INSTABILITY OF NONNEGATIVE SOLUTIONS FOR A CLASS OF SEMIPOSITONE PROBLEMS

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ABSTRACT. We consider the boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda f(u(x)), & x \in \Omega \\ Bu(x) &= 0, & x \in \partial\Omega \end{aligned}$$

where Ω is a bounded region in R^N with smooth boundary, $Bu = \alpha h(x)u + (1 - \alpha)\partial u/\partial n$ where $\alpha \in [0, 1]$ $h : \partial\Omega \rightarrow R^+$ with $h = 1$ when $\alpha = 1$, $\lambda > 0$, f is a smooth function with $f(0) < 0$ (semipositone), $f'(u) > 0$ for $u > 0$ and $f''(u) \geq 0$ for $u > 0$. We prove that every nonnegative solution is **unstable**.

INTRODUCTION

In this paper we consider the stability of the semilinear elliptic boundary value problem

$$(1.1) \quad -\Delta u(x) = \lambda f(u(x)), \quad x \in \Omega,$$

$$(1.2) \quad Bu(x) = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded region in R^N with smooth boundary, $Bu(x) = \alpha h(x)u(x) + (1 - \alpha)\partial u/\partial n$ where $\alpha \in [0, 1]$ is a constant, $h : \partial\Omega \rightarrow R^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, $\lambda > 0$ is a constant and f is a smooth function satisfying:

$$(1.3) \quad f(0) < 0 \text{ (semipositone),}$$

$$(1.4) \quad f'(u) > 0 \text{ for } u > 0,$$

$$(1.5) \quad f''(u) \geq 0 \text{ for } u > 0.$$

Recall that a solution of (1.1)–(1.2) is stable in the maximum norm if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|u_0(x) - U(x)\|_\infty < \delta$ then

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$\|u(x, t) - U\|_\infty < \varepsilon$ for all $t \geq 0$, where u satisfies the initial value problem $u_t = \Delta u + \lambda f(u)$; $x \in \Omega$, $t \geq 0$, $Bu = 0$; $x \in \partial\Omega$, $t \geq 0$ and $u(x, 0) = u_0(x)$; $x \in \Omega$.

We prove:

Theorem 1.1. *Every nonnegative solution of (1.1)–(1.5) is unstable.*

We shall prove the instability of a solution u by showing that the principal eigenvalue of the equation linearized about u is negative; the instability of u then follows from the well-known principle of linearized stability.

Very recently in [1], the boundary value problem $-\Delta u(x) = g(x)f(u(x))$; $x \in \Omega$, $Bu(x) = 0$; $x \in \partial\Omega$ was studied where in particular f is nonnegative with $f'' < 0$ and g changes sign. In fact this sign changing property of g makes it a harder question to analyze stability. However by multiplying the corresponding linearized equation and the given differential equation by appropriate terms and then subtracting an equation can be obtained involving the smallest eigenvalue μ_1 of the linearized equation but independent of $g(x)$. Then it follows that μ_1 is positive and so all solutions are stable. But here the situation is very much different with f always taking both positive and negative values for every nonnegative solution. In fact we will depend heavily on the fact that $f(0) < 0$ in proving our result and we will not be restricted to nonconstant solutions as in [1]. We will prove Theorem 1.1 in §2.

We conclude this section by noting that these semipositone problems, in particular when $\alpha = 1$, have been of considerable interest very recently and some existence, nonexistence, uniqueness and multiplicity type results can be found in [2]–[5].

2. Proof of Theorem 1.1. Let $f(u(x)) = g(u(x)) + f(0)$. Then $g'(u) = f'(u) > 0$ for $u > 0$, $g''(u) = f''(u) \geq 0$ for $u > 0$, $g(0) = 0$ and $g(u) > 0$ for $u > 0$. Now we rewrite (1.1)–(1.2) as:

$$(2.1) \quad -\Delta u(x) = \lambda \{g(u(x)) + f(0)\}, \quad x \in \Omega,$$

$$(2.2) \quad Bu(x) = 0, \quad x \in \partial\Omega,$$

and let U be any nonnegative solution. Then the linearized equation associated with (2.1)–(2.2) is:

$$(2.3) \quad -\Delta \phi(x) - \lambda g'(U(x)) \phi(x) = \mu \phi(x), \quad x \in \Omega,$$

$$(2.4) \quad B\phi(x) = 0, \quad x \in \partial\Omega,$$

Let μ_1 be the smallest (principal) eigenvalue and let $\psi(x) (\geq 0)$ be a corresponding eigenfunction. Now consider (2.1) $g'(U)\psi(x) - (2.3)g(U(x))$ and integrating over Ω we have:

$$(2.5) \quad \begin{aligned} & \int_{\Omega} \{(-\Delta U)g'(U)\psi(x) - (-\Delta \psi(x))g(U(x)) - \lambda f(0)g'(U)\psi(x)\} dx \\ &= -\mu_1 \int_{\Omega} \psi(x)g(U(x)) dx. \end{aligned}$$

But

(2.6)

$$\begin{aligned} \int_{\Omega} (-\Delta U)(g'(U)\psi(x))dx &= \int_{\Omega} \nabla(g'(U)\psi(x)) \cdot \nabla U(x)dx \\ &\quad - \int_{\partial\Omega} g'(U)\psi \left(\frac{\partial U}{\partial n} \right) ds \\ &= \int_{\Omega} g''(U)\psi(x)|\nabla U|^2 dx \\ &\quad + \int_{\Omega} g'(U)(\nabla\psi \cdot \nabla U)dx - \int_{\partial\Omega} g'(U)\psi \left(\frac{\partial U}{\partial n} \right) ds, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} (\Delta\psi(x)) g(U(x)) dx &= - \int_{\Omega} \nabla(g(U(x))) \cdot \nabla\psi(x)dx \\ &\quad + \int_{\partial\Omega} g(U) \left(\frac{\partial\psi}{\partial n} \right) ds \\ (2.7) \qquad &= - \int_{\Omega} g'(U)(\nabla U \cdot \nabla\psi)dx \\ &\quad + \int_{\partial\Omega} g(U) \left(\frac{\partial\psi}{\partial n} \right) ds. \end{aligned}$$

Thus using (2.6), (2.7) in (2.5) we obtain

(2.8)

$$\begin{aligned} (-\mu_1) \int_{\Omega} \psi(x)g(U(x)) dx &= \int_{\Omega} g''(U)\psi(x)|\nabla U|^2 dx \\ &\quad - \int_{\Omega} \lambda f(0)g'(U)\psi(x)dx \\ &\quad + \int_{\partial\Omega} \left\{ g(U) \left(\frac{\partial\psi}{\partial n} \right) - g'(U)\psi \left(\frac{\partial U}{\partial n} \right) \right\} ds. \end{aligned}$$

But since $f(0) < 0$, $\psi > 0$; $x \in \Omega$ and both $g''(U) \geq 0$, $g'(U) > 0$ for $U > 0$ we have:

$$(2.9) \qquad \int_{\Omega} g''(U)\psi(x)|\nabla U|^2 dx \geq 0$$

and

$$(2.10) \qquad - \int_{\Omega} \lambda f(0)g'(U)\psi(x)dx > 0.$$

Now when $\alpha = 1$ (then $h \equiv 1$) we have $U = 0$; $x \in \partial\Omega$ and thus $g(U) = 0$; $x \in \partial\Omega$ and also we have $\psi = 0$; $x \in \partial\Omega$. Hence

$$(2.11) \qquad \int_{\partial\Omega} \left\{ g(U) \left(\frac{\partial\psi}{\partial n} \right) - g'(U)\psi \left(\frac{\partial U}{\partial n} \right) \right\} ds = 0.$$

When $\alpha \neq 1$, we have

$$\begin{aligned} & \int_{\partial\Omega} \left\{ g(U) \left(\frac{\partial \psi}{\partial n} \right) - g'(U) \psi \left(\frac{\partial U}{\partial n} \right) \right\} ds \\ &= \int_{\partial\Omega} \left\{ \frac{-g(U) \alpha h \psi}{(1-\alpha)} + \frac{g'(U) \psi \alpha h U}{(1-\alpha)} \right\} ds \\ &= \int_{\partial\Omega} \left\{ \frac{\alpha h \psi}{(1-\alpha)} \right\} [U g'(U) - g(U)] ds. \end{aligned}$$

But $\alpha \geq 0$, $h > 0$, $\psi > 0$; $x \in \Omega$ and $U g'(U) - g(U) \geq 0$ for $U > 0$ since if $z(v) := v g'(v) - g(v)$ then $z(0) = 0$ while $z'(v) = v g''(v) \geq 0$ for $v > 0$. Thus if $\alpha \neq 1$ then

$$(2.12) \quad \int_{\partial\Omega} \left\{ g(U) \left(\frac{\partial \psi}{\partial n} \right) - g'(U) \psi \left(\frac{\partial U}{\partial n} \right) \right\} ds \geq 0.$$

Now using (2.9)–(2.12) in (2.8) we obtain

$$(2.13) \quad (-\mu_1) \int_{\Omega} \mu(x) g(U(x)) dx > 0.$$

But $\psi > 0$; $x \in \Omega$ and $g(U) > 0$ for $U > 0$. Hence from (2.13) we obtain that $\mu_1 < 0$ and the result follows (see Theorem 4.2 in [6]).

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