INTERPOLATING SEQUENCES IN A HOMEOMORPHIC PART OF H^∞

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ABSTRACT. A characterization of interpolating sequences in a homeomorphic part of the algebra of bounded analytic functions on the unit open disc is given as zero sets of some interpolating Blaschke products.

1. INTRODUCTION

Let D be the open unit disc and H^∞ be the Banach algebra of bounded analytic functions on D with the supremum norm. We denote by $M(H^\infty)$ the maximal ideal space of H^∞ with the weak-* topology. By the corona theorem, D is dense in $M(H^\infty)$. We identify a function in H^∞ with its Gelfand transform. We denote the closed unit ball of H^∞ by $\text{ball}(H^\infty)$. For points x and y in $M(H^\infty)$, the pseudo-hyperbolic metric is defined by

$$\rho(x, y) = \sup\{|h(x)|; h \in \text{ball}(H^\infty), h(y) = 0\}.$$

When z and w are points in D , we have $\rho(z, w) = |z - w|/|1 - \bar{z}w|$. A sequence $\{x_j\}_j$ in $M(H^\infty)$ is called *interpolating* if for every bounded sequence $\{a_j\}_j$ there is a function f in H^∞ such that $f(x_j) = a_j$ for every j . A characterization of an interpolating sequence $\{z_j\}_j$ in D is given by $\inf_k \prod_{j \neq k} \rho(z_j, z_k) > 0$ [3]. For a sequence $\{z_j\}_j$ in D with $\sum_{j=1}^\infty 1 - |z_j| < \infty$, put

$$B(z) = \prod_{j=1}^\infty \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z} \quad (z \in D).$$

Then B is a bounded analytic function on D and called a Blaschke product with zeros $\{z_j\}_j$. Here $\{z_j\}_j$ is called the zero sequence of B . If $\{z_j\}_j$ is an interpolating sequence, B is called interpolating. For a function h in H^∞ , put $Z(h) = \{x \in M(H^\infty); h(x) = 0\}$. For a subset E of $M(H^\infty)$, we denote by $\text{cl} E$ the weak-* closure of E in $M(H^\infty)$.

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For a point x in $M(H^\infty)$, put $P(x) = \{y \in M(H^\infty); \rho(y, x) < 1\}$ that is called a part containing x . If $P(x) \neq \{x\}$, $P(x)$ is called nontrivial. In [6], K. Hoffman proved that if $P(x)$ is nontrivial then there is an interpolating sequence $\{z_j\}_j$ in D such that x is contained in $\text{cl}\{z_j\}_j$, and there is an analytic map L_x from D onto $P(x)$, that is, $f \circ L_x \in H^\infty$ for every $f \in H^\infty$, where L_x is given by $L_x(z) = \lim_\alpha (z + z_{j_\alpha}) / (1 + \bar{z}_{j_\alpha} z)$ for a net $\{z_{j_\alpha}\}_\alpha$ in $\{z_j\}_j$ with $z_{j_\alpha} \rightarrow x$. When L_x is a homeomorphism, we call $P(x)$ a *homeomorphic part*. There exist many homeomorphic parts, and nontrivial nonhomeomorphic parts [2, 5, 6]. The open disc D is a typical homeomorphic part.

In this paper, as a generalization of interpolating sequences in D , we study interpolating sequences in homeomorphic parts. Our theorem gives a characterization of these sequences as zero sets of some interpolating Blaschke products. Let $\{x_j\}_j$ be an interpolating sequence such that all points x_j come from the same part. Then there is a function h in H^∞ with $Z(h) \supset \{x_j\}_j$ and h does not vanish on its part. Let B be a Blaschke factor of h . Generally, B is not interpolating. A problem of this paper is to find an interpolating subproduct b of B such that $Z(b) \supset \{x_j\}_j$. Here we note that our idea cannot apply for nonhomeomorphic parts. Recently, Axler and Gorkin [1] studied sequences in $M(H^\infty)$ that have interpolating subsequences. Our result concerns interpolating sequences, however, not subsequences.

2. PRELIMINARIES

Let b be an interpolating Blaschke product with zeros $\{z_j\}_j$. Put $\delta(b) = \inf_k \prod_{j \neq k} \rho(z_j, z_k)$.

Lemma 1 [4, p. 404]. *Let b be an interpolating Blaschke product with $\delta(b) \geq \delta > 0$. Let $\{z_j\}_j$ be the zero sequence of b . Then there exist $\lambda(\delta)$ and $r(\delta)$ with $0 < \lambda(\delta), r(\delta) < 1$ such that*

$$\{z \in D; |b(z)| < r(\delta)\} \subset \bigcup_{j=1}^{\infty} \{z \in D; \rho(z, z_j) < \lambda(\delta)\}.$$

Here we may take as $r(\delta) \rightarrow 1$ ($\delta \rightarrow 1$).

Since the pseudo-hyperbolic metric is lower semicontinuous on $M(H^\infty) \times M(H^\infty)$ [6, p. 103], we have

Lemma 2. *Under the same assumptions of Lemma 1,*

$$\{x \in M(H^\infty); |b(x)| < r(\delta)\} \subset \{x \in M(H^\infty); \rho(x, Z(b)) \leq \lambda(\delta)\},$$

where $\rho(x, Z(b)) = \inf\{\rho(x, y); y \in Z(b)\}$.

By [6, p. 82], we have

Lemma 3. *Let b be an interpolating Blaschke product with $b(x) = 0$ for some $x \in M(H^\infty)$. For every $0 < \delta < 1$, there is a subproduct b_1 of b such that $b_1(x) = 0$ and $\delta(b_1) > \delta$.*

Lemma 4. *Let $P(x)$ be a homeomorphic part. If B is a Blaschke product with $B(x) = 0$ and B does not vanish on $P(x)$, then there is an interpolating subproduct b of B such that $Z(b) \cap P(x) = \{x\}$.*

Proof. Since B does not vanish on $P(x)$, B has a zero of finite order at x . By [6, p. 100], x lies in the closure of interpolating subsequence $\{z_j\}_j$ of the zeros of B . Put $\{x_k\}_k = \{y \in P(x); B(y) = 0 \text{ and } y \neq x\}$. Then $\{L_x^{-1}(x_k)\}_k$ are zeros of $B \circ L_x$ in D , so that $L_x^{-1}(x_k)$ ($k \rightarrow \infty$) goes to the boundary of D . Since $P(x)$ is a homeomorphic part, $\{x_k\}_k$ does not have a cluster point in $P(x)$. Hence x is not contained in $\text{cl}\{x_k\}_k$. Take open subsets U and V of $M(H^\infty)$ such that

$$x \in U, \quad \text{cl}\{x_k\}_k \subset V, \quad \text{and} \quad U \cap V = \emptyset.$$

Let b be an interpolating Blaschke product with zeros $\{z_j\}_j \cap U$. Then b is a desired function.

Lemma 5. *Let P be a homeomorphic part and $\{x_j\}_j$ be a sequence in P . Let B be a Blaschke product such that $Z(B) \supset \{x_j\}_j$ and B does not vanish on P . Then there is a subproduct $\prod_{k=1}^\infty b_k$ of B such that*

- (1) b_k is an interpolating Blaschke product with $Z(b_k) \cap P = \{x_k\}$,
- (2) $Z(\prod_{k=1}^\infty b_k) \cap P = \{x_j\}_j$.

Proof. Put $\{y_i\}_i = Z(B) \cap P$. By Lemma 4, it is easy to get a subproduct $\prod_{k=1}^\infty \psi_k$ of B such that ψ_k is an interpolating Blaschke product with $Z(\psi_k) \cap P = \{y_k\}$. Since P is a homeomorphic part, by the same reason of the proof of Lemma 4, each point y_k is isolated in $\text{cl}\{y_i\}_i$, so that there are open subsets U and V of $M(H^\infty)$ such that

$$\{x_j\}_j \subset U, \quad \{y_i\}_i \setminus \{x_j\}_j \subset V, \quad \text{and} \quad U \cap V = \emptyset.$$

Let b_k be a subproduct of ψ_k whose zeros are contained in U . Then $\{b_k\}_k$ satisfies our assertion.

The following lemma is a key to the proof of our theorem.

Lemma 6. *Let $P(x)$ be a homeomorphic part and $\{x_j\}_j$ be a sequence in $P(x)$ with $x_j \neq x$ for every j . Suppose that*

$$\sup\{|h(x)|; h \in \text{ball}(H^\infty), Z(h) \supset \{x_j\}_j\} > \delta > 0.$$

If B is a Blaschke product such that B does not vanish on $P(x)$ and $Z(B) \supset \{x_j\}_j$, then there is a subproduct b of B such that $|b(x)| > \delta$ and $Z(b) \supset \{x_j\}_j$.

Proof. Take a function h in $\text{ball}(H^\infty)$ such that $|h(x)| > \delta$ and $Z(h) \supset \{x_j\}_j$. Then the Blaschke factor of h has the same property. Hence there is a Blaschke product ψ such that

$$|\psi(x)| > \delta \quad \text{and} \quad Z(\psi) \supset \{x_j\}_j.$$

Take δ_1 as $\delta < \delta_1 < |\psi(x)|$, and put $E = \{z \in D; |\psi(z)| > \delta_1\}$. Then x is contained in $\text{cl } E$. By Lemma 5, there is a subproduct $\prod_{k=1}^{\infty} \psi_k$ of ψ such that ψ_k is an interpolating Blaschke product and $Z(\psi_k) \cap P(x) = \{x_k\}$ for every k . We note that

$$|\psi_k(z)| \geq |\psi(z)| > \delta_1 \quad \text{for every } z \in E.$$

Also there is a subproduct $\prod_{k=1}^{\infty} b_k$ of B such that b_k is an interpolating Blaschke product and $Z(b_k) \cap P(x) = \{x_k\}$ for every k . Let $\{z_{k,i}\}_i$ be the zero sequence of b_k . Let M_k be an interpolation constant for $\{z_{k,i}\}_i$; that is, for every sequence $\{a_i\}_i$ with $|a_i| \leq 1$, there is a function g in H^∞ such that $\|g\| \leq M_k$ and $g(z_{k,i}) = a_i$ for every i . Take $\varepsilon > 0$ such that $\delta < \delta_1 - \varepsilon$, and take a sequence of positive numbers $\{\varepsilon_k\}_k$ such that

$$1 - \varepsilon \leq \prod_{k=1}^{\infty} \frac{1 - M_k \varepsilon_k \delta^{-1}}{1 + M_k \varepsilon_k}.$$

Put $\Lambda_k = \{z_{k,i}; |\psi_k(z_{k,i})| < \varepsilon_k\}$. Since $\psi_k(x_k) = 0$ and x_k is contained in $\text{cl}\{z_{k,i}\}_i$, x_k is contained in $\text{cl } \Lambda_k$. Here there is a function f_k in H^∞ such that

$$\|f_k\| \leq M_k \varepsilon_k \quad \text{and} \quad f_k(z_{k,i}) = \psi_k(z_{k,i}) \quad \text{for } z_{k,i} \in \Lambda_k.$$

Let b'_k be a subproduct of b_k with zeros Λ_k . Then $b'_k(x_k) = 0$ and $\psi_k - f_k = b'_k h_k$ for some function h_k in H^∞ . Since $\|h_k\| \leq 1 + M_k \varepsilon_k$, we have

$$\frac{|\psi_k(z)| - M_k \varepsilon_k}{1 + M_k \varepsilon_k} \leq |b'_k(z)| \quad \text{for } z \in D.$$

Consequently, for each point z in E we have

$$\begin{aligned} \left| \left(\prod_{k=1}^{\infty} b'_k \right) (z) \right| &= \prod_{k=1}^{\infty} |b'_k(z)| \\ &\geq \prod_{k=1}^{\infty} \frac{|\psi_k(z)| - M_k \varepsilon_k}{1 + M_k \varepsilon_k} \\ &= \prod_{k=1}^{\infty} |\psi_k(z)| \prod_{k=1}^{\infty} \frac{1 - M_k \varepsilon_k |\psi_k(z)|^{-1}}{1 + M_k \varepsilon_k} \\ &\geq |\psi(z)| \prod_{k=1}^{\infty} \frac{1 - M_k \varepsilon_k \delta^{-1}}{1 + M_k \varepsilon_k} \\ &> \delta_1 (1 - \varepsilon) > \delta_1 - \varepsilon > \delta. \end{aligned}$$

Hence $b = \prod_{k=1}^{\infty} b'_k$ satisfies our assertion.

As a consequence, we get

Lemma 7. Let $P(x)$ be a homeomorphic part and $y \in P(x)$, $y \neq x$. If B is a Blaschke product such that B does not vanish on $P(x)$ and $B(y) = 0$, then for

every $\varepsilon > 0$ there is a subproduct b of B such that $|b(x)| > \rho(x, y) - \varepsilon$ and $b(y) = 0$.

3. THE MAIN RESULT

Theorem. Let P be a homeomorphic part and $\{x_j\}_j$ be a distinct sequence in P . Then the following conditions are equivalent.

- (1) $\{x_j\}_j$ is an interpolating sequence.
- (2) $\inf_k \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j \text{ with } j \neq k\} > 0$.
- (3) $\inf_k \prod_{j \neq k} \rho(x_j, x_k) > 0$ and

$$\inf_k [\lim_{n \rightarrow \infty} \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j \geq n\}] > 0.$$
- (4) There is an interpolating Blaschke product b with $Z(b) \cap P = \{x_j\}_j$.
- (5) There is a sequence of interpolating Blaschke product $\{b_j\}_j$ such that $b_j(x_j) = 0$, $\prod_{j=1}^\infty b_j \in H^\infty$, and $\inf_k |(\prod_{j \neq k} b_j)(x_k)| > 0$.

Proof. (1) $\Rightarrow$ (2) is an easy consequence of the open mapping theorem.

(2) $\Rightarrow$ (3). Suppose that

$$\inf_k \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j \neq k\} > \delta > 0.$$

Fix k arbitrarily. Let h be a function in $\text{ball}(H^\infty)$ such that

$$h(x_j) = 0 \quad \text{for } j \neq k \text{ and } |h(x_k)| > \delta.$$

By Lemma 5, for $n > k$ we can write h as $h = (\prod_{j \neq k}^n b_j)h_n$, where b_j is an interpolating Blaschke product, $Z(b_j) \cap P = \{x_j\}$, and h_n is a function in $\text{ball}(H^\infty)$. Since $h_n(x_j) = 0$ for $j > n$,

$$\begin{aligned} \delta < |h(x_k)| &= \left| \prod_{j \neq k}^n b_j(x_k) \right| |h_n(x_k)| \\ &\leq \prod_{j \neq k}^n \rho(x_j, x_k) \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j > n\}. \end{aligned}$$

Hence we get (3).

(3) $\Rightarrow$ (2). Take $\sigma > 0$ such that $\prod_{j \neq k} \rho(x_j, x_k) > \sigma$ and

$$\lim_{n \rightarrow \infty} \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j \geq n\} > \sigma$$

for every k . Take positive numbers $\{\varepsilon_{j,k}\}_{j,k}$ such that

$$\prod_{j \neq k} [\rho(x_j, x_k) - \varepsilon_{j,k}] > \sigma.$$

Fix k arbitrarily. Then there is an integer n_k greater than k and a function h_k in $\text{ball}(H^\infty)$ such that $h_k(x_j) = 0$ for $j \geq n_k$ and $|h_k(x_k)| > \sigma$. Choose

functions $\{f_j\}_{j=1}^{n_k}$ in $\text{ball}(H^\infty)$ such that $f_j(x_j) = 0$ and $|f_j(x_k)| > \rho(x_j, x_k) - \varepsilon_{j,k}$ for $j \neq k$. Put

$$h = \left(\prod_{1 \leq j \neq k < n_k} f_j \right) h_k.$$

Then $h \in \text{ball}(H^\infty)$, $|h(x_k)| > \sigma^2$, and $h(x_j) = 0$ for $j \neq k$.

(2) $\Rightarrow$ (4). Suppose that

$$\inf_k \sup\{|h(x_k)|; h \in \text{ball}(H^\infty), h(x_j) = 0 \text{ for } j \neq k\} > \delta > 0.$$

By the proof of (2) $\Rightarrow$ (3), $\inf_k \prod_{j \neq k} [\rho(x_j, x_k) > \delta]$. Take positive numbers $\{\varepsilon_{j,k}\}_{j,k}$ such that

$$\varepsilon_{k,j} = \varepsilon_{j,k} \quad \text{and} \quad \inf_k \prod_{j \neq k} [\rho(x_j, x_k) - \varepsilon_{j,k}] > \delta.$$

Take a sequence $\{r_n\}_n$ such that

$$0 < r_n < 1 \quad \text{and} \quad \prod_{n=1}^{\infty} r_n > \delta.$$

Take $\delta_n > 0$ such that $r_n < r(\delta_n)$, and put $\lambda_n = \lambda(\delta_n)$, where $r(\delta_n)$ and $\lambda(\delta_n)$ are defined in Lemma 1. We may choose $\{\delta_n\}_n$ as $\delta_n < \delta_{n+1}$ for every n .

By induction, we shall construct sequences of interpolating Blaschke products $\{\phi_j\}_j$ and $\{b_j\}_j$, and a sequence of Blaschke products $\{B_j\}_j$ as follows.

- (a) $B_{j-1} = \phi_j B'_j$ for $j > 1$.
- (b) B_j is a subproduct of B'_j with $|B_j(x_j)| > \delta$ and $Z(B_j) \supset \{x_i; i > j\}$.
- (c) $\delta(\phi_j) > \delta_j$, $Z(\phi_j) \cap P = \{x_j\}$, and $\{w_{j,i}\}_i$ is a zero sequence of ϕ_j .
- (d) b_j is a subproduct of ϕ_j with zeros $\{z_{j,i}\}_i$ and $b_j(x_j) = 0$, hence $\delta(b_j) > \delta_j > \delta_1$.
- (e) $|B_j(z_{j,i})| > \delta$ for every i .
- (f) If $|\phi_p(x_j)| \leq r_p$ ($1 \leq p < j$), then $|b_p(w_{j,i})| > \rho(x_p, x_j) - \varepsilon_{p,j}$ for every i .
- (g) If $|\phi_p(x_j)| > r_p$ ($1 \leq p < j$), then $|b_p(w_{j,i})| > r_p$ for every i .
- (h) If $|\phi_p(x_i)| \leq r_p$ ($1 \leq p \leq j < i$), then $|b_j(x_i)| > \rho(x_j, x_i) - \varepsilon_{j,i}$.

1st step. Take a function h_1 in $\text{ball}(H^\infty)$ such that $h_1(x_j) = 0$ for $j > 1$ and $|h_1(x_1)| > \delta$. Let B_1 be the Blaschke factor of h_1 . Then

$$B_1(x_j) = 0 \quad \text{for } j > 1 \text{ and } |B_1(x_1)| > \delta.$$

Take an interpolating Blaschke product ϕ_1 with zeros $\{w_{1,i}\}_i$ such that

$$|B_1(w_{1,i})| > \delta \quad \text{for every } i \text{ and } \phi_1(x_1) = 0.$$

By Lemmas 3 and 4, we may assume that

$$\delta(\phi_1) > \delta_1 \quad \text{and} \quad Z(\phi_1) \cap P = \{x_1\}.$$

By Lemma 2, $\{x \in M(H^\infty); |\phi_1(x)| < r(\delta_1)\} \subset \{x \in M(H^\infty); \rho(x, Z(\phi_1)) \leq \lambda_1\}$, so that we get $|\phi_1| \geq r(\delta_1) > r_1$ on $\{y \in P; \rho(y, x_1) > \lambda_1\}$. Hence $\{x_j; |\phi_1(x_j)| \leq r_1\}$ is a finite set. By applying Lemma 7 finitely many times, there is a subproduct b_1 of ϕ_1 with zeros $\{z_{1,i}\}_i$ such that $b_1(x_1) = 0$ and $|b_1(x_i)| > \rho(x_1, x_i) - \varepsilon_{1,i}$ for x_i with $|\phi_1(x_i)| \leq r_1$ ($1 < i$).

k th step. Suppose that for $k \geq 2$, $\{\phi_j\}_{j < k}$, $\{b_j\}_{j < k}$, and $\{B_j\}_{j < k}$ are already chosen. By (b), $B_{k-1}(x_k) = 0$ and $B_{k-1}(x_{k-1}) \neq 0$. By Lemmas 3 and 4, there is an interpolating subproduct ϕ_k of B_{k-1} with zeros $\{w_{k,i}\}_i$ such that

$$(a \text{ and } c) \quad B_{k-1} = \phi_k B'_k, \quad \delta(\phi_k) > \delta_k \quad \text{and} \quad Z(\phi_k) \cap P = \{x_k\}.$$

Suppose that $|\phi_p(x_k)| \leq r_p$ for some $1 \leq p < k$. By (h),

$$|b_p(x_k)| > \rho(x_p, x_k) - \varepsilon_{p,k},$$

so that we may assume that

$$(f) \quad |b_p(w_{k,i})| > \rho(x_p, x_k) - \varepsilon_{p,k} \quad \text{for every } i.$$

Suppose that $|\phi_p(x_k)| > r_p$ for some $1 \leq p < k$. Then $|b_p(x_k)| > r_p$, so that we may also assume

$$(g) \quad |b_p(w_{k,i})| > r_p \quad \text{for every } i.$$

Since $Z(B'_k) \supset \{x_j; j > k\}$, by Lemma 6 there is a subproduct B_k of B'_k such that

$$(b) \quad |B_k(x_k)| > \delta \quad \text{and} \quad Z(B_k) \supset \{x_j; j > k\}.$$

By the same way as 1st step, $\{x_i; |\phi_p(x_i)| \leq r_p\}$ is a finite set for every p , so that

$$\Lambda_k = \{x_i; |\phi_p(x_i)| \leq r_p \text{ for some } 1 \leq p \leq k < i\}$$

is a finite set. Hence by Lemma 7, there is a subproduct b_k of ϕ_k with zeros $\{z_{k,i}\}_i$ such that

$$(d \text{ and } h) \quad b_k(x_k) = 0 \quad \text{and} \quad |b_k(x_i)| > \rho(x_k, x_i) - \varepsilon_{k,i} \quad \text{for } x_i \in \Lambda_k.$$

Since $|B_k(x_k)| > \delta$, we may assume that

$$(e) \quad |B_k(z_{k,i})| > \delta \quad \text{for every } i.$$

This completes our induction.

Put $b = \prod_{j=1}^{\infty} b_j$. Then by (d), $Z(b) \supset \{x_j\}_j$. We shall prove that b is an interpolating Blaschke product; that is, $\{z_{j,i}; j, i = 1, 2, \dots\}$ is an interpolating sequence.

$$\begin{aligned}
& \inf_{(j,i)} \prod_{(s,t) \neq (j,i)} \rho(z_{s,t}, z_{j,i}) \\
&= \inf_{(j,i)} \left[\prod_{t \neq i} \rho(z_{j,t}, z_{j,i}) \right] \left[\prod_{s \neq j} \prod_{t=1}^{\infty} \rho(z_{s,t}, z_{j,i}) \right] \\
&\geq \inf_{(j,i)} \delta(b_j) \prod_{s \neq j} |b_s(z_{j,i})| \\
&\geq \delta_1 \inf_{(j,i)} \left(\prod_{1 \leq s < j} |b_s(z_{j,i})| \right) \left(\left| \left(\prod_{j < s} b_s \right) (z_{j,i}) \right| \right) \quad \text{by (d)} \\
&\geq \delta_1 \inf_{(j,i)} \left(\prod_{1 \leq s < j} |b_s(z_{j,i})| \right) |B_j(z_{j,i})| \quad \text{by (a), (b), (d)} \\
&\geq \delta_1 \delta \inf_j \prod_{1 \leq s < j} r_s [\rho(x_s, x_j) - \varepsilon_{s,j}] \quad \text{by (e), (f), (g)} \\
&\geq \delta_1 \delta \left(\prod_{s=1}^{\infty} r_s \right) \inf_j \prod_{s \neq j} [\rho(x_s, x_j) - \varepsilon_{s,j}] \\
&\geq \delta_1 \delta^3.
\end{aligned}$$

Hence b is an interpolating Blaschke product with $Z(b) \supset \{x_j\}_j$. By Lemma 5, we get (4).

(4) $\Rightarrow$ (5). Let b be an interpolating Blaschke product with $Z(b) \cap P = \{x_j\}_j$. By Lemma 5, there is a sequence of subproducts $\{b_j\}_j$ of b such that

$$b = \prod_{j=1}^{\infty} b_j \quad \text{and} \quad Z(b_j) \cap P = \{x_j\}.$$

Let $\{z_{k,i}\}_i$ be a zero sequence of b_k . Since $x_k \in \text{cl}\{z_{k,i}\}_i$,

$$\left| \left(\prod_{j \neq k} b_j \right) (x_k) \right| \geq \inf_i \left| \left(\prod_{j \neq k} b_j \right) (z_{k,i}) \right| \geq \delta(b).$$

Hence we get (5).

(5) $\Rightarrow$ (1) Take $\{b_j\}_j$ as (5). By Lemma 3, we may assume that $\delta(b_j) \geq 1/2$ for every j . Let $\{z_{k,i}\}_i$ be a zero sequence of b_k . Put

$$\delta = \inf_k |(\prod_{j \neq k} b_j)(x_k)|.$$

We may take a subsequence $\{w_{k,i}\}_i$ of $\{z_{k,i}\}_i$ such that

$$\left| \left(\prod_{j \neq k} b_j \right) (w_{k,i}) \right| \geq \delta/2 \quad \text{and} \quad x_k \in \text{cl}\{w_{k,i}\}_i.$$

Then $\inf_{(k,i)} \prod_{(t,s) \neq (k,i)} \rho(w_{t,s}, w_{k,i}) \geq \delta/4$. Hence $\{w_{k,i}\}_{k,i}$ is an interpolating sequence. For a bounded sequence $\{a_k\}_k$, there is a function f in H^∞ such that $f(w_{k,i}) = a_k$ for every k and i . Hence we get $f(x_k) = a_k$ for every k . This completes the proof.

The referee referred the author to a preprint entitled *Homeomorphic disks in the spectrum of H^∞* by P. Gorkin, H.-M. Lingenberg, and R. Mortini. They have obtained almost the same result and proven that condition (3) in our theorem is superfluous, for only one condition $\inf_k \prod_{j \neq k} \rho(x_j, x_k) > 0$ works.

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