MAXIMUM MODULUS ALGEBRAS AND ANALYTIC VARIETIES

DONNA KUMAGAI AND ZBIGNIEW SLODKOWSKI

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ABSTRACT. Let A be a maximum modulus algebra on X , and V a maximal open subset of X such that V has the structure of one-dimensional variety on which functions from A are analytic. Then, the restriction algebra $A_{X \setminus V}$ is again a maximum modulus algebra.

Let X be a locally compact space, and A a complex algebra of continuous functions on X that contains constant functions and separates points of X . A is called a *maximum modulus algebra* on X if the following local maximum property holds: Given a compact subset N of X , for each $x_0 \in N$ and each $g \in A$,

$$|g(x_0)| \leq \max_{\partial_x N} |g|.$$

The algebra of all analytic functions on a complex manifold is an example of a maximum modulus algebra. If we let A be a uniform algebra defined on its maximal ideal space M and $X = M \setminus S(A)$, where $S(A)$ is the Silov boundary of M , then A restricted to X is a maximum modulus algebra. Conditions for existence of analytic structures associated with maximum modulus algebras have been researched extensively. See [6] for some notable results.

In the present study we show that if A is a maximum modulus algebra on X then the existence of an open analytic variety V in X guarantees that the restriction algebra $A_{X \setminus V}$ on the complement of V in X is again a maximum modulus algebra. Our method is to apply several previous results by Z. Słodkowski in [4], where he develops a higher dimensional version of Oka's analytic multivalued functions. These results in [4] are combined with a now classic theorem on analytic structure by Aupetit and Wermer [1]. An interesting result that relates to ours is obtained by T. W. Gamelin [2] in the context of uniform algebras using Hayashi's pole function.

We begin with some preliminary expositions.

Definition. Let X and Y be topological spaces. A continuous map $G: X \rightarrow Y$ is said to be proper if $G^{-1}(K)$ is compact in X for every compact $K \subset Y$.

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We make the following remarks on proper maps for later use.

Remark (i). Let N be a compact subset of X and $G: X \rightarrow Y$ be continuous. Then, the restriction of G to N is proper onto its range.

Remark (ii). Let $G: X \rightarrow Y$ be continuous and $N \subset X$ compact. Then, the restriction map $G|_{N \setminus G^{-1}(G[\partial N])}$ is proper. In general, the restriction of a proper map to the full inverse image of a set is proper.

Proof of (ii). Put $U = N \setminus G^{-1}(G[\partial N])$. Then $G(U) \cap G(\partial N) = \emptyset$. Let $K \subset G(U)$ be compact. To show that $G^{-1}(K)$ is compact it suffices to show that it is closed in N . So let $\{k_\alpha\}$ be a net in $G^{-1}(K)$ with $k_\alpha \rightarrow k \in N$. Since G is continuous, $G(k_\alpha) \rightarrow G(k)$. Since K is closed, $G(k) \in K$, i.e., $k \in G^{-1}(K)$.

Remark (iii). Let X and Y be regions in $\mathbb{C}^n$ and $\mathbb{C}^k$ respectively. Suppose that $G: X \rightarrow Y$ is holomorphic and proper. For each $w \in Y$, the fiber $G^{-1}(w) = \{\lambda \in X | G(\lambda) = w\}$ is finite.

Proof of (iii). If $w = (w_1, w_2, \dots, w_k) \in Y$ and $G = (g_1, \dots, g_k)$, then $G^{-1}(w)$ is the intersection of the zero sets of $g_i - w_i$, $1 \leq i \leq k$, and is compact since G is proper. By a well-known theorem, any compact subvariety of $\mathbb{C}^n$ is a finite set.

Remark (iv). A composition of proper maps is proper.

Remark (v). Let $G: X \rightarrow \mathbb{C}^n$ be proper onto its range. Then, $G(X)$ is locally closed; i.e., there exists an open subset U of $\mathbb{C}^n$ with $G(X) \subseteq U$ such that $\overline{G(X)} \cap U = G(X)$.

Proof of (v). For each y in $G(X)$ choose a neighborhood T_y of y such that $G^{-1}(\overline{T_y})$ is compact. Let U be a union of such neighborhoods T_y . We assert that G is proper into U . Indeed, let $K \subset U$ be compact. It is covered by a finite set $\{T_{y_1}, \dots, T_{y_s}\}$. Since $G^{-1}(\overline{T_{y_1}} \cup \dots \cup \overline{T_{y_s}})$ is compact and $G^{-1}(K)$ is closed, $G^{-1}(K)$ is compact. Next we show that $\overline{G(X)} \cap U = G(X)$. Given any $y \in \overline{G(X)} \cap U$, choose a neighborhood T of y in U such that $\overline{T}$ and hence $G^{-1}(\overline{T})$ are compact. Then $G\{G^{-1}(\overline{T})\} = \overline{T} \cap G(X)$ is compact, and therefore closed. Since $y \in \overline{G(X)}$ and T is a neighborhood of y , we have $y \in \overline{T} \cap \overline{G(X)} = \overline{T} \cap G(X) \subset G(X)$.

The first two lemmas are stated in a general setting of a locally compact system but they apply to our case as well. Recall that a pair (X, A) is called a *locally compact system* in the sense of Rickart if X is a locally compact space; A a complex algebra of continuous functions on X which contains constants and separates points of X ; and the given topology on X is the coarsest topology which makes functions in A continuous.

Lemma 1. Let (X, A) be a locally compact system and let K be a compact subset of X . Given an open covering $\{U_i\}$ of X and functions $g_1, \dots, g_r$,

from A , there exists an open subset U of X and functions $g_1, \dots, g_m$ in A such that

- (i) $K \subset U$
- (ii) $\{g_1, \dots, g_r\} \subset \{g_1, \dots, g_m\}$
- (iii) The map $G = \{g_1|_U, \dots, g_m|_U\}$ is proper onto its range.

Furthermore, for every $y \in G(U)$ the fiber above y , $G^{-1}(y)$, is contained in some U_t .

Lemma 2. Let L and K be compact subsets of $\mathbf{R}^n$ with $L \subset K \subset \mathbf{R}^n$. Suppose $u: K \rightarrow [-\infty, +\infty)$ is upper semicontinuous and $\max u > \max u|_L$. Then, there exists $x_0 \in K \setminus L$, $\varepsilon > 0$, and an affine form $l: \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$(u + l)(x_0) = 0$$

$$(u + l)(x) \leq -\varepsilon|x - x_0|^2 \quad \text{for all } x \in K.$$

See [5] for proofs of Lemmas 1 and 2.

Our main theorem is as follows:

Theorem 1. Let A be a maximum modulus algebra on X . Let $V \subseteq X$ be a maximal open subset of X such that V has the structure of one dimensional variety on which all functions from A are analytic. Denote $Y = X \setminus V$. Then the restriction algebra A_Y is again a maximum modulus algebra.

For proof of Theorem 1 we need the following lemma.

Lemma 3. Let $(\mathbf{C}^m)^*$, $m > 1$, be the set of complex linear functionals and let M be a one-dimensional variety in $\mathbf{C}^m$. Let $S = \{f \in (\mathbf{C}^m)^*: f^{-1}(w) \cap M \text{ is countable for all } w \in \mathbf{C}\}$. Then, S is dense in $(\mathbf{C}^m)^*$. In fact, S contains a dense G_δ -set.

Proof of Lemma 3. Since M is a one-dimensional variety, it is a union of at most countably many connected one-dimensional manifolds, $M_1, M_2, \dots$, and of countable singular points. For each M_n , choose a pair of distinct points, $a_n, b_n \in M_n$. Let $S_n = \{f \in (\mathbf{C}^m)^*: f(a_n) \neq f(b_n)\}$. Observe that if $f \in S_n$ then for each w in the range of f , the set $f^{-1}(w) \cap M_n$ is countable. Otherwise it would be a subvariety of M_n containing a neighborhood. Since M_n is a connected manifold, $f^{-1}(w) \supset M_n$, which implies that $f|_{M_n} = \text{constant}$, contrary to the assumption that $f(a_n) \neq f(b_n)$. Clearly, S_n is an open dense set in $(\mathbf{C}^m)^*$ and $S_n \subset S$ for each n .

Proof of Theorem 1. Suppose A_Y does not have the local maximum property, i.e., there is a function $g_0 \in A$ and a compact subset N of X such that

$$\max_{x \in N \cap Y} |g_0(x)| = 1 > \max_{x \in Y \cap \partial N} |g_0(x)|,$$

where ∂N is the topological boundary of N in X . Let λ be a point in $N \cap Y$ on which $|g_0|$ attains its maximum. Multiplying by a suitable constant we

may alter g_0 so that $g_0(\lambda) = |g_0(\lambda)| = 1$. Denote $J = \{x \in N: g_0(x) = 1\}$. Note that the map $x \rightarrow g_0(x): N \cap Y \setminus g_0^{-1}g_0(\partial N \cap Y) \rightarrow \mathbb{C}$ is proper onto its range. By Lemma 1 we can find functions $g_1, \dots, g_m$ in $\mathcal{A}$ such that $G = (g_0, g_1, \dots, g_m)$ is a proper map of some neighborhood of J in X onto its range. Without loss of generality we can shrink X and assume that X is already such a neighborhood. In view of Remark (v) on proper maps made earlier we see that $G(X)$ is locally closed and so is $G(Y)$; i.e., there is an open set $U \supset G(X)$ such that $G(X) = U \cap \overline{G(X)}$ and $G(Y) = U \cap \overline{G(Y)}$. Denote by $\xi_0, \xi_1, \dots, \xi_m$ the coordinate functionals on $\mathbb{C}^{m+1}$. Thus, if $z \in G(Y)$ then $\xi_0(z) = g_0(w)$ for some $w \in Y$. By Lemma 2 there exists $z_0 \in G(J)$, $\varepsilon > 0$, and a $\mathbb{C}$ -linear form $l: \mathbb{C}^{m+1} \rightarrow \mathbb{C}$ such that

$$\begin{aligned} (\operatorname{Re} \xi_0 + \operatorname{Re} l)(z_0) &= 0 \\ (\operatorname{Re} \xi_0 + \operatorname{Re} l)(z) &\leq -\varepsilon |z - z_0|^2 \quad \text{for all } z \in G(Y). \end{aligned}$$

It holds that

$$(\operatorname{Re} \xi_0 + \operatorname{Re} l)(z_0) > (\operatorname{Re} \xi_0 + \operatorname{Re} l)(z) \quad z \in G(Y) \setminus z_0.$$

Choose an open neighborhood H of z_0 such that $\overline{H} \subseteq U$ and $\overline{H}$ is compact. Then, there exists $\varepsilon > 0$ such that $(\operatorname{Re} \xi_0 + \operatorname{Re} l)(z_0) > 2\varepsilon + (\operatorname{Re} \xi_0 + \operatorname{Re} l)(z)$, $z \in \partial H \cap G(Y)$.

If we put $V^* = G(X) \setminus G(Y)$ then it is a one-dimensional variety, since it is the proper image of the one-dimensional variety $V \cap G^{-1}(V^*)$ under the map $G|_{G^{-1}(V^*)}$. By Lemma 3, now we find a linear functional f with the properties that (i) $f^{-1}(w) \cap V^*$ is at most countable for all $w \in \mathbb{C}$ and (ii) $\max_{z \in \overline{H} \cap G(Y)} |(\xi_0 + l)(z) - f(z)| < \varepsilon$. Consequently, $\max \operatorname{Re} f|_{H \cap G(Y)} > \max \operatorname{Re} f|_{\partial H \cap G(Y)}$. Choose a point z^* in $H \cap G(Y)$ at which $\operatorname{Re} f$ attains its maximum. If we put $w^* = f(z^*)$ then $f^{-1}(w^*) \cap V^*$ is at most countable by the choice of f . Observe that f restricted to the set $\overline{H} \cap G(Y)$ is proper onto its range and so is f restricted to the set $\{z \in H \cap G(Y): \operatorname{Re} f(z) > \max \operatorname{Re} f|_{\partial H}\}$ by Remark (ii) on proper maps. By this there is δ_0 such that f restricted to the intersection of $[G(Y) \cap \overline{H}]$ and $f^{-1}\{w: |w - f(z^*)| < \delta_0\}$ is a proper map. Consequently, $f^{-1}(w^*) \cap G(Y)$ is compact. Since $V^* = G(X) \setminus G(Y)$, we see that $f^{-1}(w^*) \cap G(X)$ is a disjoint union of a compact set $f^{-1}(w^*) \cap G(Y)$ and a countable set $f^{-1}(w^*) \cap V^*$. We shall construct a relatively compact neighborhood B_0 of $f^{-1}(w^*) \cap G(Y)$ in the hyperplane $f^{-1}(w^*)$ such that $B_0 \subset U$ and ∂B_0 is disjoint from $f^{-1}(w^*) \cap G(X)$. For each $x \in f^{-1}(w^*) \cap G(Y)$ the set of distances $\{d_n(x) = |x - a_n|: a_n \in f^{-1}(w^*) \cap V^*\}$ is countable. Pick $\varepsilon(x) > 0$ such that $\varepsilon(x) \neq d_n(x)$ for any n and the ball of radius $\varepsilon(x)$ about x is contained in U . Cover $f^{-1}(w^*) \cap G(Y)$ with a finite union of these balls and call it B_0 . Clearly, ∂B_0 does not meet $f^{-1}(w^*) \cap G(Y)$ nor $f^{-1}(w^*) \cap V^*$, i.e., $\partial B_0 \cap f^{-1}(w^*) \cap G(X) = \emptyset$. Let e be a unit vector orthogonal to the hyperplane

$f^{-1}(w^*)$. Since $G(X)$ is closed in U and ∂B_0 is compact, there is $\delta < \delta_0$ such that $\bigcup_{|w| < \delta} (\partial B_0 + we)$ is disjoint from $G(X)$. Denote $B = \bigcup_{|w| < \delta} (B_0 + we)$. It is now clear that $f: B \cap G(X) \rightarrow \mathbb{C}$ is proper onto its range. Therefore, by Remark (iii) about proper maps made earlier, for each w in $\{|w - f(z^*)| < \delta, \operatorname{Re} w > \operatorname{Re} f(z^*)\}$, the set $f^{-1}(w) \cap B \cap G(X)$ is finite. Slodkowski showed in [4] that $\forall h \in A$ the set function $\lambda \rightarrow h[(f \circ G)^{-1}(\lambda)]$ is an analytic multivalued function. By this fact and the result obtained by Yamaguchi [7], Nishino [3] et al, we can deduce that the cardinality of $(f \circ G)^{-1}(w)$ is finite for every w . The well-known result of Aupetit and Wermer [1] now shows that $G^{-1}(B)$ is a one-dimensional analytic variety; hence $G^{-1}(B) \subset V$ by the maximality of V . Thus, $Y \cap G^{-1}(B) = \emptyset$. This is a contradiction, since $z^* \in G(Y \cap G^{-1}(B))$, which holds because $z^* \in f^{-1}(w^*) \cap G(Y) \subset B$. This proves Theorem 1.

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, BERKS CAMPUS, READING, PENNSYLVANIA 19610-6009

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680