AN ALTERNATING PROCEDURE FOR OPERATORS ON UNIFORMLY CONVEX AND UNIFORMLY SMOOTH BANACH SPACES

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ABSTRACT. Let X be a real uniformly convex and uniformly smooth Banach space. For any $1 < p < \infty$, J_p , J_p^* respectively denote the duality mapping with gauge function $\varphi(t) = t^{p-1}$ from X onto X^* and X^* onto X . If $T : X \rightarrow X$ is a bounded linear operator, then $M(T) : X \rightarrow X$ is the mapping defined by $M(T) = J_q^* T^* J_p T$, where $T^* : X^* \rightarrow X^*$ is the adjoint of T and $q = (p-1)^{-1}p$. It is proved that if T_n is a sequence of operators on X such that $\|T_n\| \leq 1$ for all n , then $M(T_n, \dots, T_1)x$ strongly converges in X for any $x \in X$, with an estimate of the rate of convergence:

$$\|M(T_n \cdots T_1)x - M(x)\| \leq \sigma(x)\|x\|\psi(1 - (m(x)/\|T_n \cdots T_1 x\|)^p),$$

where $M(x) = \lim_{n \rightarrow \infty} M(T_n \cdots T_1)x$, $m(x) = \lim_{n \rightarrow \infty} \|T_n \cdots T_1 x\|$, and $\sigma : X \rightarrow R^+$, $\psi : R^+ \rightarrow R^+$ are definite, strictly increasing positive functions. The result obtained generalizes and improves on the theorem offered recently by Akcoglu and Sucheston [1].

1. INTRODUCTION

Let X be a real Banach space, with dual space X^* and associated generalized duality pairing denoted by $\langle \cdot, \cdot \rangle$. Suppose that X and X^* are both uniformly convex (therefore, reflexive). We identify $X = X^{**}$ in the usual way. For any $1 < k < \infty$, let J_k be defined by

$$J_k x := \{x^* \in X^* : \langle x^*, x \rangle = \|x^*\| \|x\|, \|x^*\| = \|x\|^{k-1}\} \quad \forall x \in X.$$

If T is a bounded linear operator on X , then $M(T)$ denotes the operator on X defined by

$$M(T)x = J_q^* T^* J_p T x \quad \forall x \in X,$$

where $T^* : X^* \rightarrow X^*$ is the adjoint operator of T , and J_p , J_q^* , respectively, are the duality mapping from X to X^* and X^* to X , with $q = (p-1)^{-1}p$ and $p > 1$ arbitrarily.

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Let $T_n : X \rightarrow X$, $n \geq 1$, be an arbitrary sequence of linear contractions (namely, $\|T_n\| \leq 1$, $n \geq 1$). In a recent paper [1], Akcoglu and Sucheston proved that $M(T_n \cdots T_1)f$ strongly converges in L_p spaces ($1 < p < \infty$) as $n \rightarrow \infty$ for every element $f \in L_p$. This result has several interesting consequences and applications [1, 2, 7] (in particular, it generalizes the well-known theorems of Stein [8], Rota [9], and Von Neumann [2]). In this note, we extend this result to the general cases of uniformly convex and uniformly smooth Banach spaces. Our argument will differ from that which Akcoglu and Sucheston used, allowing us to give an explicit estimate on the rate of convergence of $M(T_n \cdots T_1)x$.

2. PRELIMINARIES

Recall that a Banach space X is said to be uniformly convex if the modulus of X that is defined by

$$\delta_X(\varepsilon) := \inf\{1 - \|x + y\|/2 : \|x\| = \|y\| = 1, \|x - y\| \leq \varepsilon\}$$

is positive for any $\varepsilon \in (0, 2]$; X is said to be uniformly smooth (or equivalently, X^* is uniformly convex) if the modulus of smoothness of X defined by

$$\rho_X(\tau) := \sup\left\{\frac{\|x + y\|}{2} + \frac{\|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \tau\right\}$$

satisfies the equality $\lim_{\tau \rightarrow \infty} \rho_X(\tau)/\tau = 0$. It is known [3, Proposition 1.e.2] that there exists a complete duality between uniform convexity and uniform smoothness: X is uniformly convex (smooth) if and only if X^* is uniformly smooth (convex). We shall employ the following characteristics of these spaces that were recently established by the authors in [6]:

Lemma 1 [6]. *Let X be a uniformly convex Banach space and $k > 1$ be an arbitrary real number. Then there exists a positive constant c_1 such that*

$$(1a) \quad \|x + y\|^k \geq \|x\|^k + k\langle j_k x, y \rangle + \sigma_k(x, y) \quad \forall j_k x \in J_k x, \quad x, y \in X,$$

where

$$(1b) \quad \sigma_k(x, y) = c_1 \int_0^1 \frac{(\|x + ty\| \vee \|x\|)^k}{t} \delta_X\left(\frac{t\|y\|}{2(\|x + ty\| \vee \|x\|)}\right) dt.$$

Lemma 2 [6]. *Let X be a uniformly smooth Banach space. Then for any $k > 1$, the duality mapping J_k is single-valued, and there exists a positive constant c_2 such that*

$$\|J_k x - J_k y\| \leq c_2(\|x\| \vee \|y\|)^{k-1} \bar{\rho}_X\left(\frac{\|x - y\|}{\|x\| \vee \|y\|}\right) \quad \forall x, y \in X$$

where $\bar{\rho}_X(0) = 0$ and

$$(2) \quad \bar{\rho}_X(\tau) = \rho_X(\tau)/\tau \quad \forall \tau > 0.$$

We also need the following consequence of Lemmas 1 and 2:

Lemma 3. *Let X be a uniformly smooth Banach space and $S : X \rightarrow X$ be a linear contraction. Then for any $k, p \in (1, \infty)$,*

$$\|S^* J_p Sx - J_p x\| \leq \|x\|^{p-1} \varphi_{X^*}^{-1} \left(\frac{k}{c_1} \|x\|^{-p} (\|x\|^p - \|Sx\|^p) \right) \quad \forall 0 \neq x \in X$$

where

$$(3) \quad \varphi_{X^*}(t) = \int_0^{t/2} \delta_{X^*}(\xi)/\xi d\xi,$$

and c_1 is the constant determined by X^* in Lemma 1.

Proof. Since J_p is single-valued, X^* is uniformly convex, and

$$\begin{aligned} \|J_p x + t(S^* J_p Sx - J_p x)\| &= \|(1-t)J_p x + tS^* J_p Sx\| \\ &\leq \text{Max}\{\|J_p x\|, \|S^* J_p Sx\|\} \\ &\leq \|x\|^{p-1} = \|J_p x\| \quad \forall 0 \leq t \leq 1, \end{aligned}$$

we have

$$\begin{aligned} \sigma_k(J_p x, S^* J_p Sx - J_p x) &= c_1 \int_0^1 \frac{\|J_p x\|^k}{t} \delta_{X^*} \left(\frac{t\|S^* J_p Sx - J_p x\|}{2\|J_p x\|} \right) dt \\ &= c_1 \|J_p x\|^k \int_0^{\frac{\|S^* J_p Sx - J_p x\|}{2\|J_p x\|}} \delta_{X^*}(\xi)/\xi d\xi \\ &= c_1 \|x\|^{k(p-1)} \varphi_{X^*} \left(\frac{\|S^* J_p Sx - J_p x\|}{\|x\|^{p-1}} \right) \end{aligned}$$

where φ_{X^*} , as defined in (3), is a strictly increasing positive function. Then by Lemma 1, we get

$$\begin{aligned} \|S^* J_p Sx\|^k &= \|J_p x + S^* J_p Sx - J_p x\|^k \\ &\geq \|J_p x\|^k + k \langle j_k^*(J_p x), S^* J_p Sx - J_p x \rangle \\ &\quad + c_1 \|x\|^{k(p-1)} \varphi_{X^*} \left(\frac{\|S^* J_p Sx - J_p x\|}{\|x\|^{p-1}} \right). \end{aligned}$$

It is known from [4; 6, Lemma 4] that $J_k^* x^* = \|x^*\|^{k-q} J_q^* x^*$ and $J_q^* x^* = J_p^{-1} x^*$ for any $x^* \in X^*$, $k, p \in (1, \infty)$, and $q = (p-1)^{-1}p$. Therefore,

$$\begin{aligned} \|S^* J_p Sx\|^k &\geq \|J_p x\|^k + k \|J_p x\|^{k-q} \langle x, S^* J_p Sx - J_p x \rangle \\ &\quad + c_1 \|x\|^{k(p-1)} \varphi_{X^*} \left(\frac{\|S^* J_p Sx - J_p x\|}{\|x\|^{p-1}} \right) \\ &= \|x\|^{k(p-1)} + k \|x\|^{(p-1)(k-q)} (\|Sx\|^p - \|x\|^p) \\ &\quad + c_1 \|x\|^{k(p-1)} \varphi_{X^*} \left(\frac{\|S^* J_p Sx - J_p x\|}{\|x\|^{p-1}} \right). \end{aligned}$$

Since

$$\|S^* J_p S x\|^k \leq \|J_p S x\|^k = \|S x\|^{k(p-1)} \leq \|x\|^{k(p-1)},$$

it follows that

$$c_1 \|x\|^{k(p-1)} \varphi_{X^*} \left(\frac{\|S^* J_p S x - J_p x\|}{\|x\|^{p-1}} \right) \leq k \|x\|^{k(p-1)-p} (\|x\|^p - \|S x\|^p);$$

that is,

$$\|S^* J_p S x - J_p x\| \leq \|x\|^{p-1} \varphi_{X^*}^{-1} \left(\frac{k}{c_1} \|x\|^{-p} (\|x\|^p - \|S x\|^p) \right),$$

which completes the proof of the lemma. Q.E.D.

3. RESULTS

Theorem 4. Let X be a uniformly convex and uniformly smooth Banach space, and $T_n : X \rightarrow X$, $n \geq 1$, be an arbitrary sequence of (linear) contractions. Then $M(T_n \cdots T_1)x$ converges strongly in X as $n \rightarrow \infty$ for any $x \in X$. Furthermore, let $M(x) = \lim_{n \rightarrow \infty} M(T_n \cdots T_1)x$ and $m(x) = \lim_{n \rightarrow \infty} \|T_n \cdots T_1 x\|$. Then the following estimate of the rate of convergence holds:

$$\begin{aligned} M(x) &= 0, \quad x = 0, \\ \|M(T_n \cdots T_1)x - M(x)\| &\leq \sigma(x) \|x\| \psi \left(1 - \left(\frac{m(x)}{\|T_n \cdots T_1 x\|} \right)^p \right), \quad x \neq 0, \end{aligned}$$

where

$$(4.a) \quad \sigma(x) = \begin{cases} 1, & 1 < p \leq 2, \\ [\|x\|/m(x)]^{p-q}, & p \geq 2, \end{cases}$$

$$(4.b) \quad \psi(t) = c c_2 \bar{\rho}_{X^*}(\varphi_{X^*}^{-1}(pt/c_1)) \quad \forall t > 0,$$

and c is a constant determined by X^* .

Proof. For any $k > 1$, it is easily seen that $J_k(\alpha x) = \text{sign}(\alpha)|\alpha|^{k-1}J_k x$ and hence, $M(S)(\alpha x) = \alpha M(S)x$ for any linear operator S on X and for any scalar α . Therefore, we will only consider $\|x\| \leq 1$ without loss of generality. Let $R_n = T_n \cdots T_1$, $S_m = T_m \cdots T_{n+1}$, and $g_n = R_n x$ for any $m > n \geq 1$. Then $R_m = S_m R_n$, $M(T_n \cdots T_1)x = J_q^* R_n^* J_p R_n x = J_q^* R_n^* J_p g_n$, and $M(T_m \cdots T_1)x = J_q^* R_m^* J_p R_m x = J_q^* R_n^* S_m^* J_p S_m g_n = J_q^* R_n^* S_m^* J_p g_m$. Since the uniform convexity of X implies uniform smoothness of X^* , by using Lemma 2 for $x := R_n^* J_p g_n$, $y := R_n^* S_m^* J_p g_m$, and $k := q$, we obtain

$$\begin{aligned} (5) \quad \|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| &= \|J_q^* R_n^* J_p g_n - J_q^* R_n^* S_m^* J_p g_m\| \\ &\leq c_2 (\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|)^{q-1} \\ &\quad \cdot \bar{\rho}_{X^*} \left(\frac{\|R_n^* J_p g_n - R_n^* S_m^* J_p g_m\|}{\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|} \right). \end{aligned}$$

Since $\rho_{X^*}(\tau)$ is known to be positive, convex, and satisfy $\rho_{X^*}(0) = 0$ [4] and $\rho_{X^*}(\tau)/\tau^2$ to be equivalent to a decreasing function [3, Proposition 1.e.5], $\bar{\rho}_{X^*}(\tau)$ is nondecreasing and there exists an absolute constant $c < \infty$ such that

$$(6) \quad \rho_{X^*}(\tau)/\tau^2 \leq c\rho_{X^*}(\eta)/\eta^2 \quad \forall \tau > \eta > 0.$$

Hence, it follows from the contraction property of R_n^* , (5), (6), and (2) that

$$\begin{aligned} & \|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| \\ & \leq c_2(\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-1} \bar{\rho}_{X^*} \left(\frac{\|J_p g_n - S_m^*J_p g_m\|}{\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|} \right) \\ & = c_2 \frac{(\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-2}}{\|J_p g_n - S_m^*J_p g_m\|^{-1}} \\ & \quad \cdot \left[\frac{\|J_p g_n\|}{\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|} \frac{\|J_p g_n - S_m^*J_p g_m\|}{\|J_p g_n\|} \right]^{-2} \\ & \quad \cdot \rho_{X^*} \left(\frac{\|J_p g_n\|}{\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|} \frac{\|J_p g_n - S_m^*J_p g_m\|}{\|J_p g_n\|} \right) \\ & \leq cc_2(\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-2} \|J_p g_n - S_m^*J_p g_m\| \\ & \quad \cdot \left[\frac{\|J_p g_n - S_m^*J_p g_m\|}{\|J_p g_n\|} \right]^{-2} \rho_{X^*} \left(\frac{\|J_p g_n - S_m^*J_p g_m\|}{\|J_p g_n\|} \right) \\ & = cc_2(\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-2} \|J_p g_n\| \bar{\rho}_{X^*} \left(\frac{\|J_p g_n - S_m^*J_p g_m\|}{\|J_p g_n\|} \right). \end{aligned}$$

(Here, we have assumed $\|J_p g_n\| \neq 0$ for any $n \geq 1$. If there exists some n such that $\|J_p g_n\| = 0$, then it is easy to see that $M(T_n \cdots T_1)x \rightarrow 0$. Consequently the conclusion has been proved already.) By using Lemma 3 for $x := g_n$, $S := S_m$, and $k := p$, we then have

$$\begin{aligned} & \|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| \\ & \leq cc_2(\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-2} \|J_p g_n\| \\ (7) \quad & \cdot \bar{\rho}_{X^*} \left(\varphi_{X^*}^{-1} \left(\frac{p}{c_1} \|g_n\|^{-p} (\|g_n\|^p - \|S_m g_n\|^p) \right) \right) \\ & = (\|R_n^*J_p g_n\| \vee \|R_n^*S_m^*J_p g_m\|)^{q-2} \|J_p g_n\| \psi(1 - (\|g_m\|/\|g_n\|)^p), \end{aligned}$$

where $\psi(t)$, as defined in (4.b), is a strictly increasing positive function. Now we observe that $\lim_{n \rightarrow \infty} \|g_n\| = \lim_{n \rightarrow \infty} \|T_n \cdots T_1 x\| \equiv m(x)$ exists, for $\{\|T_n \cdots T_1 x\|\}$ clearly is a nonincreasing sequence of nonnegative numbers and, if $m(x) = 0$, then, obviously, $\|M(T_n \cdots T_1)x\| = \|J_q^* R_n^* J_p R_n x\| \leq \|T_n \cdots T_1 x\| \rightarrow 0$ as $n \rightarrow \infty$; if $m(x) \neq 0$, then $\|g_n\| = \|T_n \cdots T_1 x\| \geq m(x) > 0$ for any

$n \geq 1$, which in turn implies that

$$(8) \quad \begin{aligned} \|R_n^* J_p g_n\| &\geq \|x\| \|R_n^* J_p R_n x\| \geq \langle R_n^* J_p R_n x, x \rangle = \langle J_p R_n x, R_n x \rangle \\ &= \|J_p R_n x\| \|R_n x\| = \|g_n\|^p \geq [m(x)]^p > 0 \quad \forall n \geq 1; \end{aligned}$$

hence, (7) implies in this case that

$$\|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| \rightarrow 0 \quad (\text{as } m > n \rightarrow \infty).$$

Thus, in either case, $\{M(T_n \cdots T_1)x\}$ is a Cauchy sequence for any $x \in X$. Consequently, $\lim_{m \rightarrow \infty} M(T_m \cdots T_1)x \equiv M(x)$ exists for every $x \in X$. Taking the limit as $m \rightarrow \infty$ in the two sides of the inequality (7) and noting that $\|M(T_m \cdots T_1)x\| = \|R_m^* S_m^* J_p g_m\|^{q-1}$, we then obtain

$$\begin{aligned} &\|M(T_n \cdots T_1)x - M(x)\| \\ &\leq (\|R_n^* J_p g_n\| \vee [M(x)]^{p-1})^{q-2} \|J_p g_n\| \psi(1 - (m(x)/\|g_n\|)^p) \end{aligned}$$

Obviously, to complete the proof, it remains to show that

$$(9) \quad \Delta \equiv (\|R_n^* J_p g_n\| \vee [M(x)]^{p-1})^{q-2} \|J_p g_n\| \leq \sigma(x) \|x\|.$$

Distinguish between two possible cases:

Case I. If $q \geq 2$, then from the fact that $\|R_n^* J_p g_n\| \leq \|J_p g_n\| = \|g_n\|^{p-1} = \|T_n \cdots T_1 x\|^{p-1} \leq \|x\|^{p-1}$, and that $\|M(T_m \cdots T_1)x\| = \|J_q^* R_m^* J_p g_m\| = \|R_m^* J_p g_m\|^{q-1} \leq \|x\|^{(p-1)(q-1)} = \|x\|$, we get

$$\begin{aligned} \Delta &= \lim_{m \rightarrow \infty} [\|R_n^* J_p g_n\| \vee \|M(T_n \cdots T_1)x\|^{p-1}]^{q-2} \cdot \|J_p g_n\| \\ &\leq \|x\|^{(p-1)(q-2)} \|J_p g_n\| \leq \|x\|^{(p-1)(q-2)} \cdot \|x\|^{p-1} = \|x\|, \end{aligned}$$

which obviously establishes (9) in this case.

Case II. If $q < 2$, then by means of the fact that (8) implies $\|R_n^* J_p g_n\| \geq [m(x)]^p / \|x\|$ and $[M(x)]^{p-1} = \lim_{m \rightarrow \infty} \|R_m^* J_p g_m\| \geq [m(x)]^p / \|x\|$, we obtain

$$\begin{aligned} \Delta &\leq ([m(x)]^p / \|x\|)^{q-2} \|J_p g_n\| \\ &\leq ([m(x)]^p / \|x\|)^{q-2} \|x\|^{p-1} = [\|x\| / m(x)]^{p-q} \|x\|. \end{aligned}$$

Hence, (9) is justified in this case also. With this, the proof is complete. Q.E.D.

As a special case of Theorem 4, we have

Theorem 5. Let $X = l^p$, L^p , or W_m^p , $1 < p < \infty$, be the usual sequence spaces, function spaces, or Sobolev spaces, and $T_n : X \Rightarrow X$, $n \geq 1$, be an arbitrary sequence of (linear) contractions and $x \in X$. Then $M(T_n \cdots T_1)x$ converges strongly in X as $n \rightarrow \infty$, with the estimate of the rate of convergence:

$$\begin{aligned} M(x) &= 0, \quad x = 0, \\ \|M(T_n \cdots T_1)x - M(x)\| &\leq k \|x\| [1 - m(x) / \|T_n \cdots T_1 x\|]^{(k_1-1)/k_2}, \quad x \neq 0, \end{aligned}$$

where k is an absolute constant and k_1, k_2 are given by

$$(10) \quad k_1 = \begin{cases} 2, & 1 < p \leq 2, \\ q, & p \geq 2, \end{cases} \quad k_2 = \begin{cases} q, & 1 < p \leq 2, \\ 2, & p \geq 2. \end{cases}$$

Proof. It is known [3–5] that in these spaces the moduli of convexity and smoothness satisfy

$$\delta_X(\varepsilon) \geq \begin{cases} \varepsilon^2(p-1)/8, & 1 < p \leq 2 \\ (1/p)(\varepsilon/2)^p, & p \geq 2 \end{cases} \quad (X := l^p, L^p, \text{ or } W_m^p),$$

$$\rho_X(\tau) \leq \begin{cases} \tau^p/p, & 1 < p \leq 2 \\ \tau^2(p-1)/2, & p \geq 2 \end{cases} \quad (X := l^p, L^p, \text{ or } W_m^p),$$

which imply that these spaces are both uniformly convex and uniformly smooth and, in particular, there exist constants $c_1, c_2 > 0$ such that

$$\delta_{X^*}(\varepsilon) \leq c_2 \varepsilon^{k_2}, \quad \rho_{X^*}(\tau) \leq c_1 \tau^{k_1},$$

where k_1, k_2 are defined as (10). Now by making use of these estimates, it is easily seen that Lemma 2 implies

$$\|J_q^* x^* - J_q y^*\| \leq k_1 c_2 (\|x^*\| \vee \|y^*\|)^{q-k_1} \|x^* - y^*\|^{k_1-1},$$

and Lemma 3 implies

$$\|S^* J_p S x - J_p x\|^{k_2} \leq \frac{2^{k_2} k_2 k}{c_1 c_2} \|x\|^{(p-1)(k_2-q)} [\|x\|^p - \|Sx\|^p].$$

Therefore, with all notations as in Theorem 4, we now have

$$\begin{aligned} & \|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| \\ &= \|J_q^* R_n^* J_p g_n - J_q^* R_n^* S_m^* J_p g_m\| \\ &\leq c_2 k_1 (\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|)^{q-k_1} \|R_n^* J_p g_n - R_n^* S_m^* J_p g_m\|^{k_1-1} \\ &\leq c_2 k_1 (\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|)^{q-k_1} \|S_m^* J_p S_m g_n - J_p g_n\|^{k_1-1} \\ &\leq c_2 k_1 (\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|)^{q-k_1} \\ &\quad \cdot \left[\frac{2^{k_2} k_2 k}{c_1 c_2} \|g_n\|^{(p-1)(k_2-q)} (\|g_n\|^p - \|S_m g_n\|^p) \right]^{(k_1-1)/k_2} \\ &= c_2 k_1 (\|R_n^* J_p g_n\| \vee \|R_n^* S_m^* J_p g_m\|)^{q-k_1} \\ &\quad \cdot \left[\frac{2^{k_2} k_2 k}{c_1 c_2} \|g_n\|^{(p-1)k_2} \left(1 - \frac{\|S_m g_n\|^p}{\|g_n\|^p} \right) \right]^{(k_1-1)/k_2}. \end{aligned}$$

Since $k_1 \leq q \leq k_2$ from (10) and $\|R_n^* J_p g_n\| \leq \|g_n\|^{p-1} \leq \|x\|^{p-1}$, it follows that

$$\|M(T_n \cdots T_1)x - M(T_m \cdots T_1)x\| \leq k \|x\| [1 - \|S_m g_n\|^p / \|g_n\|^p]^{(k_1-1)/k_2}$$

for any $m > n \geq 1$, $x \neq 0$, and some constant $k > 0$. From this, the conclusion of Theorem 5 obviously follows. This completes the proof. Q.E.D.

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