SATURATED FAMILIES

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ABSTRACT. We will show that $V = K$ implies that there exist saturated (or completely separable) almost disjoint families on sets of any infinite cardinality.

1. INTRODUCTION

Definition. For a family $\mathcal{A} \subseteq [X]^\omega$, let $I_{\mathcal{A}}$ be the following ideal: $x \in I_{\mathcal{A}} \leftrightarrow x$ is almost covered by finitely many elements of $\mathcal{A}$. ("almost" means "except for finitely many points").

Let $I_{\mathcal{A}}^+ = \mathcal{P}(X) - I_{\mathcal{A}}$ (= the positive sets modulo this ideal).

Definition. An almost disjoint family $\mathcal{A} \subseteq [X]^\omega$ is called *saturated* if for all $Y \in I_{\mathcal{A}}^+$ there exists an $a \in \mathcal{A}$ $a \subseteq Y$.

Definition. Let $S(\kappa)$ abbreviate the statement "there exists an infinite almost disjoint saturated family $\mathcal{A} \subseteq [\kappa]^\omega$."

[BDS] exhibit several conditions that imply that every almost disjoint family on ω can be refined to a saturated family: CH (or MA) are sufficient, or even the following consequence of MA : "Every maximal almost disjoint family on ω has 2^ω many elements."

The question whether $S(\aleph_2)$ holds is Problem 37 in [EH1], but [EH2] mentions Baumgartner's result $CH \rightarrow \forall n S(\aleph_n)$.

[K] considers a related problem: Let us call an almost disjoint family $\subseteq [X]^\omega$ ω_1 -saturated if it is saturated with respect to all uncountable sets, i.e. every uncountable $Y \subseteq X$ contains an element of the family. [K] shows (from $V = L$) that $\forall \kappa < \aleph_{\omega_1}$ there exists an ω_1 -saturated family on κ .

[HJS] prove that a finite support iteration of length ω_1 of Hechler reals will yield a model in which

$$\forall \kappa S(\kappa)$$

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holds, and the question is asked, whether the same is in fact true for every model for ZFC, or even for L . We will prove that under following assumptions (1)–(3), $\forall \kappa \ S(\kappa)$ will hold:

- (1) $S(\omega)$,
- (2) $\forall \kappa : \kappa > cf(\kappa) = \omega \rightarrow \square_\kappa$,
- (3) $\forall \kappa : \kappa > cf(\kappa) = \omega \rightarrow$

$$\exists \mathcal{C} \subseteq [\kappa]^\omega, \text{card}(\mathcal{C}) = \kappa^+, \forall X \in [\kappa]^\omega \exists F \in \mathcal{C} \ X \subseteq F.$$

Remark. (1) is implied by CH . The failure of either (2) or (3) is a large cardinal assumption: If the covering lemma holds for K , (2) and (3) will be true. (For (3) it suffices to have an inner model satisfying GCH for which the covering lemma holds. Of course, for $\kappa > 2^\omega$, (3) just says $\kappa^\omega = \kappa^+$.)

The proof will be done by inductively constructing an increasing sequence of saturated families on ordinals $\omega \cdot \alpha$. Successor steps can be handled by Condition (1). Limit steps of uncountable cofinality are trivial. For limit ordinals of cofinality ω that are not cardinals, a construction will be given in §3. For cardinals of cofinality ω , we will use (2) and (3) from above in §4.

Thanks to Lajos Soukup for suggesting Condition (3) instead of $\kappa^\omega = \kappa^+$, and for correcting an error in §4.

2. DEFINITIONS, NOTATION, AND A FEW SIMPLE FACTS

2.1. Notation

$X, X_1, \dots$: infinite sets.

$x, y, \dots$: countable subsets of X , i.e. elements of $[X]^\omega$.

$\mathcal{A}, \mathcal{B}, \dots$: almost disjoint families of countable sets, usually $\subseteq [X]^\omega$.

$a, a_1, \dots, b, b_1, \dots$: elements of almost disjoint families.

$Y, Z \dots$: subsets of X .

2.2. Definition. Let $Y \subseteq X$. An almost disjoint family $\mathcal{A} \subseteq [X]^\omega$ is called saturated with respect to Y if $Y \in I_{\mathcal{A}}$ or $\exists a \in \mathcal{A} \ a \subseteq Y$.

$\mathcal{A}$ is saturated on Y if $\mathcal{A}$ is saturated with respect to all $Z \subseteq Y$.

$\mathcal{A}$ is *saturated* if $\mathcal{A}$ is saturated on $X = \bigcup \mathcal{A}$. (When we talk about a saturated family $\mathcal{A} \subseteq [X]^\omega$, it is understood that $\mathcal{A}$ is saturated on X . But the statement “ $\mathcal{A}$ is saturated on Y ” does not by itself imply $\mathcal{A} \subseteq [Y]^\omega$.)

It is easy to see that a saturated family must be a *maximal* almost disjoint family.

2.3. Fact. If $\mathcal{A}$ is saturated on X , and $\mathcal{B}$ is saturated on Y (X and Y almost disjoint), $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ then $\mathcal{C}$ is saturated on $X \cup Y$ (assuming $\mathcal{C}$ is almost disjoint).

2.4. Fact. If $\mathcal{A}$ is saturated with respect to Y , and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{B}$ is saturated with respect to Y .

2.5. Definition. For $\mathcal{A} \subseteq [X]^\omega$, $Y \subseteq X$, let

$$\mathcal{A} \upharpoonright Y = \{a \cap Y : a \in \mathcal{A}, a \cap Y \text{ infinite}\}.$$

2.6. Fact. If $\lambda < \kappa$, then $S(\kappa) \rightarrow S(\lambda)$.

Proof. Let $\mathfrak{A} \subseteq [\kappa]^\omega$ be an infinite saturated family on κ . Let X be a subset of κ of size λ such that $\mathfrak{A} \upharpoonright X$ is infinite. (If $\lambda > \omega$, any set of size λ will have that property. Otherwise, let $\mathfrak{A}' \subseteq \mathfrak{A}$ be countably infinite, and let $X = \bigcup \mathfrak{A}'$.) Then $\mathfrak{A} \upharpoonright X$ is an infinite saturated family on X .

2.7. Fact. An almost disjoint family $\mathfrak{A} \subseteq [X]^\omega$ is saturated iff it is saturated with respect to all countable subsets y of X .

Proof. Assume that $\mathfrak{A}$ is saturated with respect to all countable $y \subseteq X$. Let $Y \subseteq X$ be uncountable. Then $Y \in I_{\mathfrak{A}}^+$, so we have to find $a \subseteq Y$, $a \in \mathfrak{A}$. It is clear that $\mathfrak{A}$ is a maximal almost disjoint family. Hence there are infinitely many sets $a_0, a_1, \dots$ in $\mathfrak{A}$ that have infinite intersection with Y . Let $y = Y \cap \bigcup a_i$. y cannot be almost covered by finitely many sets $b_1, \dots, b_m \in \mathfrak{A}$ (otherwise each a_i would have infinite intersection with some b_k). Since $\mathfrak{A}$ is saturated with respect to y , y (and hence Y) contains some $a \in \mathfrak{A}$.

2.8. Fact. If $\mathfrak{A}$ is saturated with respect to X , and saturated with respect to Y , then $\mathfrak{A}$ is saturated with respect to $X \cup Y$.

2.9. Fact. If $S(\omega)$, then every partition of a countable set into infinitely many countable sets can be extended to a saturated family.

Proof. It is enough to find some saturated family extending some partition of some countable set (into infinitely many countable sets): Take any infinite saturated family $\mathfrak{A}$ on ω , let $a_0, a_1, \dots$ be distinct elements of $\mathfrak{A}$. Let $X = \bigcup a_n$, $\mathfrak{B} = \mathfrak{A} \upharpoonright X$. Get $\mathfrak{B}'$ by replacing each a_i by $a'_i = a_i - \bigcup_{j < i} a_j$. Then $\mathfrak{B}'$ is saturated (since $I_{\mathfrak{B}} = I_{\mathfrak{B}'}$).

3. EXTENDING SATURATED FAMILIES

3.1. Fact. Let λ be an ordinal, $cf(\lambda) > \omega$, let $(X_\beta)_{\beta < \lambda}$ be an increasing family of sets, and let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots \subseteq \mathfrak{A}_\beta \dots$ be saturated families on $X_0, X_1, \dots, X_\beta, \dots$ respectively (i.e. $\mathfrak{A}_\beta$ saturated with respect to all subsets of X_β , but not necessarily $\mathfrak{A}_\beta \subseteq [X_\beta]^\omega$). Then $\bigcup \mathfrak{A}_\beta$ is saturated on $X = \bigcup X_\beta$.

Proof. Use 2.7, and note that every countable set $\subseteq X$ is in fact $\subseteq X_\beta$, for some $\beta < \lambda$.

The corresponding fact is not true for $cf(\lambda) = \omega$. (Let $\lambda = \bigcup \lambda_n$, $\lambda_{n+1} > \lambda_n$ and let y be an infinite set such that $\forall n \ y \cap \lambda_n$ is finite. Then y cannot contain any set in $\bigcup \mathfrak{A}_\beta = \bigcup \mathfrak{A}_{\lambda_n}$, nor can it be covered by a finite union of such sets.)

However, the following lemma holds:

3.2. Lemma. Let $cf(\lambda) = \omega$, let $(X_\beta)_{\beta < \lambda}$ be an increasing family of sets, and let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots \subseteq \mathfrak{A}_\beta \dots$ be saturated families on $X_0, X_1, \dots, X_\beta, \dots$ respectively, and let $\forall \alpha < \lambda \ \mathfrak{A}_\alpha \subseteq \bigcup_\beta [X_\beta]^\omega$. Assume there exists a saturated family on $X = \bigcup_\beta X_\beta$. Then there exists a saturated family $\mathfrak{A}_\lambda \supseteq \bigcup_\beta \mathfrak{A}_\beta$ on X .

3.3. Corollary

$$S(\kappa) \rightarrow S(\kappa^+), \\ \forall \mu < \lambda \ S(\mu) \rightarrow S(\lambda), \quad \text{if } cf(\lambda) > \omega.$$

Proof. Build a chain of families $\mathfrak{A}_\alpha$ on $\omega \cdot \alpha$ by induction, starting with a saturated family on ω .

For successor stages, let $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha \cup \mathfrak{B}$, where $\mathfrak{B}$ is any saturated family on $\omega \cdot (\alpha + 1) - \omega \cdot \alpha$. ($\mathfrak{A}_{\alpha+1}$ will be saturated by 2.3.)

If α is a limit ordinal of uncountable cofinality, let

$$\mathfrak{A}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{A}_\beta.$$

(This works, by 3.1.)

If α is a limit ordinal of countable cofinality, let $\mathfrak{A}_\alpha$ be a saturated family extending $\bigcup_{\beta < \alpha} \mathfrak{A}_\beta$. (Lemma 3.2 applies, since the cardinality of α is $\leq \kappa$ or $< \lambda$, respectively.)

For the proof of 3.2 we may assume w.l.o.g. that $\lambda = \omega$. We may also assume that all $(X_{n+1} - X_n)$ are infinite: If $X_{n_{k+1}} - X_{n_k}$ is infinite (for all k) for some infinite increasing sequence $(n_k : k \in \omega)$, work with the family $(X_{n_k} : k \in \omega)$ instead of $(X_n : n \in \omega)$. Otherwise, there is an n_0 such that $\forall n \geq n_0$ $(X_{n+1} - X_n)$ is finite. Hence for $n \geq n_0$, $\mathfrak{A}_n = \mathfrak{A}_{n_0}$. If $X - X_{n_0}$ is finite, then $\mathfrak{A}_{n_0}$ is saturated on X . Otherwise, let $\mathfrak{A}_\omega = \mathfrak{A}_{n_0} \cup \mathfrak{B}$, where $\mathfrak{B}$ is saturated on $X - X_{n_0}$. Then $\mathfrak{A}_\omega$ is saturated on X and extends all $\mathfrak{A}_n$'s.

Some technology for the proof of 3.2:

3.4. Definition. If f is a function on $\mathfrak{A}$ ($\mathfrak{A}$ an almost disjoint family $\subseteq [X]^\omega$), and $\forall a \in \mathfrak{A}$ $f(a)$ is an almost disjoint family (not necessarily infinite) $\subseteq [a]^\omega$, then let $f_*(\mathfrak{A}) = \bigcup_{a \in \mathfrak{A}} f(a)$.

3.5. Fact. $f_*(\mathfrak{A})$ is almost disjoint. If $\mathfrak{A}$ is saturated on X , and if for all $a \in \mathfrak{A}$ $f(a)$ is saturated on a , then $f_*(\mathfrak{A})$ is saturated.

Proof. It is clear that $f_*(\mathfrak{A})$ is almost disjoint.

To show that $f_*(\mathfrak{A})$ is saturated, consider any $y \in [X]^\omega$. Since $\mathfrak{A}$ is saturated, there are two cases:

Case 1. If $y \supseteq a$, for some $a \in \mathfrak{A}$, then $y \supseteq b$, for any $b \in f(a)$. ($f(a)$ is not empty, since it is saturated on an infinite set.)

Case 2. If $y \not\supseteq a$, for all $a \in \mathfrak{A}$, then $y \subseteq^* a_1 \cup \dots \cup a_n$, for some sets $a_1, \dots, a_n \in \mathfrak{A}$. Let $\mathfrak{B} = f(a_1) \cup \dots \cup f(a_n)$. Then $\mathfrak{B}$ (and, by 2.4, also $f_*(\mathfrak{A})$) is saturated with respect to y , by the following argument:

Let $y' = y \cap (a_1 \cup \dots \cup a_n)$. Then, by 2.3, $\mathfrak{B}$ is saturated with respect to y' . If $y' \supseteq b$ for some $b \in \mathfrak{B}$, then also $y \supseteq b$, and if $y' \in I_{\mathfrak{B}}$, then also $y \in I_{\mathfrak{B}}$, since y is almost contained in y' .

For the following, we fix a family $(X_n : n \in \omega)$ as in the hypothesis of 3.2.

3.6. Definition. Let $y \in [X]^\omega$.

y is called *bounded*, if $\exists n \ y \subseteq X_n$.

y is called *thin*, if $\forall n \in \omega \ y \cap (X_{n+1} - X_n)$ is finite (or $\forall n \in \omega \ y \cap X_n$ is finite).

y is called *thick*, if y is not bounded, and $\forall n \in \omega \ y \cap (X_{n+1} - X_n)$ is either infinite or empty.

3.7. Fact. Every $y \in [X]^\omega$ is either bounded, thick, thin, or the union of a bounded set and a thin set, or the union of a thick set and a thin set. Hence, by 2.8., a family $\mathfrak{A}$ is saturated iff it is saturated with respect to all bounded, thin, and thick sets.

By assumption, $\bigcup \mathfrak{A}_n$ contains only bounded sets, and it is clear that $\bigcup \mathfrak{A}_n$ (as in the statement of 3.2) is saturated with respect to all bounded sets.

3.8. Fact. Let $\mathfrak{A}$ be a saturated family on X . Then there exists a saturated family $\mathfrak{B}$ on X containing only thin, thick and bounded sets.

Proof. For $a \in \mathfrak{A}$, let $f(a)$ be a finite saturated family on a , defined as follows: If a is thin, thick or bounded, then let $f(a) = \{a\}$. If $a = b \cup c$, where $b \cap c = \emptyset$, and b and c are thin, thick or bounded, let $f(a) = \{b, c\}$. (By 3.7, this covers all cases.) Now let $\mathfrak{B} = f_*(\mathfrak{A})$.

3.9. Lemma. Let $\mathfrak{A}$ be a saturated family on X . Then there exists a saturated family $\mathfrak{B}$ containing only thin sets and bounded sets.

Proof. By 3.8, we may assume that all sets in $\mathfrak{A}$ are bounded, thin, or thick. If $a \in \mathfrak{A}$ is bounded or thin, let $f(a) = \{a\}$. Otherwise let $f(a)$ be a saturated family extending the partition on a induced by the representation

$$a = \bigcup_{n \in \omega} (X_{n+1} - X_n) \cap a.$$

Then all sets in $f(a)$ that are not of the form $a \cap (X_{n+1} - X_n)$ are thin. Now let $\mathfrak{B} = f_*(\mathfrak{A})$.

3.10. Remark. Clearly, the family $\mathfrak{B}' =$ all thin sets in $\mathfrak{B}$ is saturated with respect to all thin subsets of X .

3.11. Lemma. Let $\mathfrak{A}$ be an almost disjoint family on X consisting only of thin sets. If $\mathfrak{A}$ is saturated with respect to all thin sets, then $\mathfrak{A}$ is saturated with respect to all thick sets.

Proof. Assume that y is an unbounded thick set that contains no set in $\mathfrak{A}$. This will yield a contradiction.

Construct a sequence $(a_n : n \in \omega)$ of sets from $\mathfrak{A}$ as follows: Given $a_0, \dots, a_{n-1}$ let $a_n \in \mathfrak{A}$ be a set different from all the a_i ($i < n$), such that $y \cap a_n$ is infinite. (a_n exists: Consider the set $y - \bigcup_{j < n} a_j$. This set is thick, so it contains some thin set x . This set x either contains a set in $\mathfrak{A}$, or is covered by finitely many of them, so in particular it meets some set $a \in \mathfrak{A}$ infinitely

often, so $a \notin \{a_0, \dots, a_{n-1}\}$. Let

$$z = \bigcup_n (a_n \cap y - X_n).$$

Then z is thin:

$$z \cap X_k = \bigcup_n (a_n \cap y - X_n) \cap X_k \subseteq \bigcup_{n < k} a_n \cap X_k = \text{finite}.$$

Since $z \subseteq y$, z contains no member of $\mathfrak{A}$. Therefore it is in $I_{\mathfrak{A}}$:

$$z \subseteq^* b_1 \cup \dots \cup b_m, \quad b_k \in \mathfrak{A}.$$

Now consider any a_n :

$$a_n \cap y - X_n \subseteq^* b_1 \cup \dots \cup b_m.$$

Since $a_n \cap y - X_n$ is infinite, for some k $(a_n \cap y - X_n) \cap b_k$ must be infinite. But then $a_n \cap b_k$ is infinite, and since both a_n and b_k are in the almost disjoint family $\mathfrak{A}$, $a_n = b_k$.

This implies that $\{a_0, a_1, \dots\}$, an infinite set, is contained in the finite set $\{b_1, \dots, b_m\}$, a contradiction.

3.12. Lemma. *Let $\mathfrak{B}$ be a saturated family on X containing only thin sets and bounded sets. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots$ be saturated on $X_0, X_1, \dots$, containing only bounded sets. Let*

$$\mathfrak{B}' = \{a \in \mathfrak{B} : a \text{ is thin}\}.$$

Then $\mathfrak{A} = \mathfrak{B}' \cup \bigcup \mathfrak{A}_n$ is saturated.

Proof. $\mathfrak{A}$ is an almost disjoint family. Clearly $\bigcup \mathfrak{A}_n$ is saturated with respect to all bounded sets, and $\mathfrak{B}'$ is saturated with respect to thin sets. By 3.11, $\mathfrak{B}'$ is also saturated with respect to thick sets. By 2.4 we get that $\mathfrak{A}$ is saturated with respect to thin, thick, and bounded sets. But every $y \in [X]^\omega$ can be written as a finite union of thin, thick and bounded sets, so by 2.8 we are done.

Proof of 3.2. Let $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots$ be saturated on $X_0, X_1, \dots$, and assume that $\mathfrak{B}$ is saturated on X . By 3.8 and 3.9 there exists a saturated family $\mathfrak{B}_1$ containing only thin and bounded sets. By 3.12 we can find a saturated family $\mathfrak{A} = \mathfrak{B}'_1 \cup \bigcup \mathfrak{A}_n$ on X extending $\bigcup \mathfrak{A}_n$.

4. REACHING CARDINALS OF COUNTABLE COFINALITY

Lemma. *Let $\lambda > \omega$, $cf(\lambda) = \omega$, and assume $\forall \mu < \lambda S(\mu)$.*

Assume that (2) and (3) from §1 hold. Then $S(\lambda)$.

Proof. Let $(C_\alpha : \alpha < \lambda^+, \alpha \text{ limit})$ be a $\square$ -sequence, i.e. assume that the following hold for all limit ordinals $\alpha < \lambda^+$:

$$C_\alpha \subseteq \alpha.$$

$$C_\alpha \text{ closed unbounded in } \alpha.$$

$$\text{card}(C_\alpha) < \lambda.$$

$$\forall \gamma \in C'_\alpha \quad C_\gamma = C_\alpha \cap \gamma,$$

where $\text{card}(C)$ is the cardinality of a set C , and C' is the set of limit points of a set C of ordinals.

Let $(\lambda_n : n \in \omega)$ be an increasing sequence of regular cardinals with limit λ . Let $k_\alpha = \min\{k : \text{card}(C_\alpha) \leq \lambda_k\}$. Let $\{x_\beta : \beta < \lambda^+\} \subseteq [\lambda]^\omega$ be a covering family, i.e.

$$\forall x \in [\lambda]^\omega \exists \beta : x \subseteq x_\beta.$$

By induction on $\alpha < \lambda^+$, we will define

$$(A_k^\alpha : k \in \omega), \quad \mathfrak{A}^\alpha \subseteq \bigcup_{k \in \omega} [A_k^\alpha]^\omega$$

such that for all α the following will hold:

- (a) $A_k^\alpha \subseteq A_{k+1}^\alpha$, $\text{card}(A_k^\alpha) = \lambda_k$, $A_k^0 \subseteq A_k^\alpha$.
- (b) $\forall \beta < \alpha : \exists k^* \forall k \geq k^* A_k^\beta \subseteq A_k^\alpha$.
- (c) α limit $\rightarrow \forall \gamma \in C'_\alpha \forall k \geq k_\alpha A_k^\gamma \subseteq A_k^\alpha$.
- (d) $\mathfrak{A}^\alpha$ is an almost disjoint family.
- (e) $\mathfrak{A}^\alpha$ is saturated with respect to all subsets of A_k^α , for all k .
- (f) $\forall \beta < \alpha : \mathfrak{A}^\beta \subseteq \mathfrak{A}^\alpha$.
- (g) $\alpha = \beta + 1 \rightarrow \exists k x_\beta \subseteq A_k^\alpha$.

We will construct these families by induction on α .

Case 1. $\alpha = 0$.

Let $A_k^0 = \lambda_k$, and let $\mathfrak{A}_k^0 \subseteq [\lambda_k - \lambda_{k-1}]^\omega$ be a saturated family on $\lambda_k - \lambda_{k-1}$ (which exists by assumption), $\mathfrak{A}^0 = \bigcup_k \mathfrak{A}_k^0$.

Remark. Since $\lambda = \bigcup_k A_k^0$, (a) implies $\lambda = \bigcup_k A_k^\alpha$ for all α .

Case 2. α a limit of uncountable cofinality.

C'_α must be unbounded in α .

Let, for $k < k_\alpha$, $A_k^\alpha = A_k^0$.

For any $\gamma_1 < \gamma_2$, $\gamma_1, \gamma_2 \in C'_\alpha$ we have $\gamma_1 \in C'_{\gamma_2}$. Hence by induction hypothesis (c)

$$A_k^{\gamma_1} \subseteq A_k^{\gamma_2}$$

for all $k \geq k_{\gamma_2}$. (Notice that $k_{\gamma_2} \leq k_\alpha$, since $C_{\gamma_2} \subseteq C_\alpha$.)

Let for $k \geq k_\alpha$

$$A_k^\alpha = \bigcup_{\gamma \in C'_\alpha} A_k^\gamma,$$

then $\text{card}(A_k^\alpha) \leq \text{card}(C'_\alpha) \cdot \lambda_k \leq \lambda_{k_\alpha} \cdot \lambda_k = \lambda_k$. Hence (a) holds.

Let

$$\mathfrak{A}^\alpha = \bigcup_{\gamma \in C'_\alpha} \mathfrak{A}^\gamma = \bigcup_{\beta < \alpha} \mathfrak{A}^\beta.$$

By 3.1, $\mathfrak{A}^\alpha$ is saturated on every A_k^α , so (e) holds.

- (d) follows from the induction hypotheses (d) and (f).
 (c) and (f) are true by construction.
 (b) follows from (c) and the induction hypothesis.

Case 3. α a limit, $cf(\alpha) = \omega$, and C'_α is cofinal in α .

As before, let for $k < k_\alpha$, $A_k^\alpha = A_k^0$, and for $k \geq k_\alpha$ let

$$A_k^\alpha = \bigcup_{\gamma \in C'_\alpha} A_k^\gamma,$$

then $\text{card}(A_k^\alpha) \leq \text{card}(C'_\alpha) \cdot \lambda_k = \lambda_k$. Then (a)–(c) hold.

Let $\gamma_0 < \gamma_1 < \dots$ be an increasing sequence in C'_α , cofinal in α . Let $B_n = A_{k_\alpha + n}^{\gamma_n}$. Then

$$(*) \quad B_0 \subseteq B_1 \subseteq \dots, \quad \lambda = \bigcup_{n < \omega} B_n.$$

Let

$$\mathfrak{A} = \bigcup_{k \in \omega} \mathfrak{A}^{\gamma_k} = \bigcup_{\beta < \alpha} \mathfrak{A}^\beta,$$

then $\mathfrak{A}$ is saturated with respect to all bounded subsets of λ , and $\mathfrak{A}$ contains only bounded subsets of λ . (“bounded”, “thin” and “thick” in this paragraph always refers to the representation (*).) There exists a saturated family $\mathfrak{B}_k \subseteq [A_k^\alpha - A_{k-1}^\alpha]^\omega$ (since $\text{card}(A_k^\alpha) < \lambda$, by (a)). Use 3.9 and 3.10 to find a family $\mathfrak{B}'_k$ of thin subsets of $A_k^\alpha - A_{k-1}^\alpha$ that is saturated with respect to all thin subsets of $A_k^\alpha - A_{k-1}^\alpha$. Let

$$\mathfrak{A}^\alpha = \mathfrak{A} \cup \bigcup_{k \in \omega} \mathfrak{B}'_k.$$

$\mathfrak{A}^\alpha$ is almost disjoint (hence (d)), because sets in $\mathfrak{A}$ are bounded and sets in $\mathfrak{B}'_k$ are thin. As in 3.12 we can use 3.11 to show that $\mathfrak{A}^\alpha$ is saturated with respect to all subsets of all A_k^α 's. This implies (e). (f) is clear.

Case 4. α limit, C'_α bounded in α

Let $\beta_0 = \sup C'_\alpha$, and let $\beta_0 < \beta_1 < \dots$ be increasing with limit α .

Let $k_0 = 0$, and for $n > 0$ let k_n be the first number such that

$$k_n > k_{n-1},$$

$$(*1) \quad \forall k \geq k_n : A_k^{\beta_{n-1}} \subseteq A_k^{\beta_n}.$$

(Such a k_n exists, by (b)).

Note that this implies:

$$(*2) \quad \forall m \leq n \forall k \geq k_n : A_k^{\beta_m} \subseteq A_k^{\beta_n}.$$

Now let, for $k_n \leq k < k_{n+1}$, $A_k^\alpha = A_k^{\beta_n}$, and let

$$\mathfrak{A}^\alpha = \bigcup_{\beta < \alpha} \mathfrak{A}^\beta = \bigcup_{n \in \omega} \mathfrak{A}^{\beta_n}.$$

Then we have

$$(*3) \quad \forall k \geq k_m : A_k^{\beta_m} \subseteq A_k^\alpha.$$

Proof. Let $k_n \leq k < k_{n+1}$. Then $n \geq m$. Now apply (*2).

We have to check that (a)–(f) are satisfied for α .

Since any $\gamma \in C'_\alpha$ is $\leq \beta_0$, we have for all $k \geq k_0 (= 0)$, $A_k^\gamma \subseteq A_k^{\beta_0} \subseteq A_k^\alpha$. This proves (c).

For any $\beta < \alpha$, let $\beta < \beta_n$, then (by (*3)) for sufficiently large k , $A_k^\beta \subseteq A_k^{\beta_n} \subseteq A_k^\alpha$. This proves (b).

To show (a), note that $A_k^\alpha \subseteq A_{k+1}^\alpha$ follows easily from the construction if $k+1$ is not of the form k_{n+1} . If $k+1 = k_{n+1}$, then

$$A_k^\alpha = A_k^{\beta_n} \subseteq A_{k+1}^{\beta_n} \subseteq A_{k+1}^{\beta_{n+1}} = A_{k+1}^\alpha,$$

where the second inclusion follows from (*1).

$\mathfrak{A}^\alpha$ is saturated on each A_k^α , since $A_k^\alpha = A_k^\beta$ for some $\beta < \alpha$, so we can apply the induction hypothesis (e) for β . Hence (e) holds.

(d) and (f) are clear.

Case 5. α a successor. Let $\alpha = \beta + 1$.

If for some k $x_\beta \subseteq A_k^\beta$, then let $A_k^\alpha = A_k^\beta$ for all k , and $\mathfrak{A}^\alpha = \mathfrak{A}^\beta$.

Otherwise, let $\mathfrak{B} \subseteq [x_\beta]^\omega$ be a saturated family that contains all infinite sets of the form $x_\beta \cap (A_k^\beta - A_{k-1}^\beta)$. Let $\mathfrak{B}'$ be the family of all thin sets in $\mathfrak{B}$, with respect to

$$x_\beta = \bigcup_k (A_k^\beta \cap x_\beta).$$

Let

$$\begin{aligned} \forall k \quad A_k^\alpha &= A_k^\beta \cup x_\beta, \\ \mathfrak{A}^\alpha &= \mathfrak{A}^\beta \cup \mathfrak{B}'. \end{aligned}$$

Then $\mathfrak{A}^\alpha$ is an almost disjoint family, and it is saturated with respect to all subsets of x_β .

This finishes the construction of the families $(\mathfrak{A}^\alpha)$. Let $\mathfrak{A} = \bigcup \mathfrak{A}^\alpha$. By (f), $\mathfrak{A}$ will be an almost disjoint family. To show that $\mathfrak{A}$ is saturated, consider any $x \in [\lambda]^\omega$: For some $\beta < \lambda^+$, $x \subseteq x_\beta$. By the last sentence in Case 5, $\mathfrak{A}$ is saturated with respect to x .

Conclusion

$$V = K \rightarrow \forall \kappa S(\kappa)$$

and in fact

$$(1)-(3) \text{ from } \S 1 \rightarrow \forall \kappa S(\kappa).$$

Proof. Assume not, and let $\lambda = \min\{\kappa : \neg S(\kappa)\}$. Clearly λ is an uncountable cardinal. By the result in §3, λ cannot be a successor cardinal, and it cannot have uncountable cofinality. By CH, $\lambda > \omega$. Since by definition of λ we have $\forall \mu < \lambda S(\mu)$, the construction in this section shows that $S(\lambda)$, a contradiction.

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