CONVERGENCE OF DIFFERENCE APPROXIMATIONS AND NONLINEAR SEMIGROUPS

SHIGEO KANDA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a convergence theorem for the difference approximation for the evolution equation $(d/dt)u(t) \in Au(t)$ and a generation theorem of nonlinear semigroups for *directed* dissipative operators A in a real Hilbert space.

INTRODUCTION

In this paper we consider the evolution equation:

$$(DE) \quad (d/dt)u(t) \in Au(t), \quad t \in [0, T),$$

for a given (multivalued) nonlinear operator A in a real Hilbert space.

After a pioneering work of Kōmura [4], many authors have treated the generation of nonlinear semigroups having dissipative or ω -dissipative operator as generators. Our main purpose is to introduce a notion of directed L -dissipative operators as a generalization of ω -dissipative operators and to give a convergence theorem for difference approximations for (DE) and a generation theorem of nonlinear semigroups through it. In §1 we state definitions and a convergence theorem for difference approximation for (DE) and the proofs are given in §2. In §3 we give a generation theorem of nonlinear semigroups through the convergence theorem in §1.

1. DEFINITIONS AND CONVERGENCE OF DIFFERENCE APPROXIMATIONS

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let A be a (multivalued) operator in H with domain $D(A)$ and range $R(A)$. We say that A is directed L -dissipative if the following conditions are satisfied:

- (i) for every $\lambda > 0$, $(1 - \lambda A)^{-1}$ is a single-valued operator.
- (ii) $\langle x' - y', x - y \rangle \leq L(\|x - y\|)\|x - y\|$ for $x, y \in D(A)$, $x' \in Ax$, and $y' \in Ay$, where L is an increasing function such that $L(0)$ equals zero and satisfies the condition, $\int_0^1 dt/L(t) = \infty$.

Received by the editors January 26, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47H10; Secondary 34A34.

Key words and phrases. Dissipative, evolution equation, semigroup, Hilbert space.

©1991 American Mathematical Society
 0002-9939/91 \$1.00 + \$.25 per page

- (iii) $\langle x' - x'', \lambda x' \rangle \leq 0$ for $\lambda > 0$, $x \in D(A)$, $x' \in Ax$, $x - \lambda x' \in D(A)$, and $x'' \in A(x - \lambda x')$.

We note that condition (iii) is an analogue to that of *time-like analyticity* in Furuya [2]. Our main theorem is the following:

Theorem 1. *Let A be a directed L -dissipative operator in H satisfying the range condition*

$$(R1) \quad R(1 - \lambda A) \supset D(A) \quad \text{for all } \lambda > 0.$$

Let $T > 0$ and $T = nh_n$, where n is a positive integer. Let $u_0 \in D(A)$. We define the simple functions u_n on $[0, T]$ by

$$u_n(t) = (1 - h_n A)^{-k} u_0 \quad \text{for } t \in ((k-1)h_n, kh_n] \text{ and } u_n(0) = u_0.$$

Then $\{u_n(t)\}$ converges uniformly on $[0, T]$ as $n \rightarrow \infty$ and if we let u be the limit function of $\{u_n\}$, then u is a Lipschitz continuous function with Lipschitz constant $|Au_0| \equiv \inf\{\|z\| \mid z \in Au_0\}$, that is,

$$(1) \quad \|u(t) - u(s)\| \leq |Au_0| |t - s| \quad \text{for } t, s \in [0, T].$$

2. PROOF OF THEOREM 1

We begin with the following lemma.

Lemma 1.1. *Let A be a directed L -dissipative operator in H satisfying the range condition (R1). Let $x_0, y_0 \in D(A)$. Let h, k be positive numbers such that $h = lk$, where l is a positive integer. Put $x_{n+1} = (1 - hA)^{-1}x_n$ and $y_{m+1} = (1 - kA)^{-1}y_m$ for $n, m = 0, 1, 2, \dots$. Let $M = \max(|Ax_0| + |Ay_0|, |Ay_0|^2)$, where $|Ax| \equiv \inf\{\|x'\| \mid x' \in Ax\}$. Let ε be a given positive number. If $\|x_n - y_{ln}\| \geq \varepsilon$ and $Mh < \min(\varepsilon/2, L(\varepsilon)\varepsilon)$, then the following estimate holds:*

$$(2) \quad \|x_{n+1} - y_{l(n+1)}\| - \|x_n - y_{ln}\| \leq 3hL(\|x_n - y_{ln}\| + Mh).$$

Proof. Let $x'_{n+1} \in Ax_{n+1}$. Let p be a nonnegative integer. Choose $y'_{l(n+1)-p} \in Ay_{l(n+1)-p}$ such that

$$y_{l(n+1)-p} = y_{l(n+1)-p-1} + ky'_{l(n+1)-p}.$$

Then

$$\begin{aligned} \|x_{n+1} - pkx'_{n+1} - y_{l(n+1)-p}\|^2 &= \|x_{n+1} - (p+1)kx'_{n+1} - y_{l(n+1)-p-1} \\ &\quad + k(x'_{n+1} - y'_{l(n+1)-p})\|^2 \\ &= \|x_{n+1} - (p+1)kx'_{n+1} - y_{l(n+1)-p-1}\|^2 \\ &\quad + 2k\langle x'_{n+1} - y'_{l(n+1)-p}, x_{n+1} - y_{l(n+1)-p} \rangle \\ &\quad - (p+1)k^2\|x'_{n+1} - y'_{l(n+1)-p}\|^2 - pk^2\|x'_{n+1}\|^2 \\ &\quad + pk^2\|y'_{l(n+1)-p}\|^2 \quad \text{for } p = 0, 1, 2, \dots, l-1. \end{aligned}$$

Choose $x'_{n+1} \in Ax_{n+1}$ such that $x_n = x_{n+1} - hx'_{n+1}$. Then, for any $x'_n \in Ax_n = A(x_{n+1} - hx'_{n+1})$,

$$0 \geq \langle x'_{n+1} - x'_n, x'_{n+1} \rangle = \|x'_{n+1}\|^2 - \langle x'_n, x'_{n+1} \rangle \geq \|x'_{n+1}\|^2 - \|x'_n\| \|x'_{n+1}\|.$$

Thus the following estimates hold:

$$(4) \quad \|x'_{n+1}\| \leq \|x'_n\| \leq |Ax_0|;$$

$$(5) \quad \|x_{n+1} - x_n\| = \|hx'_{n+1}\| \leq h|Ax_0|;$$

$$(6) \quad \begin{aligned} \|x_{n+1} - y_{l(n+1)-p}\| - \|x_n - y_{ln}\| &\leq \|x_{n+1} - x_n\| + \sum_{j=1}^{l-p} \|y_{ln+j} - y_{ln+j-1}\| \\ &\leq h|Ax_0| + (l-p)k|Ay_0| \\ &\leq Mh \quad \text{for } p = 0, 1, 2, \dots, l. \end{aligned}$$

From (3), using (4), (5), (6), and (ii) we have

$$\begin{aligned} \|x_{n+1} - pkx'_{n+1} - y_{l(n+1)-p}\|^2 - \|x_{n+1} - (p+1)kx'_{n+1} - y_{l(n+1)-p-1}\|^2 \\ \leq 2kL(\|x_n - y_{ln}\| + Mh)(\|x_n - y_{ln}\| + Mh) + Mpk^2. \end{aligned}$$

Adding these inequalities for $p = 0, 1, 2, \dots, l-1$, we get

$$\begin{aligned} \|x_{n+1} - y_{l(n+1)}\|^2 - \|x_n - y_{ln}\|^2 \\ \leq 2hL(\|x_n - y_{ln}\| + Mh)(\|x_n - y_{ln}\| + Mh) + Ml(l-1)k^2/2 \\ \leq 2hL(\|x_n - y_{ln}\| + Mh)(\|x_n - y_{ln}\| + Mh) + Mh^2. \end{aligned}$$

In addition, if $\|x_n - y_{ln}\| \geq \varepsilon > 2Mh$, then using (6) we have

$$\begin{aligned} \|x_{n+1} - y_{l(n+1)}\|^2 - \|x_n - y_{ln}\|^2 \\ = (\|x_{n+1} - y_{l(n+1)}\| - \|x_n - y_{ln}\|)(\|x_{n+1} - y_{l(n+1)}\| + \|x_n - y_{ln}\|) \\ \geq (\|x_{n+1} - y_{l(n+1)}\| - \|x_n - y_{ln}\|)(2\|x_n - y_{ln}\| - Mh) \\ \geq (\|x_{n+1} - y_{l(n+1)}\| - \|x_n - y_{ln}\|)(\|x_n - y_{ln}\| + Mh). \end{aligned}$$

Thus, if $\|x_n - y_{ln}\| \geq \varepsilon > 2Mh$ and $Mh < L(\varepsilon)\varepsilon$, then

$$\begin{aligned} \|x_{n+1} - y_{l(n+1)}\| - \|x_n - y_{ln}\| &\leq 2hL(\|x_n - y_{ln}\| + Mh) + Mh^2/\varepsilon \\ &\leq 2hL(\|x_n - y_{ln}\| + Mh) + hL(\varepsilon) \\ &\leq 3hL(\|x_n - y_{ln}\| + Mh). \end{aligned}$$

Lemma 1.2. Let A , x_0 , and y_0 be as in Lemma 1.1. Let $T > 0$ and $T = nh_n$, where n is a positive integer. Then for any $\varepsilon > 0$ there exist a positive number $\delta = \delta(\varepsilon)$ and a positive integer $N = N(\varepsilon, x_0, y_0)$ such that

$$\left\| \left(1 - \frac{t}{n}A\right)^{-n} x_0 - \left(1 - \frac{t}{ln}A\right)^{-ln} y_0 \right\| \leq \varepsilon$$

for $\|x_0 - y_0\| \leq \delta$, $n \geq N$, $l = 1, 2, 3, \dots$, and $t \in [0, T]$.

Proof. Given $\varepsilon > 0$, choose $\eta = \eta(\varepsilon) > 0$ such that $\int_{\eta}^{\varepsilon} dy/L(y) \geq 4T$. Setting $\delta = \delta(\varepsilon) = \eta/3$, choose $x_0, y_0 \in D(A)$ such that $\|x_0 - y_0\| \leq \delta$. Moreover, choose a positive integer $N = N(\varepsilon, x_0, y_0)$ such that $Mh_N = MT/N < \min(\eta/2, L(\varepsilon)\varepsilon)$. For simplicity, put

$$\alpha_k^n(t) = \left\| \left(1 - \frac{t}{n}A\right)^{-k} x_0 - \left(1 - \frac{t}{n}A\right)^{-lk} y_0 \right\| \quad \text{for } t \in [0, T].$$

Let $n \geq N$ and assume that $\alpha_n^n(t) > \varepsilon$ for some $t \in [0, T]$. Because $\alpha_0^n(t) = \|x_0 - y_0\| \leq \eta/3$ and by noting (6) we have

$$\begin{aligned} \alpha_{n-1}^n(t) &= \alpha_n^n(t) + \alpha_{n-1}^n(t) - \alpha_n^n(t) \\ &\geq \varepsilon - Mt/n \geq \varepsilon - Mh_N \\ &> \eta - \eta/2 = \eta/2, \end{aligned}$$

there exists a nonnegative integer $k_0 < k \leq n$ such that

$$\alpha_{k_0}^n(t) < \eta/2 \text{ and } \alpha_k^n(t) \geq \eta/2 \text{ for } k_0 < k \leq n.$$

For $t \in (0, T]$ and putting $h = t/n$ we have

$$\begin{aligned} 4T &\leq \int_{\eta}^{\varepsilon} \frac{dy}{L(y)} \leq \int_{\alpha_{k_0}^n(t)+Mh}^{\alpha_n^n(t)+Mh} \frac{dy}{L(y)} \\ &\leq \sum_{k=k_0}^{n-1} \frac{\alpha_{k+1}^n(t) + Mh - (\alpha_k^n(t) + Mh)}{L(\alpha_k^n(t) + Mh)} \\ &\leq \sum_{k=k_0}^{n-1} 3h = 3(n - k_0)h \leq 3t \leq 3T, \quad \text{by Lemma 1.1.} \end{aligned}$$

This is a contradiction. Hence $\alpha_n^n(t) \leq \varepsilon$ for $n \geq N$ and $t \in [0, T]$.

Proof of Theorem 1. Given $\varepsilon > 0$, put $\delta(\varepsilon) = \varepsilon/2(|Au_0| + 1)$. Put $x_k^n = (1 - h_n A)^{-k} u_0$. Let $n, m \geq 2T|Au_0|/\varepsilon \equiv N_1(\varepsilon)$. In the case $t \in [0, \delta(\varepsilon)]$, taking the integer k_n so that $t \in ((k_n - 1)h_n, k_n h_n]$, we obtain

$$\begin{aligned} \|u_n(t) - u_0\| &= \|x_{k_n}^n - u_0\| = \left\| \sum_{j=1}^{k_n} (x_j^n - x_{j-1}^n) \right\| \\ &\leq \sum_{j=1}^{k_n} \|x_j^n - x_{j-1}^n\| \leq \sum_{j=1}^{k_n} h_n |Au_0| \\ &\leq k_n h_n |Au_0| \leq (t + h_n) |Au_0| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Similarly we have

$$\|u_m(t) - u_0\| \leq \varepsilon.$$

Thus

$$\begin{aligned}\|u_n(t) - u_m(t)\| &\leq \|u_n(t) - u_0\| + \|u_0 - u_m(t)\| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon\end{aligned}$$

for $t \in [0, \delta(\varepsilon)]$ and $n, m \geq N_1(\varepsilon)$.

In the case $t \in [\delta(\varepsilon), T]$, by Lemma 1.2 there exists a positive integer $N_2(\varepsilon)$ such that

$$\left\| \left(1 - \frac{t}{k_n} A\right)^{-k_n} u_0 - \left(1 - \frac{t}{mk_n} A\right)^{-mk_n} u_0 \right\| \leq \varepsilon \quad \text{and} \quad h_n |Au_0| \leq \varepsilon$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \geq N_2(\varepsilon)$.

Thus

$$\begin{aligned}\|u_n(t) - u_{nm}(t)\| &= \|(1 - h_n A)^{-k_n} u_0 - (1 - h_{nm} A)^{-k_{nm}} u_0\| \\ &\leq \left\| \left(1 - \frac{k_n h_n}{k_n} (A)\right)^{-k_n} u_0 - \left(1 - \frac{k_n h_n}{mk_n} A\right)^{-mk_n} u_0 \right\| \\ &\quad + \|(1 - h_{nm} A)^{-mk_n} u_0 - (1 - h_{nm} A)^{-k_{nm}} u_0\| \\ &\leq \varepsilon + \|(1 - h_{nm} A)^{-mk_n} u_0 - (1 - h_{nm} A)^{-k_{nm}} u_0\| \\ &= \varepsilon + \left\| \sum_{j=k_{nm}+1}^{mk_n} (x_j^{nm} - x_{j-1}^{nm}) \right\| \\ &\leq \varepsilon + \sum_{j=k_{nm}+1}^{mk_n} \|x_j^{nm} - x_{j-1}^{nm}\| \\ &\leq \varepsilon + mh_{nm} |Au_0| = \varepsilon + h_n |Au_0| \leq 2\varepsilon\end{aligned}$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \geq N_2(\varepsilon)$. Similarly we have

$$\|u_{nm}(t) - u_m(t)\| \leq 2\varepsilon$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \geq N_2(\varepsilon)$.

Putting $N(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon))$,

$$\|u_n(t) - u_m(t)\| \leq 4\varepsilon$$

for $t \in [0, T]$ and $n, m \geq N(\varepsilon)$. Hence $\{u_n(t)\}$, converges uniformly on $[0, T]$. Put $u(t) = \lim_{n \rightarrow \infty} u_n(t)$ for $t \in [0, T]$. For $0 < s < t \leq T$, choose integers j, k such that $s \in ((j-1)h_n, jh_n]$ and $t \in ((k-1)h_n, kh_n]$. Then

$$\begin{aligned}\|u_n(t) - u_n(s)\| &= \|x_k^n - x_j^n\| \leq \sum_{i=j}^{k-1} \|x_{i+1}^n - x_i^n\| \leq \sum_{i=j}^{k-1} h_n |Au_0| \\ &\leq (k-j)h_n |Au_0| \leq (t-s+h_n) |Au_0|.\end{aligned}$$

Thus

$$\|u(t) - u(s)\| \leq (t-s) |Au_0| \quad \text{as } n \rightarrow \infty.$$

This holds for $s = 0$. Hence,

$$\|u(t) - u(s)\| \leq \|Au_0\| |t - s| \quad \text{for all } t, s \in [0, T].$$

3. GENERATION OF NONLINEAR SEMIGROUPS

Let A be a directed L -dissipative operator in H satisfying the condition (R1). Following Benilan [1], we define an integral solution u to the Cauchy problem:

$$(CP_T; u_0) \begin{cases} (d/dt)u(t) \in Au(t) & \text{for } t \in [0, T], \quad 0 < T \leq \infty, \\ u(0) = u_0 \end{cases}$$

as a continuous function on $[0, T]$ with $u(0) = u_0$ satisfying the inequality

$$(7) \quad \begin{aligned} & \|u(t) - x_0\|^2 - \|u(s) - x_0\|^2 \\ & \leq 2 \int_s^t \{ \langle y_0, u(\tau) - x_0 \rangle + L(\|u(\tau) - x_0\|) \|u(\tau) - x_0\| \} d\tau \end{aligned}$$

for $x_0 \in D(A)$, $y_0 \in Ax_0$, and $0 \leq s \leq t < T$.

It is easy to see that the limit function u as in Theorem 1 is an integral solution to $(CP_T; u_0)$, and satisfies

$$(8) \quad \|v(t) - u(t)\| - \|v(s) - u(s)\| \leq \int_s^t L(\|v(\tau) - u(\tau)\|) d\tau,$$

$0 \leq s \leq t < T$ for any integral solution $v(t)$ to $(CP_T; v_0)$. Such a solution is called a mild solution of $(CP_T; u_0)$.

Lemma 3.1. *Let $u(t) = u(t, u_0)$ be the limit function as in Theorem 1, and $v(t) = v(t, v_0)$ a mild solution to $(CP_T; v_0)$, then*

$$(9) \quad \lim_{\|v_0 - u_0\| \rightarrow 0} \|v(t, v_0) - u(t, u_0)\| = 0 \quad \text{uniformly on } [0, T].$$

In particular, if $v_0 = u_0$ then $v(t) \equiv u(t)$.

Proof. Putting $s = 0$ in (8),

$$(10) \quad \|v(t) - u(t)\| \leq \|v_0 - u_0\| + \int_0^t L(\|v(\tau) - u(\tau)\|) d\tau.$$

Putting $f(t) = \|v(t) - u(t)\|$ and $F(t) = \|v_0 - u_0\| + \int_0^t L(\|v(\tau) - u(\tau)\|) d\tau$, (10) is expressed as

$$(10') \quad f(t) \leq F(t).$$

Thus $F'(t) = L(f(t)) \leq L(F(t))$. Hence,

$$\int_0^{F(t)} \frac{ds}{L(s)} = \int_0^t \frac{F'(\tau)}{L(F(\tau))} d\tau \leq \int_0^t d\tau = t.$$

Since $\int_0^1 ds/L(s) = \infty$, $\lim_{\|v_0 - u_0\| \rightarrow 0} F(t) = 0$ uniformly on $[0, T]$. So (9) is obtained.

Lemma 3.2. *Let $u_0 \in \overline{D(A)}$. Then there exists a unique mild solution to $(CP_T; u_0)$.*

Proof. Since $u_0 \in \overline{D(A)}$, there exists a sequence $\{x_n\} \subset D(A)$ which goes to u_0 . Let $u(t, x_n)$ be the limit function with initial value x_n . By (9) we have

$$\lim_{n, m \rightarrow \infty} \|u(t, x_n) - u(t, x_m)\| = 0 \quad \text{uniformly on } [0, T].$$

Putting $u(t) = \lim_{n \rightarrow \infty} u(t, x_n)$, it is clear that $u(t)$ is a mild solution to $(CP_T; u_0)$.

Next, letting $v(t)$ be any mild solution to $(CP_T; u_0)$, using (9) again,

$$\lim_{n \rightarrow \infty} \|v(t) - u(t, x_n)\| = 0.$$

Thus

$$\|v(t) - u(t)\| \leq \|v(t) - u(t, x_n)\| + \|u(t, x_n) - u(t)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $v(t) \equiv u(t)$.

From Lemma 3.2, it is easily seen that there exists a unique mild solution $u(t, x)$ to $(CP_\infty; x)$ for any $x \in \overline{D(A)}$. Then we have the following theorem.

Theorem 2. *Let A be a directed L -dissipative operator satisfying the condition (R1). For each $x \in \overline{D(A)}$, let $u(t, x)$ be the unique mild solution to $(CP_\infty; x)$. Let $T(t)$ be an operator on $\overline{D(A)}$ such that $T(t)x = u(t, x)$. Then $\{T(t) | t \geq 0\}$ has semigroup property.*

Proof. It is obvious that $T(t)x = x$ and $T(t+s)x = T(t)T(s)x$ for $t, s \geq 0$ and $x \in \overline{D(A)}$. We show that $u(t, x): [0, \infty) \times \overline{D(A)} \rightarrow \overline{D(A)}$ is continuous. Assume that $(t_n, x_n), (t, x) \in [0, \infty) \times \overline{D(A)}$ and $(t_n, x_n) \rightarrow (t, x)$ as $n \rightarrow \infty$. Choose a sequence $\{x'_n\} \subset D(A)$ which converges to x . Then by (1),

$$\begin{aligned} \|u(t_n, x_n) - u(t, x)\| &\leq \|u(t_n, x_n) - u(t_n, x'_n)\| + \|u(t_n, x'_n) - u(t_n, x'_m)\| \\ &\quad + \|u(t_n, x'_m) - u(t, x'_m)\| + \|u(t, x'_m) - u(t, x)\| \\ &\leq \|u(t_n, x_n) - u(t_n, x'_n)\| + \|u(t_n, x'_n) - u(t_n, x'_m)\| \\ &\quad + |Ax'_m| |t_n - t| + \|u(t, x'_m) - u(t, x)\|. \end{aligned}$$

By (9), for any $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$\begin{aligned} \|u(t_n, x_n) - u(t_n, x'_n)\| &< \varepsilon/3, \\ \|u(t_n, x'_n) - u(t_n, x'_m)\| &< \varepsilon/3, \\ \|u(t, x'_m) - u(t, x)\| &< \varepsilon/3 \end{aligned}$$

for $n, m \geq N$. Thus

$$\begin{aligned} \|u(t_n, x_n) - u(t, x)\| &\leq \varepsilon/3 + \varepsilon/3 + |Ax'_m| |t_n - t| + \varepsilon/3 \\ &= \varepsilon + |Ax'_m| |t_n - t|. \end{aligned}$$

Hence

$$\overline{\lim}_{n \rightarrow \infty} \|u(t_n, x_n) - u(t, x)\| \leq \varepsilon.$$

Remark. Let X be a real Banach space. In the case A is a continuous mapping from a subset of $[a, b) \times X$ ($a < b \leq \infty$) into X , Iwamiya has given a result that guarantees existence and uniqueness of solutions under a weaker condition than (ii) in §1 (see [3] in detail) for the nonautonomous differential equation in X

$$(CP; \tau, z) \begin{cases} (d/dt)u(t) = A(t, u(t)), \\ u(t) = z, \end{cases}$$

giving (τ, z) in $[a, b) \times X$.

ACKNOWLEDGMENT

The author would like to express his hearty thanks to Professor Y. Kōmura for his invaluable advice and constant encouragement and Professors N. Tanaka and K. Kobayasi for their helpful advice.

REFERENCES

1. Ph. Benilan, *Équations d'évolution dans un espace de Banach quelconque et applications*, Thèse, Orsay, 1972.
2. K. Furuya, *Analyticity of nonlinear semigroups*, Israel J. Math. **68** (1989), 89–101.
3. T. Iwamiya, *Global existence of solutions to nonautonomous differential equations in Banach spaces*, Hiroshima Math. J. **13** (1983), 65–81.
4. Y. Kōmura, *Nonlinear semi-groups in Hilbert space*, J. Math. Soc. Japan **19** (1967), 493–507.

DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, 1-6-1 NISHI-WASEDA, SHINJUKU-KU, TOKYO 169, JAPAN