ON INTERSECTIONS OF COMPACTA IN EUCLIDEAN SPACE

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ABSTRACT. Let X be a codimension-three tame compactum in Euclidean space E^n . If $\dim X \times Y < n$, then every map $f: Y \rightarrow E^n$ can be approximated by map g with $X \cap \text{Im } g = \emptyset$.

1. INTRODUCTION

This paper is a sequel to the recent series of works [M-Ru1-2], [K-L], [K], [Sp1-2] [D-S], [D-R-S1-2], [D2].

Suppose a compactum X lies in Euclidean space E^n . Under what conditions can every map $f: Y \rightarrow E^n$ of compactum Y be approximated by a map whose image does not intersect X ? This paper suggests a solution of the problem in terms of the dimension of the product. Denote by $C(X, Z)$ the space of all continuous maps of X into Z .

Theorem 1. *Let X be a codimension-three tame compactum in Euclidean space E^n . Suppose that $\dim(X \times Y) < n$ for some compactum Y . Then the space $C(Y, E^n - X)$ is dense in $C(Y, E^n)$.*

By the tameness of X we understand the equality $\text{dem } X = \dim X$, where dem is Shtanko's embedding dimension [Sh], [E].

This theorem with the additional restriction $\dim X + \dim Y \leq n$ was proved in [D-Š] and [D-R-Š]. The restriction $\dim X + \dim Y \leq n$ is very strong because some recent results in dimension theory [D1] imply that there exist compacta X and Y with $\dim(X \times Y) < n$ and $\dim X + \dim Y = m$ for given $m < 2n - 3$ (of course, $m > \max\{\dim X, \dim Y\}$).

Corollary 1 [D2]. *Let X and Y be compacta with $\dim X + \dim Y \leq 4/3n - 1$. Suppose that $\dim(X \times Y) < n$. Then every pair of maps $f: X \rightarrow E^n$ and $j: Y \rightarrow E^n$ has unstable intersection.*

A pair of maps $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ of compacta X and Y into a metric space Z is said to have an unstable intersection [D-R-Š2] if, for every

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$\varepsilon > 0$, there exists a pair of maps $f': X \rightarrow Z$ and $g': Y \rightarrow Z$ satisfying the conditions that $d(f, f') < \varepsilon$ and $d(g, g') < \varepsilon$ and with the property $\text{Im } f' \cap \text{Im } g' = \emptyset$.

Corollary 1 with the restriction $\dim X + \dim Y \leq n$ was proved in [D-R-Š2] and [Sp2]. It was also proved without this restriction in recent work [Sp-T].

Corollary 2 [D2]. *Let X and Y be compacta with $\dim(X \times Y) < n$. Then there exists a number m such that every pair of maps $f: X \rightarrow E^{n+m}$ and $g: Y \times I^m \rightarrow E^{n+m}$ has an unstable intersection.*

Here I^m is an m -dimensional cube.

Conjecture. The inequality $\dim X \times Y < n$ for a pair of compacta implies the instability of intersections for every pair of maps $f: X \rightarrow E^n$ and $g: Y \rightarrow E^n$.

The positive answer to the questions in any of the following problems implies the conjecture.

Problem 1. Let X and Y be compacta with $\dim(X \times Y) < n$. Can a given map $f: X \rightarrow E^n$ be approximated by f' with the property $\dim(\text{Im } f' \times Y) < n$?

Problem 2. Let X be a compactum with $\dim X \leq n - 2$, and let G be an Abelian group. Can an arbitrary map $f: X \rightarrow E^n$ be approximated by f' with the property $c\text{-dim}_G \text{Im } f' \leq c\text{-dim}_G X$?

Recall that the cohomological dimension of a compactum Z with respect to G as coefficient, $c\text{-dim}_G Z$, is the largest number m for which there exists a closed subset $A \subset Z$ with $\check{H}^m(Z, A; G) \neq 0$, where $\check{H}^*$ is a Čech cohomology group [Ku].

Problem 3. Let X and Y be compacta. Suppose that a pair of maps $f: X \rightarrow E^n$ and $g: Y \rightarrow E^n$ has a stable intersection. Does it imply that the pair $f: X \rightarrow E^n \times \{0\} \hookrightarrow E^{n+1}$ and $g \times \text{id}_I: Y \times I \rightarrow E^{n+1}$, where $I = [-1, 1]$, has a stable intersection in E^{n+1} ?

As a consequence of the proof of Theorem 1, a generalization of the Alexandrov theorem [A] for an arbitrary group is obtained.

2. THE MAIN LEMMA

A homology group $H_k(X; Z)$ with the integers as coefficients is denoted by $H_k(X)$.

Lemma 1. *Let M be a simply connected CW-complex and Y be a compactum. If $c\text{-dim}_{H_k(M)} Y \leq k$ for all k , then $c\text{-dim}_{\pi_k(M)} Y \leq k$ for every k .*

We will denote the homotopy group $\pi_k(M)$ by π_k and the homology group $H_k(M)$ by H_k .

Let us consider a Postnikov tower for the space M [Spa]: $* \leftarrow E_2 \leftarrow E_3 \leftarrow \dots \leftarrow E_n \leftarrow \dots$.

Assertion 1. For each n and arbitrary G there exist the equalities $H_n(E_n; G) = H_n(E_{n+1}; G) = \dots = H_n(M; G)$.

Proof. It is easy to see that E_n is homotopy equivalent to CW -complex T , which is obtained from M by attaching cells of dimension $\geq n+2$. Hence $H_n(E_n; G) = H_n(T; G) = H_n(T^{(n+1)}; G) = H_n(M^{(n+1)}; G) = H_n(M; G)$. The other equalities can be obtained similarly.

Recall that $Z = Z/pZ$ and $Z_{p^\infty} = \varinjlim Z_{p^k}$.

Assertion 2. Let π be an Abelian group with no element of order p , and suppose that each element of π is divisible by p . Then $\tilde{H}_*(K(\pi, n); G) = 0$ for all n if $G = Z_p$ or $G = Z_{p^\infty}$.

Here $K(\pi, n)$ is Eilenberg-McLane complex.

Proof. It is easy to verify that $\pi \otimes Z[\frac{1}{p}] = \pi$. Let $X_{[p]}$ denote a localization of space X out of the prime number p . Then [Su] $K(\pi, n)_{[p]} = K(\pi_{[p]}, n) = K(\pi \otimes Z[\frac{1}{p}], n) = K(\pi, n)$. Hence

$$H_m(K(\pi, n)) = H_m(K(\pi, n)_{[p]}) = H_m(K(\pi, n)) \otimes Z\left[\frac{1}{p}\right].$$

According to the universal coefficients formula we have $0 \rightarrow H_m(K(\pi, n)) \otimes Z[\frac{1}{p}] \otimes G \rightarrow H_m(K(\pi, n); G) \rightarrow (H_{m-1}(K(\pi, n)) \otimes Z[\frac{1}{p}]) * G \rightarrow 0$. In the case $G = Z_{p^\infty}$ or $G = Z_p$, both the right and the left sides of the sequence are zero.

Assertion 3. Suppose that H_k has no p -torsion and each element of H_k is divisible by p for all $k < n$. Then $\tilde{H}_*(E_k; G) = 0$ for all $k < n$ if $G = Z_p$ or $G = Z_{p^\infty}$.

Proof. We will use induction on k .

If $k = 1$, it is true by virtue of $E_1 = *$. Assume that the assertion is proved for some $k < n$. Let us consider the spectral sequence of the fibration $E_{k+1} \rightarrow E_k$ in the Postnikov tower:

$$E_{m,l}^2 = H_m(E_k; H_l(K(\pi_{k+1}, k+1); G)).$$

The universal coefficient formula gives us the sequence

$$\begin{aligned} 0 \rightarrow H_m(E_k) \otimes H_l(K(\pi_{k+1}, k+1); G) &\rightarrow E_{m,l}^2 \\ \rightarrow H_{m-1}(E_k) * H_l(K(\pi_{k+1}, k+1); G) &\rightarrow 0. \end{aligned}$$

Assertion 2 implies that $E_{m,l}^2 = 0$ if $l > 0$. If $l = 0$ and $m > 0$ then $E_{m,l}^2 = 0$ by the assumption of induction. Since $E_{m,l}^2 = 0$ if $(m, l) \neq (0, 0)$ then $\tilde{H}_*(E_{k+1}; G) = 0$.

Assertion 4. Let $f: E \rightarrow B$ be a fibration whose fibre F is $(n-1)$ -connected. Suppose that $\tilde{H}_*(B; G) = 0$. Then the inclusion $F \hookrightarrow E$ induces an isomorphism $H_n(F; G) \rightarrow H_n(E; G)$.

Proof. Consider the spectral sequence of f .

Assertion 5. Let $f: \mathbf{E} \rightarrow \mathbf{B}$ be a fibration with $(n-1)$ -connected fibre $\mathbf{F}$. Suppose that $\tilde{H}_*(\mathbf{B}; \mathbf{Z}_{p^\infty}) = 0$. Then the inclusion $j: \mathbf{F} \hookrightarrow \mathbf{E}$ induces an isomorphism $j_*: H_{n+1}(\mathbf{F}; \mathbf{Z}_{p^\infty}) \rightarrow H_{n+1}(\mathbf{E}; \mathbf{Z}_{p^\infty})$.

Proof. Consider the spectral sequence of the fibration f . It is enough to show that $E_{1,n}^2 = E_{2,n}^2 = 0$.

Indeed $E_{m,n}^2 = H_m(\mathbf{B}; H_n(\mathbf{F}); \mathbf{Z}_{p^\infty}) = H_m(\mathbf{B}; H_n(\mathbf{F}) \otimes \mathbf{Z}_{p^\infty}) = 0$. The last equality follows from the universal coefficients formula and stipulation

$$\tilde{H}^*(\mathbf{B}; \mathbf{Z}_{p^\infty}) = 0.$$

Denote by $\mathcal{G}$ Bokshtein's family of groups. It consists of the rationals $\mathbf{Q}$, $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, p -primary quasi-cyclic groups $\mathbf{Z}_{p^\infty}$, and localizations $\mathbf{Z}_{(p)}$ of the integers in p where p runs on all prime numbers.

Following Bokshtein, define a set $\mathcal{G}(G)$ for an arbitrary Abelian group G :

- (1) $\mathbf{Q} \in \mathcal{G}(G) \Leftrightarrow \mathbf{Q} \otimes G \neq 0$,
- (2) $\mathbf{Z}_{(p)} \in \mathcal{G}(G) \Leftrightarrow \mathbf{Z}_{p^\infty} \otimes G \neq 0$,
- (3) $\mathbf{Z}_p \in \mathcal{G}(G) \Leftrightarrow G \otimes \mathbf{Z}_{p^\infty} = 0$ and $G \otimes \mathbf{Z}_p \neq 0$,
- (4) $\mathbf{Z}_{p^\infty} \in \mathcal{G}(G) \Leftrightarrow \mathbf{Z}_{p^\infty} \otimes G = 0$, $\mathbf{Z}_p \otimes G = 0$, and $\mathbf{Z}_p * G = 0$.

Bokshtein's Theorem [Ku]. For arbitrary compactum $\mathbf{X}$ and arbitrary Abelian group G there is the equality

$$c\text{-dim}_G \mathbf{X} = \max_{\mathbf{H} \in \mathcal{G}(G)} c\text{-dim}_\mathbf{H} \mathbf{X}.$$

Proof of Lemma 1. Denote by δ_k the set $\{G: c\text{-dim}_G Y \leq k\}$. By the conditions of Lemma 1, we have $\mathbf{H}_k \in \delta_k$ and $\delta_k \subset \delta_{k+1}$ for all k . All we need to prove is the inclusion $\pi_k \in \delta_k$ for each k . We do it by induction on k .

Since $\pi_1 = 0$, then $\pi_1 \in \delta_1$. Assume that, for $k < n$, inclusions $\pi_k \in \delta_k$ hold. We must prove the inclusion $\pi_n \in \delta_n$. By virtue of Bokshtein's Theorem, $c\text{-dim}_{\pi_n} Y = c\text{-dim}_G Y$ for some $G \in \mathcal{G}(\pi_n)$. Thus, it is sufficient to prove that $\mathcal{G}(\pi_n) \subset \delta_n$. We have to consider four cases: when $G = \mathbf{Q}$, $\mathbf{Z}_{(p)}$, $\mathbf{Z}_p$, and $\mathbf{Z}_{p^\infty}$.

Suppose that $G \in \mathcal{G}(\pi_n)$. Then, if $G \in \delta_{n-1}$, it follows that

- (*) $G \in \delta_{n-1} \subset \delta_n$. Hence we can assume for each G that $G \notin \delta_{n-1}$.

Suppose that $G = \mathbf{Q} \in \mathcal{G}(\pi_n)$. Then (*) and the condition $\mathbf{H}_k \in \delta_k$ imply that $\cup_{i < n} \mathbf{H}_i$ has no element of infinite order. Let us consider a localization $\mathbf{M}_{(0)}$ of $\mathbf{M}$ in 0. Then $\mathbf{H}_k(\mathbf{M}_{(0)}) = \mathbf{H}_k \otimes \mathbf{Q} = 0$ for all $k < n$. Hurewicz's theorem implies $\mathbf{H}_n(\mathbf{M}_{(0)}) = \pi_n(\mathbf{M}_{(0)})$. Since $\mathbf{H}_n \otimes \mathbf{Q} = \mathbf{H}_n(\mathbf{M}_{(0)}) = \pi_n(\mathbf{M}_{(0)}) = \pi_n \otimes \mathbf{Q} \neq 0$, the group $\mathbf{H}_n$ contains an element of infinite order. Hence $\mathbf{Q} \in \mathcal{G}(\mathbf{H}_n)$, by the definition of $\mathcal{G}(\mathbf{H}_n)$. Therefore, $\mathbf{Q} \in \delta_n$.

Suppose that $\mathbf{Z}_{(p)} \in \mathcal{G}(\pi_n)$. There are three possibilities: (1) $\mathbf{Q}$ and some p -torsion group are both $\in \delta_{n-1}$; (2) $\mathbf{Q} \notin \delta_{n-1}$; (3) $\mathbf{Q} \in \delta_{n-1}$ and there is no p -torsion group in δ_{n-1} .

- (1) From Bokshtein's inequality [Ku],

$$c - \dim_{\mathbf{Z}_{(p)}} \mathbf{Y} \leq \max\{c - \dim_{\mathbf{Q}} \mathbf{Y}, \quad c - \dim_{\mathbf{Z}_{p^\infty}} \mathbf{Y} + 1\},$$

it follows that $\mathbf{Z}_{(p)} \in \delta_n$.

- (2) Consider a localization of $\mathbf{M}$ in prime p , and apply the above argument. By using modulo p -torsions and Hurewicz's theorem [Spa], we will obtain an equality $\mathbf{H}_n \otimes \mathbf{Z}_{(p)} = \pi_n \otimes \mathbf{Z}_{(p)}$. Since $\mathbf{Z}_{(p)} \otimes \mathbf{Z}_{p^\infty} = \mathbf{Z}_{p^\infty}$, this equality together with $\mathbf{Z}_{(p)} \in \mathcal{G}(\pi_n)$ implies $\mathbf{Z}_{(p)} \in \mathcal{G}(\mathbf{H}_n)$. Hence $\mathbf{Z}_{(p)} \in \delta_n$.
- (3) In this case all elements of $\mathbf{H}_i$ for $i < n$ are divisible by p . Apply Assertions 3 and 4 to obtain $\mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_{p^\infty}) = \mathbf{H}_n(\mathbf{K}(\pi_n, n); \mathbf{Z}_{p^\infty})$. The universal coefficient formula implies that $\mathbf{H}_n \otimes \mathbf{Z}_{p^\infty} = \pi_n \otimes \mathbf{Z}_{p^\infty} \neq 0$. Therefore, $\mathbf{Z}_{(p)} \in \mathcal{G}(\mathbf{H}_n)$ and $\mathbf{Z}_{(p)} \in \delta_n$.

Suppose that $\mathbf{G} = \mathbf{Z}_p \in \mathcal{G}(\pi_n)$. There are two possibilities: (1) $\mathbf{Z}_{p^\infty} \in \delta_{n-1}$ and (2) $\mathbf{Z}_{p^\infty} \notin \delta_{n-1}$. In the first case, Bokshtein's inequality

$$c - \dim_{\mathbf{Z}_p} \mathbf{Y} \leq c - \dim_{\mathbf{Z}_{p^\infty}} \mathbf{Y} + 1$$

implies the inclusion $\mathbf{Z}_p \in \delta_n$. In the second case, due to Bokshtein's inequality

$$c - \dim_{\mathbf{Z}_p} \mathbf{Y} \leq c - \dim_{\mathbf{Z}_{(p)}} \mathbf{Y},$$

we can assume that $\mathbf{Z}_{(p)} \notin \delta_n$. Apply Assertions 1, 3, and 4 to obtain equalities $\mathbf{H}_n(\mathbf{M}; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{K}(\pi_n, n); \mathbf{Z}_p) = \pi_n \otimes \mathbf{Z}_p \neq 0$. From the other side, $\mathbf{H}_n(\mathbf{M}; \mathbf{Z}_p) = \mathbf{H}_n(\mathbf{E}_n; \mathbf{Z}_p) = \mathbf{H}_n \otimes \mathbf{Z}_p$. Hence $\mathbf{H}_n \otimes \mathbf{Z}_p \neq 0$. Since $\mathbf{Z}_{(p)} \notin \delta_n$ and $\mathbf{H}_n \in \delta_n$, $\mathbf{Z}_{(p)} \notin \mathcal{G}(\mathbf{H}_n)$, and hence $\mathbf{H}_n \otimes \mathbf{Z}_{p^\infty} = 0$. Then $\mathbf{Z}_p \in \mathcal{G}(\mathbf{H}_n) \subset \delta_n$.

Suppose at least that $\mathbf{G} = \mathbf{Z}_{p^\infty} \in \mathcal{G}(\pi_n)$. By virtue of Bokshtein's inequalities [Ku],

$$c - \dim_{\mathbf{Z}_{p^\infty}} \mathbf{Y} \leq c - \dim_{\mathbf{Z}_p} \mathbf{Y} \leq c - \dim_{\mathbf{Z}_{(p)}} \mathbf{Y},$$

we can assume that $\mathbf{Z}_p, \mathbf{Z}_{(p)} \notin \delta_n$.

There are two cases: (1) $\mathbf{Q} \notin \delta_{n-1}$ and (2) $\mathbf{Q} \in \delta_{n-1}$. In the first case we consider the Serre class of Abelian groups $\mathbf{C}$ which consists of torsion groups having no element of order p [Spa]. Hurewicz's modulo- $\mathbf{C}$ theorem implies a modulo- $\mathbf{C}$ equality $\mathbf{H}_n = \pi_n$. Hence $\mathbf{Z}_{p^\infty} \in \mathcal{G}(\mathbf{H}_n) \subset \delta_n$ by the definition of $\mathcal{G}(\mathbf{H}_n)$. In the second case, apply Assertions 3, 4, and 5 to obtain the isomorphism

$$j_* : \mathbf{H}_{n+1}(\mathbf{K}(\pi, n); \mathbf{Z}_{p^\infty}) \rightarrow \mathbf{H}_{n+1}(\mathbf{E}_n; \mathbf{Z}_{p^\infty}).$$

Since $\mathbf{H}_n(\mathbf{K}(\pi_n, n)) * \mathbf{Z}_{p^\infty} = \pi_n * \mathbf{Z}_{p^\infty} \neq 0$ we can assume that $\mathbf{H}_{n+1}(\mathbf{E}_n; \mathbf{Z}_{p^\infty}) \neq 0$. The spectral sequence of the fibration $\mathbf{E}_{n+1} \rightarrow \mathbf{E}_n$ implies $\mathbf{H}_{n+1}(\mathbf{E}_{n+1}; \mathbf{Z}_{p^\infty}) \neq 0$. Therefore, $\mathbf{H}_{n+1} \otimes \mathbf{Z}_{p^\infty} \neq 0$ or $\mathbf{H}_n * \mathbf{Z}_{p^\infty} \neq 0$. The condition $\mathbf{H}_{n+1} \otimes \mathbf{Z}_{p^\infty} \neq 0$ implies $\mathbf{Z}_{(p)} \in \mathcal{G}(\mathbf{H}_{n+1}) \subset \delta_{n+1}$. By virtue of Bokshtein's inequality

ity $c\text{-dim}_{Z_{p^\infty}} Y \leq \max\{c\text{-dim}_Q Y, c\text{-dim}_{Z_{(p)}} Y - 1\}$, we can conclude that $Z_{p^\infty} \in \delta_n$. The condition $H_n * Z_{p^\infty} \neq 0$ combined with the fact that $Z_p, Z_{(p)} \notin \mathcal{G}(H_n)$ implies that $Z_{p^\infty} \in \mathcal{G}(H_n)$.

3. PROOF OF THEOREM 1

Notation (Kuratowski). $X\tau Z \Leftrightarrow$ for every closed subset $A \subset X$ and every map $f: A \rightarrow Z$ there exists an extension $\bar{f}: X \rightarrow Z$ of f .

Recall that $X\tau K(G, n)$ is equivalent to $c\text{-dim}_G X < n$ [Ku].

Assertion 6. *Let Y and Z be homotopy-equivalent ANE-spaces. Then $X\tau Y$ if and only if $X\tau Z$.*

Proof. Apply the homotopy extension theorem.

Assertion 7. *Let $f: E \rightarrow B$ be a locally trivial fibration with fibre F , and assume that $E, F \in \text{ANE}$ and that B is a polyhedron. Suppose that $X\tau B$ and $X\tau F$ for some compactum X . Then $X\tau E$.*

Proof. Consider a map $\varphi: A \rightarrow E$ where A is a closed subset of X . By $X\tau B$ there exists an extension $\psi: X \rightarrow B$ of the map $f \circ \varphi$. For each point $x \in B^{(0)} - f\varphi(A)$ choose a point $x' \in f^{-1}(x)$. Here $B^{(i)}$ denotes the i -dimensional skeleton of B . By using $X\tau F$, extend the lifting of φ to $\varphi': A \cup \psi^{-1}(\psi(A) \cap B^{(0)}) \rightarrow E$. Define a map $\varphi_0: \psi^{-1}(B^{(0)}) \cup A \rightarrow E$ by the formula

$$\varphi_0(x) = \begin{cases} \varphi'(x) & \text{if } x \in \psi^{-1}(\psi(A) \cap B^{(0)}) \cup A \\ (\psi(x))' & \text{if } x \in \psi^{-1}(B^{(0)} - \psi(A)). \end{cases}$$

Assume that there is a map $\varphi_k: \psi^{-1}(B^{(k)}) \cup A \rightarrow E$ such that $\varphi_k|_A = \varphi$ and $f \circ \varphi_k = \psi|_{\dots}$. Consider an arbitrary $(k+1)$ -dimensional simplex $\sigma \subset B^{(k+1)}$. The local triviality of f implies that there exists a fibre-preserving homeomorphism $h_\sigma: f^{-1}(\sigma) \rightarrow \sigma \times F$. Let $\pi_F: \sigma \times F \rightarrow F$ be the projection. By virtue of $X\tau F$ there exists an extension $\eta_\sigma: \psi^{-1}(\sigma) \rightarrow B$ of the map $\xi_\sigma = \pi_F \circ h_\sigma \circ \varphi_k|_{\dots}: \psi^{-1}(\sigma^{(k)}) \rightarrow B$. The composition of diagonal product $(\psi|_{\psi^{-1}(\sigma)})\Delta\eta_\sigma$ and h_σ^{-1} gives us a map $\beta_\sigma: \psi^{-1}(\sigma) \rightarrow f^{-1}(\sigma)$. The union of all β_σ defines an extension $\varphi_{k+1}: \psi^{-1}(B^{(k+1)}) \cup A \rightarrow E$ of φ_k . By virtue of the compactness of X there will be an inclusion $X \subset \psi^{-1}(B^{(k)})$ for some k . In that case, φ_k will be a required extension.

Lemma 2. *Let X be a finite-dimensional compactum with the property*

$$c\text{-dim}_{\pi_k(M)} X \leq k$$

for all k where M is a simply connected CW-complex. Then $X\tau M$.

Proof. Let m be the dimension of X . Consider the Postnikov tower of space $M: * \leftarrow E_2 \leftarrow E_3 \leftarrow \dots \leftarrow E_m \leftarrow \dots$. Let N_m be a homotopy equivalent to

E_m CW-complex which is obtained by attaching to M cells of the dimensions $\geq m + 2$. It is easy to see that $X\tau N_m$ implies $X\tau M$. By Assertion 6, $X\tau E_m$ implies $X\tau N_m$. Thus, it is sufficient to prove that $X\tau E_m$. Applying the Milnor construction of universal bundles $[M]$, we can assume that the Postnikov tower consists of polyhedra with locally trivial fibrations. Apply Assertion 7 and use induction to conclude that $X\tau E_m$.

Lemmas 1, 2 imply the following theorem:

Theorem 2. *For finite-dimensional compacta, an inequality $c\text{-dim}_G Y \leq n$ implies the property $Y\tau M(G, n)$, where $M(G, n)$ is a Moor space.*

Remark. Since $\dim Y \leq n \Leftrightarrow Y\tau S^n$, Theorem 2 is an extension for arbitrary G of the well-known Alexandroff theorem $[A]$: for finite-dimensional compacta the inequality $c\text{-dim}_Z Y \leq n$ implies $\dim Y \leq n$.

Proof of Theorem 1. Let $f: Y \rightarrow E^n$ be an arbitrary map and let $\varepsilon > 0$ be given. We can approximate f by g with a polyhedron K as the image of g . Suppose that $\rho(f, g) < 1/2\varepsilon$. We assume that K is a subpolyhedron in E^n with respect to some triangulation τ . Suppose that mesh $\tau < 1/4\varepsilon$. Consider the second barycentric subdivision $\beta^2\tau$ of τ , and denote by B_σ the star of each simplex $\sigma \in \tau$ with respect to $\beta^2\tau$. Then for every pair σ_1, σ_2 we have $B_{\sigma_1 \cap \sigma_2} = B_{\sigma_1} \cap B_{\sigma_2}$. It is easy to see that B_σ is homeomorphic to the n -dimensional cell.

By $K^{(i)}$ we denote the i -skeleton of K with respect to τ . For every $\sigma \in K^{(0)}$, choose a point $x_\sigma \in \text{Int } B_\sigma - X$. This defines a map $g_0: g^{-1}(K^{(0)}) \rightarrow E^n - X$. Assume that we can define a map $g_i: g^{-1}(K^{(i)}) \rightarrow E^n - X$ such that, for every simplex $\sigma \subset K^{(i)}$, there is the inclusion $g_i(g^{-1}(\sigma)) \subset \text{Int } B_\sigma$. Let σ be an arbitrary $i + 1$ -dimensional simplex in K . Denote by M the set $\text{Int } B_\sigma - X$. The equality $\dim X = \dim Y \leq n - 3$ implies that M is simply connected. Since $\dim X \times Y < n$, for any open subset $V \subset Y$, the equality $H_c^n((\text{Int } B_\sigma \cap X) \times V; \mathbb{Z}) = 0$ holds. By virtue of the Künnet formula, $H_c^k(V; H_c^{n-k}(\text{Int } B_\sigma \cap X)) = 0$ for $k \leq n$. Apply the Alexander duality to obtain $H_c^k(V; H_{k-1}(M)) = 0$ for $k \leq n$. It means $[Ku]$ that $c\text{-dim}_{H_k(M)} Y \leq k$ for all k . Apply Lemmas 1 and 2 to obtain the property $Y\tau M$. Hence there exists a map $q_\sigma: g^{-1}(\sigma) \rightarrow \text{Int } B_\sigma - X = M$ such that the restriction of q_σ onto $g^{-1}(\sigma^{(i)})$ coincides with restriction g_i onto $g^{-1}(\sigma^{(i)})$. The union of q_σ defines a map $g_{i+1}: g^{-1}(K^{(i+1)}) \rightarrow E^n - X$ with the property $g_{i+1}(g^{-1}(\sigma)) \subset \text{Int } B_\sigma$ for each $\sigma \subset K^{(i+1)}$. Note that $Y = g_n^{-1}(K^{(n)})$. The map $g_n: Y \rightarrow E^n - X$ has the property $\rho(g_n, f) < \varepsilon$.

4. PROOF OF COROLLARIES

We called $[D-R\text{-}\S 2]$ a map $f: X \rightarrow E^n$ regular branched if for every $k \geq 0$ $\dim B_k(f) \leq k \dim X - (k - 1)n$, where $B_k(f) = \{z \in E^n : \# f^{-1}(z) \geq k\}$.

Denote by $\mathbf{R}(\mathbf{X}, \mathbf{E}^n)$ the set of all regular branched maps from $\mathbf{X}$ to $\mathbf{E}^n$.

Theorem 3 [D-R-Š2, Theorem 3.1]. *For every compactum $\mathbf{X}$, the set $\mathbf{R}(\mathbf{X}, \mathbf{E}^n)$ is of the second Baire category in $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.*

Lemma 3. *Let $\mathbf{X}$ and $\mathbf{Y}$ be compacta such that $\dim \mathbf{X} \leq \dim \mathbf{Y}$, $\dim(\mathbf{X} \times \mathbf{Y}) < n$ and $\dim \mathbf{X} + \dim \mathbf{Y} \leq (4/3)n - 1$. Then $\mathbf{N} = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f \times \mathbf{Y}) < n \text{ and } \dim(\operatorname{Im} f) \leq \dim \mathbf{X}\}$ is dense in $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.*

Proof. The set $\mathbf{N}_2 = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f) \leq \dim \mathbf{X}\}$ is of the second Baire category. This fact is well known and follows, for example, from [D-R-Š2, Corollary 3.3]. It suffices to prove that $\mathbf{R}(\mathbf{X}, \mathbf{E}^n) \subset \mathbf{N}_1 = \{f \in \mathbf{C}(\mathbf{X}, \mathbf{E}^n) : \dim(\operatorname{Im} f \times \mathbf{Y}) < n\}$. Then $\mathbf{N} = \mathbf{N}_1 \cap \mathbf{N}_2$ will be dense in $\mathbf{C}(\mathbf{X}, \mathbf{E}^n)$.

Since $\dim \mathbf{X} < \frac{2}{3}n$, it follows that $\mathbf{B}_k(f) = \emptyset$ for $k > 2$.

Consider the map $f \times \operatorname{id}_{\mathbf{Y}} : \mathbf{X} \times \mathbf{Y} \rightarrow \operatorname{Im} f \times \mathbf{Y}$. Then, for every $k > 2$, $\mathbf{B}_k(f \times \operatorname{id}_{\mathbf{Y}}) = \emptyset$. Since $\dim \mathbf{X} \leq \frac{2}{3}n - 1/2$, $\dim \mathbf{B}_2(f \times \operatorname{id}_{\mathbf{Y}}) = \dim(\mathbf{B}_2(f) \times \mathbf{Y}) \leq \dim \mathbf{B}_2(f) + \dim \mathbf{Y} \leq 2 \dim \mathbf{X} - n + \dim \mathbf{Y} = \dim \mathbf{X} + \dim \mathbf{Y} + \dim \mathbf{X} - n \leq \frac{4}{3}n - 1 + \frac{2}{3}n - 1/2 - n = n - \frac{3}{2}$. Apply the Zarelua-Skordev two-to-one mapping theorem [Z], [Sk] to conclude that

$$\dim(\operatorname{Im} f \times \mathbf{Y}) \leq \max\{\dim(\mathbf{X} \times \mathbf{Y}), \dim \mathbf{B}_2(f \times \operatorname{id}_{\mathbf{Y}}) + 1\} < n.$$

Proof of Corollary 1. Suppose that $\dim \mathbf{X} < \dim \mathbf{Y}$. According to Lemma 3, there exists a map $f' : \mathbf{X} \rightarrow \mathbf{E}^n$ $1/2\varepsilon$ -close to f and with $\dim(\operatorname{Im} f' \times \mathbf{Y}) < n$. Since $\dim \mathbf{X} \leq n - 3$, the Shtanko theorem [Sh] implies that there exists a reembedding $\varphi : \operatorname{Im} f' \rightarrow \mathbf{E}$ of the set $\operatorname{Im} f'$ which is $\varepsilon/2$ -close to the initial embedding and has the tameness property. Theorem 1 applied to $\varphi(\operatorname{Im} f')$ and $\mathbf{Y}$ implies that there exists a map $g' : \mathbf{Y} \rightarrow \mathbf{E}^n - \varphi(\operatorname{Im} f')$ ε -close to a given map g . Thus, the maps $\varphi \circ f$ and g' are ε -close to f and g , respectively, and have disjoint images.

Proof of Corollary 2. Choose a number m so large that the inequality $\dim \mathbf{X} + \dim \mathbf{Y} + m \leq \frac{4}{3}(n + m) - 1$ holds, and then apply Corollary 1.

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