 2.3, $\mathcal{A}_0 = \overline{H}$. We can see that $\|1_V - h\|_\infty \geq 1$ for any $h \in \overline{H}$. Indeed, if there is $h \in \overline{H}$ such that $\|1_V - h\|_\infty < 1 - \epsilon$ for some $\epsilon > 0$, then we can find $x_i \in G$ and $\lambda_i > 0$ ($i = 1, 2, \dots, n$) such that $\sum_{i=1}^n \lambda_i = 1$ and $\|\sum_{i=1}^n \lambda_{ix_i} h\|_\infty \leq \epsilon/2$. It follows that

$$\|1_V - h\|_\infty \geq \left\| \sum_{i=1}^n \lambda_{ix_i} (1_V - h) \right\|_\infty \geq \left\| \sum_{i=1}^n \lambda_{ix_i} 1_V \right\|_\infty - \left\| \sum_{i=1}^n \lambda_{ix_i} h \right\|_\infty \geq 1 - \epsilon/2$$

which is impossible. Hence there is $m \in LIM(L^\infty(G))$ such that $m(1_V) = 1$, i.e. $m \notin TLIM(L^\infty(G))$. $\square$

3. AMENABLE LOCALLY COMPACT GROUPS

For a locally compact group G , it is well known that if G_d is amenable, then $\mathcal{U} = \|\cdot\|_\infty$ -closed linear span $\{x f - f : x \in G, f \in L^\infty(G)\} \cup \mathbb{C}$ (see [7]). Rosenblatt and Yang in [7] proved that if G is discrete and contains F_2 , the free group on two generators, then $\mathcal{U}$ is not a subspace. They asked if $\mathcal{U}$ is a subspace when G is not amenable. Our Theorem 3.4 answers this problem negatively.

The existence and uniqueness of the left invariant mean on $L^\infty(G)$ have been discussed in many papers. It is natural to ask whether there exists a largest admissible subspace S_M of $L^\infty(G)$ with a unique left invariant mean (see [7, p. 5, Problem (d)]). It is proved in [7] that such space does not exist for any discrete group containing F_2 . Our Theorem 3.6 answers this problem completely.

We need the following lemma, probably known, for which we were unable to find a reference.

A set E of G is called a permanently positive subset (P.P.) if $\bigcap_{i=1}^n x_i E$ is not locally null for any $x_1, x_2, \dots, x_n \in G$.

Lemma 3.1. *Let E be a P.P. set in G . Then there exists $m \in LIM(S_{1_E})$ with $m(1_E) = 1$.*

Proof. Put $m(1) = m(x 1_E) = 1$ for any $x \in G$ and linearly extend m to S_{1_E} . Then m is well defined. Indeed, let $h = \alpha_0 + \sum_{i=1}^n \alpha_{ix_i} 1_E = 0$ ($x_i \in G$, $i = 1, 2, \dots, n$). Since $\bigcap_{i=1}^n x_i^{-1} E$ is not locally null and $h = \sum_{i=0}^n \alpha_i$ on $\bigcap_{i=1}^n x_i^{-1} E$, $\sum_{i=0}^n \alpha_i = 0$. Hence $m(h) = \sum_{i=0}^n \alpha_i = 0$. Similarly, if $h \geq 0$ and $h \in S_{1_E}$, then $m(h) \geq 0$. Since m is nonnegative and $m(1) = 1$, m is bounded. Therefore $m \in LIM(S_{1_E})$. $\square$

Lemma 3.2. *For any infinite locally compact group G , there is a subset A in G such that both A and A^c are P.P. sets, where $A^c = G \setminus A$.*

Proof. If G is discrete or G is σ -compact and nondiscrete, then there is a subset A in G such that both A and A^c are P.P. sets in G (see [7, p. 5] and [8, Proposition 3.4]). Let G be non- σ -compact and nondiscrete. We can find an open and closed σ -compact subgroup G_0 of G (see [6, Proposition 22.24]). Let

A_0 be a subset of G_0 such that both A_0 and $A_0^c = G_0 \sim A_0$ are P.P. sets in G_0 . Suppose $\{x_\alpha G_0 : \alpha \in \Lambda\}$ is all the left cosets of G_0 in G . Let $A = \bigcup_{\alpha \in \Lambda} x_\alpha A_0$. Since for any compact subset K of G , there are $\alpha_1, \alpha_2, \dots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup_{i=1}^n x_{\alpha_i} G_0$, $K \cap A = \bigcup_{i=1}^n x_{\alpha_i} A_0 \cap K$ is measurable. Hence A is measurable (see [5, (11.31) and (15.8)]). For any $g_i \in G$ ($i = 1, 2, \dots, n$), since $G = \bigcup_{\alpha \in \Lambda} G_0 x_\alpha^{-1}$, there is $\alpha_i \in \Lambda$ such that $g_i = y_i x_{\alpha_i}^{-1}$ for some $y_i \in G_0$ ($i = 1, 2, \dots, n$). Hence

$$\bigcap_{i=1}^n g_i A \supseteq \bigcap_{i=1}^n y_i x_{\alpha_i}^{-1} x_{\alpha_i} A_0 = \bigcap_{i=1}^n y_i A_0.$$

Since $\bigcap_{i=1}^n y_i A_0$ is a P.P. set in G_0 , $\bigcap_{i=1}^n g_i A$ is a P.P. set in G . Similarly, $A^c = \bigcup_{\alpha \in \Lambda} x_\alpha A_0^c$ is a P.P. set in G , where $A_0^c = G_0 \sim A_0$. $\square$

Lemma 3.3. *If $\mathcal{U}$ is a subspace, then $\mathcal{U} \supseteq H_s$ where*

$$H_s = \text{span}\{x f - f : x \in G, f \in L^\infty(G) \text{ is a simple function}\}.$$

Proof. It suffices to show that for each measurable subset E of G and $x \in G$, $x 1_E - 1_E \in \mathcal{U}$. Assume that G is infinite. By Lemma 3.2, we can find a subset A in G such that both A and A^c are P.P. sets. Put $E_A = (E \cap A) \cup A^c$, then E_A is a P.P. set. By Lemma 3.1, $LIM(S_{1_{E_A}}) \neq \phi$ and $LIM(S_{1_{A^c}}) \neq \phi$. Put $\xi_1 = x 1_{E_A} - 1_{E_A}$ and $\xi_2 = x 1_{A^c} - 1_{A^c}$, then $S_{\xi_1} \subseteq S_{1_{E_A}}$ and $S_{\xi_2} \subseteq S_{1_{A^c}}$. Hence $LIM(S_{\xi_i}) \neq \phi$ ($i = 1, 2$). Also $\xi_1, \xi_2 \in \mathcal{A}_0$, hence $m(\xi_i) = 0$ for all $m \in LIM(S_{\xi_i})$ ($i = 1, 2$), i.e. $\xi_1, \xi_2 \in \mathcal{U}$. Since $\mathcal{U}$ is a subspace and

$$\xi_1 - \xi_2 = (x 1_{E_A} - 1_{E_A}) - (x 1_{A^c} - 1_{A^c}) = x 1_{E \cap A} - 1_{E \cap A}.$$

$x 1_{E \cap A} - 1_{E \cap A} \in \mathcal{U}$. Similarly, $x 1_{E \cap A^c} - 1_{E \cap A^c} \in \mathcal{U}$. Note that

$$(x 1_{E \cap A} - 1_{E \cap A}) + (x 1_{E \cap A^c} - 1_{E \cap A^c}) = x 1_E - 1_E.$$

Hence $x 1_E - 1_E \in \mathcal{U}$. $\square$

The following answers question (b) in the Remark following Corollary 1.4 of [7].

Theorem 3.4. *If $\mathcal{U}$ is a subspace, then G is amenable.*

Proof. By Lemma 3.3, $\mathcal{U} \supseteq H_s$. If G is not amenable, $\overline{H_s} = \overline{H} = L^\infty(G)$. For each $f_0 \in L^\infty(G)$, there exist $f_n \in H_s$ ($n = 1, 2, \dots$) such that $f_n \rightarrow f_0$ in $\|\cdot\|_\infty$. Since $f_n \in H_s \subseteq \mathcal{U}$, $LIM(S_{f_n}) \neq \phi$. So

$$1 = m_{f_n}(1 - h_{f_n}) \leq \|1 - h_{f_n}\|_\infty \quad (n = 1, 2, \dots)$$

for $m_{f_n} \in LIM(S_{f_n})$ and $h_{f_n} \in H_{f_n}$, where

$$H_{f_n} = \text{span}\{x f_n - f_n : x \in G\} \quad (n = 0, 1, 2, \dots).$$

If $h_{f_0} = \sum_{i=1}^m \alpha_i(x_i f_0 - f_0) \in H_{f_0}$ for $\alpha_i \in \mathbb{C}$ and $x_i \in G$ ($i = 1, 2, \dots, m$), put $h_{f_n} = \sum_{i=1}^m \alpha_i(x_i f_n - f_n)$, then $h_{f_n} \in H_{f_n}$ ($n = 1, 2, \dots$) and $h_{f_n} \rightarrow h_{f_0}$ in

$\|\cdot\|_\infty$. Hence $\|1 - h_{f_n}\|_\infty \rightarrow \|1 - h_{f_0}\|_\infty$ i.e. $\|1 - h_{f_0}\|_\infty \geq 1$ for any $h_{f_0} \in H_{f_0}$. But G is not amenable, by [3, Theorem 2.12], there exists $f_0 \in UCB(G)$, the set of uniformly continuous function on G , and $t_i, s_i \in G$ ($i = 1, 2, \dots, N$) such that

$$\sum_{i=1}^N t_i f_0 - f_0 = \sum_{i=1}^N (t_i f_0 - f_0) + (f_0 - s_i f_0) \geq 1$$

i.e. there exists $h_{f_0} \in H_{f_0}$ such that $h_{f_0} \geq 1$. Take $h_{f_0}^* = (2\|h_{f_0}\|_\infty)^{-1} h_{f_0}$ then $h_{f_0}^* \in H_{f_0}$ and

$$(2\|h_{f_0}\|_\infty)^{-1} \leq h_{f_0}^* \leq 1/2$$

so $\|1 - h_{f_0}^*\|_\infty < 1$ which is impossible. $\square$

Open problem. Does amenability of G imply that $\mathcal{U}$ is a subspace? (This is the case when G is discrete (see [7, Theorem 1.1]).

Corollary 3.5. *If G is a discrete group, the following statements are equivalent*

- (a) G is amenable,
- (b) $\mathcal{U}$ is a subspace,
- (c) $\mathcal{A}$ is a subspace,
- (d) $\mathcal{A}_0 = \overline{H}$,
- (e) $\mathcal{U}_0 = \overline{H}$ where $\mathcal{U}_0 = \{f \in \mathcal{U} : m(f) = 0 \text{ for all } m \in LIM(S_f)\}$.

Proof. By Theorem 2.3 and Lemma 2.1, (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). By Theorem 3.4 and Theorem 1.1 of [7], (a) $\Leftrightarrow$ (b).

It is clear that if $\mathcal{U}_0$ is a subspace, then $\mathcal{U}$ is a subspace. Hence (e) $\Rightarrow$ (b). To see that (d) $\Rightarrow$ (e). Let $f \notin \overline{H}$, there exists $m \in LIM(L^\infty(G))$ such that $m(f) \neq 0$, i.e. $f \notin \mathcal{U}_0$ and $\mathcal{U}_0 \subseteq \overline{H}$. Since $LIM(L^\infty(G)) \neq \phi$ by Theorem 2.3, $\overline{H} = \mathcal{A}_0 \subseteq \mathcal{U}_0$. Therefore $\mathcal{U}_0 = \overline{H}$. $\square$

Now we come to discuss the existence of the largest admissible subspace of $L^\infty(G)$ with a unique left invariant mean.

Theorem 3.6. *There is a largest admissible subspace S_M in $L^\infty(G)$ with a unique left invariant mean if and only if G is amenable. In this case, $S_M = \overline{H} + \mathbb{C}$.*

Proof. Suppose that such S_M exists. Note that if $\xi \in \mathcal{A}_0$ and $LIM(S_\xi) \neq \phi$, then $LIM(S_\xi)$ is a singleton. Using this fact and the same proof of Lemma 3.3, we have $S_M \supseteq H_s$ (see Lemma 3.3 for H_s). Let $m \in LIM(S_M)$, we can extend m to $\overline{S}_M$ such that $m \in LIM(\overline{S}_M)$ which is also a singleton. Hence S_M is closed and $S_M \supseteq \overline{H}_s = \overline{H}$, i.e. $S_M \supseteq \overline{H} + \mathbb{C}$. For any $x \in G$ and $f \in L^\infty(G)$, since ${}_x f - f \in \mathcal{A}_0$, $m({}_x f - f) = 0$, i.e. $m(h) = 0$ for all $h \in H$. Therefore

$$1 = m(1 - h) \leq \|1 - h\|_\infty$$

for any $h \in H$. Hence G is amenable.

Conversely, let G be amenable. Take $S_M = \overline{H} + \mathbb{C}$. Let an admissible subspace S with a unique left invariant mean m be given. For each $f \in S$, $f - m(f) \in \overline{H}$, i.e. $S \subseteq \overline{H} + \mathbb{C}$. Indeed, if $f - m(f) \notin \overline{H}$, there exists $M \in LIM(L^\infty(G))$ such that $M(f - m(f)) \neq 0$. Note that $M|_S \in LIM(S)$ and

$M(f) = M|S(f) \neq m(f)$ which is impossible. For any $m \in LIM(S_M)$, since ${}_xf - f \in \mathcal{A}_0$ ($x \in G$, $f \in L^\infty(G)$), $m(h) = 0$ for any $h \in \overline{H}$. Consequently, S_M must have a unique left invariant mean. $\square$

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