COMPLETELY BOUNDED MAPS BETWEEN THE PREDUALS OF VON NEUMANN ALGEBRAS

HIROYUKI OSAKA

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ABSTRACT. Let M and N be von Neumann algebras. Then $B[M_*, N_*] = CB[M_*, N_*]$ if and only if either M is strictly finite of type I or N is finite-dimensional.

0. INTRODUCTION

In the recent development of the theory of operator algebras, it has been recognized that completely bounded maps play an important role in the study of the matricial structure of C^* -algebras (cf. [2], [12], [14]). It is of interest to know the difference between complete boundedness and mere boundedness. Smith showed in [9] that every bounded map from a C^* -algebra A to a C^* -algebra B is completely bounded if and only if either A is finite-dimensional or B is subhomogeneous. Huruya and Tomiyama showed the same result in a slightly different formulation in [3]. Recently, Effros and Ruan have systematically investigated matricially normed spaces and shown that any bounded linear map from the dual of a C^* -algebra A into a C^* -algebra B is completely bounded [1]. On the other hand, Ruan has shown that the only completely bounded map from a C^* -algebra A into the dual of a C^* -algebra B is the zero map [8]. Compared with the case of completely positive maps [10], these are contrasting results.

In this paper we shall investigate the relationship of the set of bounded maps to the subspace of completely bounded maps between the preduals of von Neumann algebras.

1. PRELIMINARIES

We denote by M_n the $n \times n$ matrix algebra over the complex number field $\mathbb{C}$. Let A be a C^* -algebra and M a von Neumann algebra. Let $M_n(A)$ be the C^* -algebra of all $n \times n$ matrices $a = [a_{i,j}]$ with entries in A . The $n \times n$

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matrix space over the dual space A^* , $M_n(A^*) = \{f = [f_{i,j}]; f_{i,j} \in A^*\}$ is regarded as the dual of $M_n(A)$ by

$$\langle [a_{i,j}], [f_{i,j}] \rangle = \sum_{i,j=1}^n f_{i,j}(a_{i,j}),$$

where $[a_{i,j}] \in M_n(A)$ and $[f_{i,j}] \in M_n(A^*)$. Let M_* be the predual of M .

Let E and F be C^* -algebras, the duals of C^* -algebras, or the preduals of von Neumann algebras. For a linear map $\varphi: E \rightarrow F$, we define $\varphi \otimes \text{id}_n: M_n(E) \rightarrow M_n(F)$ by $(\varphi \otimes \text{id}_n)([a_{i,j}]) = [\varphi(a_{i,j})]$. If $\sup_n \|\varphi \otimes \text{id}_n\| < \infty$, we say that φ is completely bounded. If φ is completely bounded, we put the norm $\|\varphi\|_{cb} = \sup_n \|\varphi \otimes \text{id}_n\|$. We denote by $B[E, F]$ and $CB[E, F]$ the set of bounded linear maps and the subspace of completely bounded maps of E to F , respectively.

2. MAIN RESULT

Let M and N be von Neumann algebras. The following lemma is a slight modification of [3, Lemma 1].

Lemma 2.1. *Let M be a von Neumann algebra containing a sequence $\{a_i\}_{i=1}^\infty$ of positive elements with $\|a_i\| = 1$ and $a_i a_j = a_j a_i = 0$ ($i \neq j$). Let B and C denote von Neumann subalgebras generated by $\{a_i\}_{i=1}^n$ and $\{a_i\}_{i=n+1}^\infty$, respectively. Then, for the integer $m = 2^n$, there exist an element $b_n \in B \otimes M_m$ and a σ -weakly continuous linear map $\Phi_n: M \rightarrow M_m$ such that $\|b_n\| \leq 1$, $\|\Phi_n\| \leq n^{-1/8}$, $\Phi_n|_C \equiv 0$, and $\|(\Phi_n \otimes \text{id}_m)(b_n)\| \geq 2^{-3/2} n^{3/8}$.*

Proof. At first we choose elements $c_1, \dots, c_n$ of M_m such that $\|c_i\| = 1$ and $\|\sum_{i=1}^n \alpha_i c_i\| \leq (2 \sum_{i=1}^n |\alpha_i|^2)^{1/2}$ for any α_i in $\mathbb{C}$, $1 \leq i \leq n$ [6], [16]. We define the map $\varphi_n: \ell^\infty(n) \rightarrow M_m$ by

$$\varphi_n(x) = \sum_{i=1}^n x(i) / (2n^{5/4})^{1/2} c_i, \quad x \in \ell^\infty(n),$$

where $x(i)$ means the i th component of x . By the above inequality, we have

$$\|\varphi_n(x)\| \leq \left(2 \sum_{i=1}^n |x(i)|^2 / (2n^{5/4}) \right)^{1/2} \leq n^{-1/8} \|x\|,$$

so that $\|\varphi_n\| \leq n^{-1/8}$.

Let $d_n = (1/2) \sum_{i=1}^n \delta_i \otimes c_i$, where δ_i is the element of $\ell^\infty(n)$ such that $\delta_i(k) = \delta_{i,k}$. Then $\|d_n\| = 1/2$ and

$$(\varphi_n \otimes \text{id}_m)(d_n) = 2^{-3/2} n^{-5/8} \sum_{i=1}^n c_i \otimes c_i.$$

Here we make use of the unit vector z in $\mathbb{C}^m \otimes \mathbb{C}^m$ such that $(c_i \otimes c_i)(z) = z$ for all i constructed in Loeb [6]. Hence we have

$$\|\varphi_n\|_{cb} \geq \|(\varphi_n \otimes \text{id}_m)(d_n)\| \geq 2^{-3/2} n^{3/8}.$$

Suppose that M acts on Hilbert space $\mathcal{H}$. Then $s(a_i)s(a_j) = s(a_j)s(a_i) = 0$ ($i \neq j$), where $s(a_i)$ is the support projection of a_i in the von Neumann algebra generated by $\{a_i\}_{i=1}^\infty$. Since $a_i \geq 0$ and $\|a_i\| = 1$ are assured, there exists a unit vector $\xi_i \in s(a_i)\mathcal{H}$ such that $1/2 < \omega_{\xi_i}(a_i) \leq 1$ and $\omega_{\xi_i}(a_j) = 0$ ($i \neq j$), where $\omega_{\xi_i}(a_i) = (a_i\xi_i|\xi_i)$. If $\omega_{\xi_i}(a_i) \neq 1$, we put $\eta_i = \xi_i/(\omega_{\xi_i}(a_i))^{1/2}$; then $\omega_{\eta_i}(a_i) = 1$, and $1 \leq \|\omega_{\eta_i}\| < 2$. Hence there exists a family of normal linear functionals $\{\rho_i\}_{i=1}^n$ on M such that $\rho_i \geq 0$, $\|\rho_i\| \leq 2$, and $\rho_i(a_j) = \delta_{i,j}$ ($1 \leq i \leq n$ and $1 \leq j < \infty$).

We now define a completely positive linear map $\varphi: \ell^\infty(n) \rightarrow M$ and a σ -weakly continuous linear map $\psi: M \rightarrow \ell^\infty(n)$ by

$$\varphi(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \alpha_i a_i \quad \text{and} \quad \psi(a) = \frac{1}{2}(\rho_1(a), \dots, \rho_n(a)).$$

Both maps are contractive and, $2\psi\varphi$ is the identity map on $\ell^\infty(n)$.

Put $\Phi_n = \varphi_n \psi$ and $b_n = (\varphi \otimes \text{id}_m)(2d_n) \in B \otimes M_m$. Then we have $\|b_n\| \leq 1$, $\|\Phi_n\| \leq \|\varphi_n\| \|\psi\| \leq n^{-1/8}$, $\Phi_n|C \equiv 0$, and

$$\begin{aligned} 2^{-3/2} n^{3/8} &\leq \|(\varphi_n \otimes \text{id}_m)(d_n)\| \\ &= \|(\varphi_n(2\psi\varphi) \otimes \text{id}_m)(d_n)\| \\ &= \|(\varphi_n \psi \otimes \text{id}_m)(\varphi \otimes \text{id}_m)(2d_n)\| \\ &= \|(\Phi_n \otimes \text{id}_m)(b_n)\|. \quad \square \end{aligned}$$

Definition 2.2. A finite type I von Neumann algebra N is called *strictly finite* if $N = \bigoplus_{i=1}^n N_i$, where N_i is of type $I_{n(i)}$ and $\sup_i n(i) < \infty$.

Remark 2.3. From the above definition, if N is not a strictly finite von Neumann algebra of type I, then we can embed $\bigoplus_{n=1}^\infty M_{2^n}$ into N as a von Neumann subalgebra (cf. [13]).

Lemma 2.4. Suppose that M is not a strictly finite von Neumann algebra of type I and N is an infinite-dimensional von Neumann algebra. Then there exists a σ -weakly continuous bounded linear map Φ of N into M which is not completely bounded.

Proof. Let a be a self-adjoint element of N with infinite spectrum [7] and let $W^*(a)$ be the von Neumann subalgebra generated by a . By an elementary spectral argument, we can find a commuting sequence $\{a_{i,j}\}_{i,j=1}^\infty$ of positive element of $W^*(a)$ with disjoint supports and $\|a_{i,j}\| = 1$. For each $n \in \mathbb{N}$, let A_n be the von Neumann subalgebra generated by $\{a_{n,i}\}_{i=1}^n$ and let the integer $m = 2^n$. By Lemma 2.1, there exist an element $b_n \in A_n \otimes M_m$ and a σ -weakly continuous linear map Φ_n of N into M_m such that $\|b_n\| \leq 1$, $\|\Phi_n\| \leq n^{-1/8}$, $\Phi_n|A_k \equiv 0$ ($k \neq n$), and $\|(\Phi_n \otimes \text{id}_m)(b_n)\| \geq 2^{-3/2} n^{3/8}$.

Now we put $\Phi = \bigoplus_{n=1}^\infty \Phi_n$; then $\|\Phi\| = \sup \|\Phi_n\| \leq 1$. Hence Φ is a σ -weakly continuous bounded linear map N into $\bigoplus_{n=1}^\infty M_m$. Since M is not

strictly finite of type I, we can embed $\bigoplus_{n=1}^{\infty} M_m$ into M as a von Neumann subalgebra. Hence Φ is a σ -weakly continuous bounded linear map of N into M .

If Φ is completely bounded, we have $\|\Phi\|_{cb} \geq \|\Phi|W^*(a)\|_{cb}$. For each $k \in \mathbb{N}$, it is easy to see that $\Phi \otimes \text{id}_k = \bigoplus_{n=1}^{\infty} \Phi_n \otimes \text{id}_k$, so $\|\Phi \otimes \text{id}_k\| = \sup_n \|\Phi_n \otimes \text{id}_k\|$. Since $\|\Phi_n \otimes \text{id}_m\| \geq 2^{-3/2} n^{3/8}$ is assured by Lemma 2.1, we have

$$\|\Phi\|_{cb} \geq \|\Phi|A_n\|_{cb} \geq 2^{-3/2} n^{3/8} \quad \text{for each } n \in \mathbb{N}.$$

Hence, Φ is not completely bounded. $\square$

Theorem 2.5. *Let M and N be two von Neumann algebras. Then the following assertions are equivalent:*

- (1) $B[M_*, N_*] = CB[M_*, N_*]$
- (2) *Either M is strictly finite of type I or N is finite-dimensional.*

Proof. (1) $\rightarrow$ (2): It follows from the previous lemma.

(2) $\rightarrow$ (1): It follows from Smith's result. $\square$

Corollary 2.6. *Let A and B be C^* -algebras. Then $B[A^*, B^*] = B[A^*, B^*]$ if and only if either A is subhomogeneous or B is finite-dimensional.*

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKASAWA, SETAGAYA-KU 158, TOKYO, JAPAN