THE NEAT EMBEDDING PROBLEM AND THE NUMBER OF VARIABLES REQUIRED IN PROOFS

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ABSTRACT. By constructing special relation algebras we show that if $3 < \alpha < \omega$, then

$$\text{SNr}_3\text{CA}_\alpha \neq \text{SNr}_3\text{CA}_{3\alpha-7}$$

and there is a logically valid first-order sentence containing at most three variables with a proof in which every sentence has at most $3\alpha - 7$ variables, but no proof in which every sentence has at most α variables.

The *neat embedding problem* is Problem 2.12 of [HMT71], which reads: For $2 < \gamma < \omega$, is there a $\kappa < \omega$ such that $\text{SNr}_\gamma\text{CA}_{\gamma+\kappa} = \text{SNr}_\gamma\text{CA}_{\gamma+\kappa+1}$? It is reasonable to conjecture that the answer is “no.” (See [Mo69, p. 342²⁸].) The key case is $\gamma = 3$, for if the answer could be shown to be “no” for this case, then the same would be true for all other values of γ .

For every finite $\alpha > 3$, let $\phi(\alpha)$ be the least β such that $\text{SNr}_3\text{CA}_\alpha \neq \text{SNr}_3\text{CA}_\beta$. Then $\phi(\alpha)$ always exists, by [Mo69]. It is known that $\phi(4) = 5$ and $\phi(5) = 6$. The results in [Mo69] show that $\phi(\alpha) \leq 3\alpha!$. This can also be shown by the method of [Ma89], which uses the same combinatorial principle as [Mo69]. If the answer to Problem 2.12 is “no,” then $\phi(\alpha) = \alpha + 1$. Theorem 1 below shows that $\phi(\alpha) \leq 3\alpha - 7$. The proof is based on the fact that, as the cardinality of a set X increases, the number of 3-element subsets of X grows faster than any constant times the number of 2-element subsets of X .

Let $\mathcal{M}$ be a first-order language with at least countably many relation symbols of rank at least two. It is reasonable to expect that if $3 \leq \alpha < \omega$, then there is a logically valid 3-sentence X (only three variables are used in X , although each may appear many times) which is α -provable (has a proof in which every sentence is an α -sentence) but not $(\alpha + 1)$ -provable. If this could be proved it would also settle Problem 2.12. Using the methods and results of [Mo69], [Mo71], and [J69] (or [Ma89]), it is possible to prove that there is a 3-sentence which is $(3\alpha!)$ -provable but not α -provable. Using the algebras constructed in the proof of Theorem 1, we can show if $3 < \alpha < \omega$, then there is a 3-sentence X which is $(3\alpha - 7)$ -provable but not α -provable. For $\alpha = 4$ this

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yields the known result ([TG87, p. 93⁵–93⁸]) that there is a 3-sentence which is 5-provable but not 4-provable. For $\alpha = 5$ it yields a 3-sentence which is 8-provable but not 5-provable. However, by another construction we know there is a 6-provable 3-sentence which is not 5-provable. This construction also yields the result $\phi(5) = 6$ mentioned above.

Theorem 1. *Let $3 < \alpha < \omega$. Then $\text{SNr}_3 \text{CA}_\alpha \neq \text{SNr}_3 \text{CA}_{3\alpha-7}$. In fact, there is a simple finite relation algebra $\mathfrak{A}$ such that*

- (i) $B_\alpha \mathfrak{A}$ is an α -dimensional cylindrical basis for $\mathfrak{A}$,
- (ii) $\mathfrak{A}$ has no $(3\alpha - 7)$ -dimensional relational basis,
- (iii) $\mathfrak{A} \in \mathfrak{RA}^* \text{CA}_\alpha$,
- (iv) $\mathfrak{CA} B_3 \mathfrak{A} \in \text{Nr}_3 \text{CA}_\alpha$,
- (v) $\mathfrak{A} \notin \text{SRA}^* \text{CA}_{3\alpha-7}$,
- (vi) $\mathfrak{CA} B_3 \mathfrak{A} \notin \text{SNr}_3 \text{CA}_{3\alpha-7}$.

Proof. Let $\beta = 3\alpha - 7$. Let $\mathfrak{B} = \langle B, +, - \rangle$ be a finite Boolean algebra which has the following atoms:

d_κ and e_κ for $\kappa < \beta$,

$a_{\kappa\lambda}^\Gamma$, where $\kappa, \lambda < \beta$, $\kappa \neq \lambda$, $\Gamma \subseteq \beta$, and either $\Gamma = \emptyset$ or $|\Gamma| = \alpha - 2$.

Let $\text{At}\mathfrak{B}$ be the set of atoms of $\mathfrak{B}$. We need some terminology for the indices used in the notation of the atoms. The *first index* of the atoms d_κ , e_κ , and $a_{\kappa\lambda}^\Gamma$ is κ , and the *second index* of d_λ , e_λ , and $a_{\kappa\lambda}^\Gamma$ is λ . (It may be perhaps clearer to use the notation $d_{\kappa\kappa}$ and $e_{\kappa\kappa}$ in place of d_κ and e_κ .) Define an operation $\sim$ on $\text{At}\mathfrak{B}$ as follows.

$$\check{d}_\kappa = d_\kappa, \check{e}_\kappa = e_\kappa, \quad \text{and} \quad (a_{\kappa\lambda}^\Gamma)^\sim = a_{\lambda\kappa}^\Gamma.$$

Extend this operation to all the elements of $\mathfrak{B}$:

$$\check{x} = \sum \{ \check{a} : x \geq a \in \text{At}\mathfrak{B} \}.$$

Let C be the set consisting of all the following triples of atoms of $\mathfrak{B}$:

$$\begin{aligned} &\langle d_\kappa, d_\kappa, d_\kappa \rangle, \langle e_\kappa, e_\kappa, e_\kappa \rangle, \langle d_\kappa, e_\kappa, e_\kappa \rangle, \langle e_\kappa, d_\kappa, e_\kappa \rangle, \langle e_\kappa, e_\kappa, d_\kappa \rangle, \\ &\langle d_\kappa, a_{\kappa\lambda}^\Gamma, a_{\kappa\lambda}^\Gamma \rangle, \langle a_{\kappa\lambda}^\Gamma, d_\lambda, a_{\kappa\lambda}^\Gamma \rangle, \langle a_{\kappa\lambda}^\Gamma, a_{\lambda\kappa}^\Gamma, d_\kappa \rangle, \\ &\langle e_\kappa, a_{\kappa\lambda}^\Gamma, a_{\kappa\lambda}^\Delta \rangle, \langle a_{\kappa\lambda}^\Gamma, e_\lambda, a_{\kappa\lambda}^\Delta \rangle, \langle a_{\kappa\lambda}^\Gamma, a_{\lambda\kappa}^\Delta, e_\kappa \rangle, \\ &\langle a_{\kappa\lambda}^\Gamma, a_{\lambda\mu}^\Delta, a_{\kappa\mu}^\Sigma \rangle, \text{ where } \lambda \in \Sigma \text{ or } \kappa \in \Delta \text{ or } \mu \in \Gamma, \\ &\langle a_{0\lambda}^\emptyset, a_{\lambda 1}^\emptyset, a_{01}^\emptyset \rangle, \langle a_{1\lambda}^\emptyset, a_{\lambda 0}^\emptyset, a_{10}^\emptyset \rangle, \langle a_{01}^\emptyset, a_{1\lambda}^\emptyset, a_{0\lambda}^\emptyset \rangle, \langle a_{10}^\emptyset, a_{0\lambda}^\emptyset, a_{1\lambda}^\emptyset \rangle, \\ &\langle a_{\lambda 0}^\emptyset, a_{01}^\emptyset, a_{\lambda 1}^\emptyset \rangle, \langle a_{\lambda 1}^\emptyset, a_{10}^\emptyset, a_{\lambda 0}^\emptyset \rangle, \text{ where } 1 < \lambda < \beta. \end{aligned}$$

Then define a binary operation $;$ on $\mathfrak{B}$ by

$$x ; y = \sum \{ c : x \geq a \in \text{At}\mathfrak{B}, y \geq b \in \text{At}\mathfrak{B}, \langle a, b, c \rangle \in C \}.$$

Set $1' = \sum_{\kappa < \beta} d_\kappa$, and let $\mathfrak{A} = \langle B, +, -, ;, \sim, 1' \rangle$.

Next we will prove that $\mathfrak{A} \in \text{WA}$. From the way that $\sim$ and $\cdot$ are extended to all the elements of $\mathfrak{B}$, it follows that $\mathfrak{A}$ is isomorphic to the complex algebra of the relational structure $\langle \text{At}\mathfrak{B}, C, \sim, \{d_\kappa : \kappa < \beta\} \rangle$. Theorem 2.2 of [Ma82] gives necessary and sufficient conditions for the complex algebra of such a structure to be an algebra in WA , so we need only verify those conditions. It is easy to see, from an inspection of the triples in C , that if $\langle x, y, z \rangle \in C$, then $\langle \check{x}, z, y \rangle \in C$ and $\langle z, \check{y}, x \rangle \in C$. Thus conditions (a) and (b) of Theorem 2.2 hold. Furthermore, if $\langle d_\kappa, x, y \rangle \in C$ then $x = y$, and for any $x \in \text{At}\mathfrak{B}$, there is some $\kappa < \beta$ such that $\langle d_\kappa, x, x \rangle \in C$. Thus condition (c) of Theorem 2.2 holds, and we conclude that $\mathfrak{A} \in \text{NA}$. To prove that condition (d) holds, and that therefore $\mathfrak{A} \in \text{WA}$, we need only assume $\langle d_\kappa, x, x \rangle, \langle x, y, z \rangle \in C$ and show that $\langle d_\kappa, z, z \rangle \in C$. Notice that all the triples in C have the property that the first index of the first atom in a triple is the same as the first index of the last atom in that triple. Thus the first index of d_κ , namely κ , is the same as the first index of x , which is in turn the same as the first index of z . So the first index of z is κ , and either $z = d_\kappa$, or $z = e_\kappa$, or $z = a_{\kappa\lambda}^\Gamma$ for some λ, Γ . In all three cases it happens that $\langle d_\kappa, z, z \rangle \in C$. Thus $\mathfrak{A} \in \text{WA}$.

Next we show that $B_\alpha \mathfrak{A}$ is an α -dimensional cylindrical basis for $\mathfrak{A}$. To do so we must verify conditions (C_0) , (C_1) , and (C_2) of Definition 4 in [Ma89]. First, condition (C_0) follows from the fact that $\mathfrak{A} \in \text{WA}$, by Lemma 5 of [Ma89]. Some very simple computations serve to show that condition (C_2) holds when $M = B_\alpha \mathfrak{A}$. Thus what remains to show is (C_1) . To do so, we assume $u, w \in B_\alpha \mathfrak{A}$, $\kappa, \lambda < \alpha$, $\kappa \neq \lambda$, and u agrees with w off of κ, λ . Then we must show that there is some v in $B_\alpha \mathfrak{A}$ which agrees with u off of κ , and agrees with w off of λ . These required conditions determine all the values of v except $v_{\kappa\lambda}$ and $v_{\lambda\kappa}$ as follows:

$$v_{\mu\nu} = \begin{cases} u_{\mu\nu} = w_{\mu\nu} & \text{if } \kappa, \lambda \neq \mu, \nu < \alpha, \\ u_{\mu\nu} & \text{if } \kappa \neq \mu, \nu < \alpha, \\ w_{\mu\nu} & \text{if } \lambda \neq \mu, \nu < \alpha. \end{cases}$$

Then $v_{\kappa\lambda}$ and $v_{\lambda\kappa}$ may be chosen as follows. In each case, the verification that $v \in B_\alpha \mathfrak{A}$ is routine.

Case 1. If $w_{\kappa\mu} \leq 1'$ for some $\mu \in \alpha \sim \{\kappa, \lambda\}$, then $v_{\kappa\lambda} = u_{\mu\lambda}$ and $v_{\lambda\kappa} = u_{\lambda\mu}$.

Case 2. If $u_{\mu\lambda} \leq 1'$ for some $\mu \in \alpha \sim \{\kappa, \lambda\}$, then $v_{\kappa\lambda} = w_{\kappa\mu}$ and $v_{\lambda\kappa} = w_{\mu\kappa}$.

Case 3. Assume $w_{\kappa\mu} + u_{\mu\lambda} \leq 0'$ whenever $\mu \in \alpha \sim \{\kappa, \lambda\}$. Let ρ be the first index of $u_{\kappa\kappa}$. (Since $u_{\kappa\kappa} \leq 1'$, ρ is also the second index of $u_{\kappa\kappa}$.) For every $\mu \in \alpha \sim \{\kappa, \lambda\}$, we have $u_{\kappa\mu} \leq u_{\kappa\kappa}; u_{\kappa\mu}$ since $u \in B_\alpha \mathfrak{A}$, hence $\langle u_{\kappa\kappa}, u_{\kappa\mu}, u_{\mu\mu} \rangle \in C$. It follows that ρ is the first index of $u_{\kappa\mu}$. Let σ be the second (and first) index of $w_{\lambda\lambda}$. By similar reasoning, we get that σ is the second index of $w_{\mu\lambda}$ for every $\mu \in \alpha \sim \{\kappa, \lambda\}$. If $\sigma = \rho$, we simply let $v_{\kappa\lambda} = v_{\lambda\kappa} = e_\sigma = e_\rho$. We may therefore assume $\sigma \neq \rho$. We will let $v_{\kappa\lambda} = a_{\rho\sigma}^\Gamma$

and $v_{\lambda\kappa} = a_{\sigma\rho}^\Gamma$ for some appropriately chosen Γ , to which we now turn.

Let $\mu \in \alpha \sim \{\kappa, \lambda\}$. Then $u_{\kappa\mu} \leq u_{\kappa\mu}; u_{\mu\mu}$, so $\langle u_{\kappa\mu}, u_{\mu\mu}, u_{\kappa\mu} \rangle \in C$. Notice that all the triples in C have the property that the second index of the second atom in a triple is the same as the second index of the last atom in that triple. Thus $u_{\mu\mu}$ and $u_{\kappa\mu}$ have the same second index. Similarly, $w_{\mu\mu}$ and $w_{\mu\lambda}$ have the same first index. But the first and second indices of $u_{\mu\mu}$ and $w_{\mu\mu}$ are all the same, since $u_{\mu\mu} = w_{\mu\mu} \leq 1$. Thus the second index of $u_{\kappa\mu}$ is the same as the first index of $w_{\mu\lambda}$. Let that common index be τ . Then we have one of the following three possibilities:

$$\begin{aligned} u_{\kappa\mu} &= a_{\rho\tau}^\Delta & \text{and} & & w_{\mu\lambda} &= a_{\tau\sigma}^\Sigma, \\ u_{\kappa\mu} &= a_{\rho\tau}^\Delta & \text{and} & & w_{\mu\lambda} &= e_\tau = e_\sigma, \\ u_{\kappa\mu} &= e_\rho = e_\tau & \text{and} & & w_{\mu\lambda} &= a_{\tau\sigma}^\Sigma, \end{aligned}$$

A comparison of these possibilities with the definition of C reveals that if $\tau \in \Gamma$, then $\langle u_{\kappa\mu}, w_{\mu\lambda}, a_{\sigma\rho}^\Gamma \rangle \in C$. Now the index τ depends on the choice of $\mu \in \alpha \sim \{\kappa, \lambda\}$, but there are only $\alpha - 2$ such choices, so there are at most $\alpha - 2$ such indices. Take Γ to be any subset of α which contains all the indices which arise in this way. Set $v_{\kappa\lambda} = a_{\sigma\rho}^\Gamma$ and $v_{\lambda\kappa} = a_{\rho\sigma}^\Gamma$. It follows that $v_{\kappa\lambda} = u_{\kappa\mu}; w_{\mu\lambda}$ for every $\mu \in \alpha \sim \{\kappa, \lambda\}$, and from this it is easy to show that $v \in B_\alpha \mathfrak{A}$. Thus (i) holds. Applying (i) with $\alpha = 4$, we get that $B_4 \mathfrak{A}$ is a 4-dimensional cylindrical basis, which implies that $\mathfrak{A}$ is a relation algebra.

So far the argument works for all $\alpha \geq 3$. We need $\alpha > 3$ in the proof of (ii), since the conclusion of (ii) contradicts (i) when $\alpha = 3$. We prove (ii) by contradiction. Suppose that $M \subseteq B_\beta \mathfrak{A}$ is a β -dimensional relational basis for $\mathfrak{A}$. By (R_0) of Definition 3, [Ma89], there is some $u \in M$ such that $u_{01} = a_{01}^\emptyset$. Since $a_{01}^\emptyset \leq \prod_{1 < \lambda < \beta} a_{0\lambda}^\emptyset; a_{\lambda 1}^\emptyset$, we can repeatedly apply (R_1) , obtaining some $v \in M$ such that

$$(1) \quad v_{01} = a_{01}^\emptyset, \quad v_{0\lambda} = a_{0\lambda}^\emptyset, \quad v_{\lambda 1} = a_{\lambda 1}^\emptyset, \quad \text{for } 1 < \lambda < \beta.$$

Note that v is designed so that the first and second indices of $v_{\kappa\lambda}$ are κ and λ , respectively. By the definition of C , if $\kappa, \lambda \in \{2, \dots, \beta - 1\}$ and $\kappa \neq \lambda$ then there is some nonempty Γ such that $v_{\kappa\lambda} = a_{\kappa\lambda}^\Gamma$ and $0, 1 \in \Gamma$. For every 3-element subset $\{\kappa, \lambda, \mu\}$ of $\{2, \dots, \beta - 1\}$, we have $v_{\kappa\lambda}; v_{\lambda\mu} \geq v_{\kappa\mu}$, hence $\{\kappa, \lambda, \mu\}$ contains some 2-element subset, say $\{\kappa, \lambda\}$, such that $\mu \in \Gamma$, where $v_{\kappa\lambda} = a_{\kappa\lambda}^\Gamma$. Whenever this happens, we say that $\{\kappa, \lambda\}$ accounts for $\{\kappa, \lambda, \mu\}$. Every 3-element subset of $\{2, \dots, \beta - 1\}$ must be accounted for by one of its 2-element subsets. Since $|\Gamma| = \alpha - 2$ and $0, 1 \in \Gamma$, every 2-element subset of $\{2, \dots, \beta - 1\}$ can account for only $\alpha - 4$ of the 3-element subsets of $\{2, \dots, \beta - 1\}$. But $(\alpha - 4) \binom{\beta - 2}{2} < \binom{\beta - 2}{3}$, so we have a contradiction. Therefore $\mathfrak{A}$ has no β -dimensional relational basis, i.e., (ii) holds.

From the fact that $B_\alpha \mathfrak{A}$ is an α -dimensional cylindrical basis for $\mathfrak{A}$, we conclude that $\mathfrak{C} \alpha B_\alpha \mathfrak{A} \in \mathfrak{CA}_\alpha$ by Theorem 10(i), [Ma89]. Now $\mathfrak{A}$ is complete

since it is finite, so we also have $\mathfrak{A} \mathfrak{C} \mathfrak{A} B_\alpha \mathfrak{A} \cong \mathfrak{A}$ by Theorem 10(iii), [Ma89]. Thus $\mathfrak{A} \cong \mathfrak{A} \mathfrak{C} \mathfrak{A} B_\alpha \mathfrak{A} \in \mathfrak{A} \mathfrak{C} \mathfrak{A} \mathfrak{C} \mathfrak{A}_\alpha$, so (iii) holds. It is straightforward to check that $\mathfrak{C} \mathfrak{A} B_\gamma \mathfrak{A} \cong \mathfrak{N} r_\gamma \mathfrak{C} \mathfrak{A} B_\alpha \mathfrak{A}$ whenever $3 \leq \gamma \leq \alpha$. In particular,

$$\mathfrak{C} \mathfrak{A} B_3 \mathfrak{A} \cong \mathfrak{N} r_3 \mathfrak{C} \mathfrak{A} B_\alpha \mathfrak{A} \in \mathfrak{N} r_3 \mathfrak{C} \mathfrak{A}_\alpha,$$

so (iv) holds.

Now we prove (v) by contradiction. Suppose $\mathfrak{A} \in \mathbf{S} \mathfrak{A} \mathfrak{C} \mathfrak{A}_\beta$. Hence there is some $\mathfrak{C}$ such that

$$(2) \quad \mathfrak{A} \subseteq \mathfrak{A} \mathfrak{C} \mathfrak{C}, \quad \mathfrak{C} \in \mathfrak{C} \mathfrak{A}_\beta.$$

It turns out that (2) implies that $\mathfrak{A}$ has a β -dimensional relational basis. However, we will only show that (2) implies the existence of a $v \in B_3 \mathfrak{A}$ satisfying (1), which leads to a contradiction, as shown above.

We have

$$\begin{aligned} 0 &\neq a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} a_{0\lambda}^\emptyset \cdot a_{\lambda 1}^\emptyset \\ &= a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} c_\lambda (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset) \\ &= c_{(\beta \sim 2)} \left(a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset) \right), \end{aligned}$$

so

$$(3) \quad 0 \neq a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset).$$

Define $\sigma_{\kappa\lambda}(x)$ for all $\kappa, \lambda < \beta$ and all $x \in \mathfrak{N} r_2 \mathfrak{B}$ as follows:

$$\sigma_{\kappa\lambda}(x) = \begin{cases} s_\kappa^0 s_\lambda^1 x & \text{if } \lambda \neq 0, \\ s_0^1 s_\kappa^0 x & \text{if } \lambda = 0, \kappa \neq 1, \\ s_0^2 s_1^0 s_2^1 x & \text{if } \lambda = 0, \kappa = 1. \end{cases}$$

Notice that $\sigma_{\kappa\lambda}(1) = 1$ and $\sigma_{\kappa\lambda}$ is completely additive. Since $\mathfrak{A} t \mathfrak{A}$ is finite, we have

$$1 = \sum^{(\mathfrak{B})} \mathfrak{A} t \mathfrak{A} = \prod_{\kappa, \lambda < \beta} \sigma_{\kappa\lambda} \left(\sum \mathfrak{A} t \mathfrak{A} \right) = \sum_{v \in \beta \times \beta \mathfrak{A} t \mathfrak{A}} \left(\prod_{\kappa, \lambda < \beta} \sigma_{\kappa\lambda}(v_{\kappa\lambda}) \right),$$

so, by (3),

$$\begin{aligned} 0 &\neq a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset) \cdot 1 \\ &= a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset) \cdot \left(\sum_{v \in \beta \times \beta \mathfrak{A} t \mathfrak{A}} \left(\prod_{\kappa, \lambda < \beta} \sigma_{\kappa\lambda}(v_{\kappa\lambda}) \right) \right) \\ &= \sum_{v \in \beta \times \beta \mathfrak{A} t \mathfrak{A}} \left(a_{01}^\emptyset \cdot \prod_{1 < \lambda < \beta} (s_\lambda^1 a_{0\lambda}^\emptyset \cdot s_\lambda^0 a_{\lambda 1}^\emptyset) \cdot \prod_{\kappa, \lambda < \beta} \sigma_{\kappa\lambda}(v_{\kappa\lambda}) \right). \end{aligned}$$

It follows that there is some $v \in {}^{\beta \times \beta} At \mathfrak{A}$ such that

$$(4) \quad 0 \neq a_{01}^{\emptyset} \cdot \prod_{1 < \lambda < \beta} (s_{\lambda}^1 a_{0\lambda}^{\emptyset} \cdot s_{\lambda}^0 a_{\lambda 1}^{\emptyset}) \cdot \prod_{\kappa, \lambda < \beta} \sigma_{\kappa\lambda}(v_{\kappa\lambda}).$$

By routine cylindric-algebraic computations it is now possible to show from (4) that $v \in B_{\beta} \mathfrak{A}$ and (1) holds. The proof is given in [Ma78, pp. 170⁸–172⁵], for a slightly different definition of $\sigma_{\kappa\lambda}$. The existence of such a v is contradictory, so (v) holds.

Finally, we prove (vi) by contradiction. Suppose $\mathfrak{C}aB_3\mathfrak{A} \in \text{SNr}_3 CA_{\beta}$. Then $\mathfrak{R}a\mathfrak{C}aB_3\mathfrak{A} \in \mathfrak{R}a^* \text{SNr}_3 CA_{\beta}$, but $\mathfrak{R}a\mathfrak{C}aB_3\mathfrak{A} \cong \mathfrak{A}$ by Theorem 10(iii), [Ma89], and $\mathfrak{R}a^* \text{SNr}_3 CA_{\beta} = \text{S}\mathfrak{R}a^* \text{Nr}_3 CA_{\beta} = \text{S}\mathfrak{R}a^* CA_{\beta}$ by [HMT85, 5.3.13], so $\mathfrak{A} \in \text{S}\mathfrak{R}a^* CA_{\beta}$ contradicting (v). Thus (vi) holds. $\square$

For every natural number n we let $\mathcal{M}^{(n)}$ be the first-order formalism defined in [TG87], §7.1. In more detail, $\mathcal{M}^{(n)}$ is a first-order language with equality symbol $\mathbf{1}$ and $n+1$ binary relation symbols $F_0, \dots, F_n$. $\mathbf{T} = \{v_{\kappa} : \kappa < \omega\}$, $\Phi^{(n)}$, and $\Sigma^{(n)}$ are the sets of variables, formulas, and sentences of $\mathcal{M}^{(n)}$, respectively. $[X]$ is the universal closure of $X \in \Phi^{(n)}$. The logical axiom schemata of $\mathcal{M}^{(n)}$ are (AI)–(AIX) ([TG87, p. 8]). $\Theta\eta^{(n)}\Psi$ is the $\mathcal{M}^{(n)}$ -theory generated by Ψ . $\mathcal{M}^{(n)+}$ is the extended formalism obtained from $\mathcal{M}^{(n)}$ by adding certain operators on predicates and an additional equality symbol $=$. $\Pi^{(n)}$, $\Phi^{(n)+}$, and $\Sigma^{(n)+}$ are the sets of predicates, formulas, and sentences of $\mathcal{M}^{(n)+}$, respectively. The logical axiom schemata of $\mathcal{M}^{(n)+}$ are (AI)–(AIX) and also (DI)–(DV) ([TG87, p. 25]). For $3 \leq \alpha < \omega$, the formalisms $\mathcal{M}_{\alpha}^{(n)}$ and $\mathcal{M}_{\alpha}^{(n)+}$ are the fragments of $\mathcal{M}^{(n)}$ and $\mathcal{M}^{(n)+}$ obtained by restricting to only the first α variables $v_0, \dots, v_{\alpha-1}$. Their definitions require some additional axiom schemata, namely (AIX'), (AIX''), (AX), and (AX') ([TG87, pp. 68, 70]). The sets of formulas and sentences of these fragments are $\Phi_{\alpha}^{(n)}$, $\Phi_{\alpha}^{(n)+}$, $\Sigma_{\alpha}^{(n)}$, and $\Sigma_{\alpha}^{(n)+}$. The theories in $\mathcal{M}_{\alpha}^{(n)}$ and $\mathcal{M}_{\alpha}^{(n)+}$ which are generated by Ψ are $\Theta\eta_{\alpha}^{(n)}\Psi$ and $\Theta\eta_{\alpha}^{(n)+}\Psi$. As noted in [TG87], all the results for $\mathcal{L}$ and $\mathcal{L}^+$ carry over to $\mathcal{M}^{(n)}$ and $\mathcal{M}^{(n)+}$ with only minor notational modifications.

Theorem 2. *If $4 \leq \alpha < \omega$ then $\emptyset \neq \Sigma_3^{(n)} \cap (\Theta\eta_{3\alpha-7}^{(n)}\emptyset \sim \Theta\eta_{\alpha}^{(n)}\emptyset)$ and $\emptyset \neq \Sigma_3^{(n)+} \cap (\Theta\eta_{3\alpha-7}^{(n)+}\emptyset \sim \Theta\eta_{\alpha}^{(n)+}\emptyset)$.*

Proof. Set $\beta = 3\alpha - 7$. By Theorem 1 there is a finite relation algebra $\mathfrak{A}$ with an α -dimensional cylindrical basis but no β -dimensional relational basis. Let $n+1 = |At \mathfrak{A}|$. Let $f = \langle f_i : i \leq n \rangle$ be an enumeration of the atoms of $\mathfrak{A}$. Let $\mathfrak{M} = \langle \mathfrak{A}, f, B_{\alpha} \mathfrak{A} \rangle$. Then $\mathfrak{M}$ is an α -dimensional algebraic realization of both $\mathcal{M}^{(n)}$ and $\mathcal{M}^{(n)+}$, in the sense of Definition 16, [Ma89], since $B_{\alpha} \mathfrak{A}$ is an α -dimensional cylindrical basis and clearly (C_3) of Definition 16 holds. (Note that Definitions 16 and 17 should be slightly modified to account for the fact that $\mathcal{M}^{(n)}$ and $\mathcal{M}^{(n)+}$ have many atomic predicates instead of just one.

All the results of [Ma89] continue to hold without any further changes.) By Theorem 22(iii), [Ma89], every $X \in \Theta\eta_\alpha^{(n)+}\emptyset$ is algebraically true of $\mathfrak{M}$.

Consider the statement

(5) every $X \in \Theta\eta_\beta^{(n)+}\emptyset$ is algebraically true of $\mathfrak{M}$.

It can be shown (by imitating the proof of Lemma 4.3.22 of [HMT85]) that (5) implies (2) for some $\mathfrak{C}$. Since (2) implies the existence of a $v \in B_\beta\mathfrak{A}$ such that (1), we have a contradiction, and the proof is complete. However, we can also obtain a sentence $X \in \Sigma_3^{(n)}$ which is algebraically false in $\mathfrak{M}$ (hence $X \notin \Theta\eta_\alpha^{(n)+}\emptyset$), such that one can show directly (by constructing a β -proof) that $X \in \Theta\eta_\beta^{(n)+}\emptyset$. Let Y be the conjunction of the following sentences:

$$\forall v_0 \forall v_1 (F_i v_0 v_1 \rightarrow \neg F_j v_0 v_1) \quad \text{for } i, j \leq n, i \neq j,$$

$$\forall v_0 \forall v_1 \left(\bigvee_{i \leq n} F_i v_0 v_1 \right),$$

$$\forall v_0 \forall v_1 (F_i v_0 v_1 \leftrightarrow F_j v_1 v_0) \quad \text{whenever } f_i^\sim = f_j, i, j \leq n,$$

$$\forall v_0 \forall v_1 \left(v_0 \overset{\circ}{\mathbf{1}} v_1 \leftrightarrow \bigvee_{f_i \leq 1} F_i v_0 v_1 \right),$$

$$\begin{aligned} \forall v_0 \forall v_1 (F_i v_0 v_1 \rightarrow \exists v_2 (F_j v_0 v_2 \wedge F_k v_2 v_1)) \\ \text{whenever } i, j, k \leq n \text{ and } f_i \leq f_j ; f_k, \end{aligned}$$

$$\begin{aligned} \forall v_0 \forall v_1 (F_i v_0 v_1 \rightarrow \neg \exists v_2 (F_j v_0 v_2 \wedge F_k v_2 v_1)) \\ \text{whenever } i, j, k \leq n \text{ and } f_i \cdot f_j ; f_k = 0. \end{aligned}$$

Y encodes the structure of $\mathfrak{A}$ in such a way that if Y has a model $\mathfrak{U} = \langle U, F_i \rangle_{i \leq n}$ then there is an embedding of $\mathfrak{A}$ into the relation algebra of all binary relations on U mapping each atom f_i to the interpretation F_i of F_i . Let $X = \neg Y$. By a tedious decipherment of Definition 17, [Ma89], one can show that Y is algebraically true of $\mathfrak{M}$, and therefore X is not algebraically true of $\mathfrak{M}$. On the other hand, $\mathfrak{A}$ is not representable (since it is finite and has no β -dimensional relational basis), so Y is inconsistent and X is logically valid. The fact that $\mathfrak{A}$ has no β -dimensional relational basis implies that there is a proof of X in $\mathcal{M}_\beta^{(n)}$. To see why this is so, let us suppose that $\mathfrak{A}$ actually coincides with the algebra constructed in the proof of Theorem 1. Then a proof of X in $\mathcal{M}_\beta^{(n)}$ can be modeled on part of the proof of Theorem 1. To get started, first note that Y has the following consequence:

$$\begin{aligned} \exists v_0 \exists v_1 \left(F_i v_0 v_1 \wedge \bigwedge \{ \exists v_2 (F_j v_0 v_2 \wedge F_k v_2 v_1) : \right. \\ \left. 1 < \lambda < \beta \ \& \ a_{0\lambda}^\emptyset = f_j \ \& \ a_{\lambda 1}^\emptyset = f_k \} \right). \end{aligned}$$

This sentence can be put into prenex normal form with quantifier prefix $\exists v_0 \exists v_1 \dots \exists v_{\beta-1}$ and a matrix corresponding to (1). The conjuncts of Y can then be used to show that the matrix is inconsistent. The combinatorial reason for this inconsistency is expressed by the part of the proof of Theorem 1 concerning the 2-element and 3-element subsets of $\{2, \dots, \beta - 1\}$. Of course, the counting argument used there cannot be translated directly into a proof in $\mathcal{M}_\beta^{(n)}$. The cases must be listed and individually ruled out. Thus the proof will be straightforward but of enormous length. $\square$

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