A UNIQUENESS RESULT FOR A SEMILINEAR REACTION-DIFFUSION SYSTEM

M. ESCOBEDO AND M. A. HERRERO

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ABSTRACT. Let $(u(t, x), v(t, x))$ and $(\bar{u}(t, x), \bar{v}(t, x))$ be two nonnegative classical solutions of

$$(S) \quad \begin{cases} u_t = \Delta u + v^p, & p > 0 \\ v_t = \Delta v + u^q, & q > 0 \end{cases}$$

in some strip $S_T = (0, T) \times \mathbb{R}^N$, where $0 < T \leq \infty$, and suppose that

$$u(0, x) = \bar{u}(0, x), \quad v(0, x) = \bar{v}(0, x),$$

where $u(0, x)$ and $v(0, x)$ are continuous, nonnegative, and bounded real functions, one of which is not identically zero. Then one has

$$u(t, x) = \bar{u}(t, x), \quad v(t, x) = \bar{v}(t, x) \quad \text{in } S_T.$$

If $pq \geq 1$, the result is also true if $u(0, x) = v(0, x) = 0$. On the other hand, when $0 < pq < 1$, the set of solutions of (S) with zero initial values is given by

$$u(t; s) = c_1(t-s)_+^{(p+1)/(1-pq)}, \quad v(t; s) = c_2(t-s)_+^{(q+1)/(1-pq)},$$

where $0 \leq s \leq t$, c_1 and c_2 are two positive constants depending only on p and q , and $(\xi)_+ = \max\{\xi, 0\}$.

1. INTRODUCTION

In this article we shall concern ourselves with the following initial-value problem:

$$(1.1a) \quad u_t = \Delta u + v^p; \quad t > 0, \quad x \in \mathbb{R}^N$$

$$(1.1b) \quad v_t = \Delta v + u^q; \quad t > 0, \quad x \in \mathbb{R}^N,$$

with $N \geq 1$, $p > 0$, $q > 0$ and

$$(1.2a) \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^N$$

$$(1.2b) \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}^N,$$

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where $u_0(x)$ and $v_0(x)$ are nonnegative, continuous, and bounded real functions. Equations (1.1) represent a simple example of a reaction-diffusion system, and they can be viewed as a model to describe heat propagation in a two-component combustible mixture. System (1.1) has been analyzed by several authors in the case of bounded and unbounded domains (cf. [7], [8], [5], [4], [3], etc.). In particular, it has been shown in [3] that problem (1.1), (1.2) always has a classical solution in some strip $S_T = [0, T) \times \mathbb{R}^N$ with $0 < T \leq \infty$. By this we shall mean a pair of nonnegative functions, often denoted in the abridged way $(u(t), v(t))$, such that they belong to $C^{2,1}(S_T)$, satisfy (1.1) and (1.2), and remain bounded in any closed strip $\bar{S}_\tau = [0, \tau] \times \mathbb{R}^N$ with $0 < \tau < T$. A number of properties of solutions of (1.1), (1.2) were derived in [3]. In particular (cf. [3, Theorems 2 and 3]) if $pq > 1$ and

$$(1.3) \quad \frac{\gamma + 1}{pq - 1} \geq \frac{N}{2},$$

where $\gamma = \max\{p, q\}$, every nontrivial solution blows up in finite time, in the sense that it becomes unbounded at some $t = T^* < +\infty$. On the other hand, if $pq > 1$ and (1.3) fails, solutions might be bounded in any strip S_T or have a finite blow-up time, according to the size of their initial values u_0, v_0 . To complete this picture, let us point out that solutions are global in time if $0 < pq \leq 1$ (cf. [3, Theorem 1]).

For the scalar equation

$$(1.4) \quad u_t = \Delta u + u^p; \quad t > 0, \quad x \in \mathbb{R}^N,$$

it is well known that there is a finite-time blow-up of nontrivial solutions if

$$(1.5) \quad 1 < p \leq 1 + \frac{2}{N},$$

whereas global continuation and blow-up are both possible if $p > 1 + \frac{2}{N}$ (cf., for instance, [8], [2], [9]). Indeed, every solution is global if $0 < p \leq 1$. Notice that (1.3) coincides with (1.5) when $p = q$, in which case (1.1), (1.2) reduce to the Cauchy problem for (1.4) if $u_0 = v_0$.

As recalled in [3], local existence for problem (1.1), (1.2) is rather standard, as can be seen by, say, fixed-point arguments. Uniqueness, however, is not a priori clear except in the straightforward case $p \geq 1$ and $q \geq 1$, and was left open in that paper. For instance, one readily checks that, if $0 < pq < 1$, functions

$$(1.6a) \quad u_1(t) = c_1 t^\alpha \quad \text{with } \alpha = \frac{p+1}{1-pq}, \quad c_1^{1-pq} = (1-pq)^{p+1} (p+1)^{-1} (q+1)^{-p}$$

and

$$(1.6b) \quad v_1(t) = c_2 t^\beta \quad \text{with } \beta = \frac{q+1}{1-pq}, \quad c_2 \beta = c_1^q$$

solve (1.1) and are such that $u_1(0) = v_1(0) = 0$. We show here the following

result:

Theorem. Assume that p and q are different from zero and $p < 1$ or $q < 1$. Then

- (a) If $0 < pq < 1$ and $(u_0, v_0) \neq (0, 0)$, problem (1.1), (1.2) has a unique solution.
- (b) If $0 < pq < 1$ and $(u_0, v_0) = (0, 0)$, the set of nontrivial nonnegative solutions of (1.1), (1.2) is given by

$$u(t; s) = c_1(t-s)_+^\alpha, \quad v(t; s) = c_2(t-s)_+^\beta,$$

where $(r)_+ = \max\{r, 0\}$, s is any nonnegative real constant, and c_1, c_2, α , and β are as in (1.6).

- (c) If $pq \geq 1$, there is a unique solution of (1.1), (1.2).

2. PROOF OF THE THEOREM

We begin by recalling an auxiliary result already proved in [3, Lemma 2.4]:

Lemma 1. Let $(u_0, v_0) \neq (0, 0)$, and let $(u(t, x), v(t, x))$ be a solution of (1.1), (1.2). Then for any $\tau > 0$ there exist constants $c > 0$ and $\alpha > 0$ such that

$$(2.1a) \quad u(\tau, x) \geq c \exp(-\alpha|x|^2),$$

$$(2.1b) \quad v(\tau, x) \geq c \exp(-\alpha|x|^2).$$

Proof. We just sketch, for completeness, the case where $u_0 \neq 0$ and $q \geq 1$. Solutions of (1.1), (1.2) satisfy

$$(2.2a) \quad u(t) = S(t)u_0 + \int_0^t S(t-s)v^p(s) ds$$

$$(2.2b) \quad v(t) = S(t)v_0 + \int_0^t S(t-s)u^q(s) ds,$$

where

$$(2.3) \quad S(t)f \equiv S(t)f(x) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-\xi|^2}{4t}\right) f(\xi) d\xi.$$

We may assume without loss of generality that $u_0 > 0$ in some ball centered at the origin. Then there exists $R > 0$ such that $\nu = \inf\{u_0(\xi) : |\tau| \leq R\} > 0$ and, by (2.2a) and (2.3),

$$u(t, x) \geq S(t)u_0 \geq \nu \exp\left(-\frac{|x|^2}{2t}\right) (4\pi t)^{-N/2} \int_{|y| \leq R} \exp\left(-\frac{|y|^2}{2t}\right) dy,$$

whence (2.1a) with $\alpha = \frac{1}{2\tau}$ and $c = \nu(4\pi\tau)^{-N/2} \int_{|y| \leq R} \exp(-\frac{|y|^2}{2\tau}) dy$. On the other hand, using (2.2b) and Jensen's inequality,

$$\begin{aligned} v(t, x) &\geq \int_0^t S(t-s)(S(s)u_0)^q ds \geq \int_0^t (S(t-s)S(s)u_0)^q ds \\ &= \int_0^t (S(t)u_0)^q ds, \end{aligned}$$

whence

$$(2.4) \quad v(t, x) \geq t(S(t)u_0)^q.$$

From this (2.1b) follows, with perhaps a different choice of c and α .

We now specialize to the case $0 < pq < 1$.

Lemma 2. *Let $(u(t, x), v(t, x))$ be a nontrivial solution of (1.1) with $0 < pq < 1$. Then*

$$(2.5a) \quad u(t, x) \geq c_1 t^\alpha$$

$$(2.5b) \quad v(t, x) \geq c_2 t^\beta,$$

where c_1, c_2, α , and β are given in (1.6).

Proof. Assume first that $u_0(x) \geq c \exp(-\alpha|x|^2)$ for some $c > 0$ and $\alpha > 0$. Suppose, for definiteness, that $0 < p, q < 1$. Since

$$(2.6) \quad S(t) \exp(-\alpha|x|^2) = (1 + 4\alpha t)^{-N/2} \exp\left(-\frac{\alpha|x|^2}{1 + 4\alpha t}\right),$$

it follows that

$$u(t, x) \geq S(t)u_0 \geq c(1 + 4\alpha t)^{-N/2} \exp\left(-\frac{\alpha|x|^2}{1 + 4\alpha t}\right)$$

and, by (2.2b),

$$v(t, x) \geq \int_0^t S(t-s)(u(s, x))^q ds,$$

whence

$$\begin{aligned} v(t, x) &\geq c^q \int_0^t (1 + 4\alpha s)^{-N/2(q-1)} (1 + 4\alpha s + 4\alpha q(t-s))^{-N/2} \\ &\quad \times \exp\left(-\frac{\alpha q|x|^2}{1 + 4\alpha s + 4\alpha q(t-s)}\right) ds \\ &\geq c^q (1 + 4\alpha t)^{-N/2} \exp\left(-\frac{\alpha q|x|^2}{1 + 4\alpha q t}\right) t. \end{aligned}$$

We substitute this inequality in (2.2a) and use (2.6) to get

$$\begin{aligned} u(t, x) &\geq c^{pq} \int_0^t (1 + 4\alpha s)^{-Np/2} s^p (1 + 4\alpha q s + 4\alpha p q(t-s))^{-N/2} (1 + 4\alpha q s)^{N/2} \\ &\quad \times \exp\left(-\frac{\alpha p q|x|^2}{1 + 4\alpha q s + 4\alpha p q(t-s)}\right) ds, \end{aligned}$$

whence

$$u(t, x) \geq c^{pq} (1 + 4\alpha t)^{-Np/2} (1 + 4\alpha q t)^{-N/2} \exp\left(-\frac{\alpha p q|x|^2}{1 + 4\alpha p q t}\right) \frac{t^{p+1}}{p+1}.$$

By induction, we obtain, for $j = 0, 1, 2, \dots$,

$$(2.7) \quad u(t, x) \geq c^{(pq)^{j+1}} f_j(t) \exp \left(-\frac{\alpha(pq)^{j+1} |x|^2}{1 + 4\alpha(pq)^{j+1} t} \right) B_{j+1} A_{j+1} t^{(p+1)} \\ \times [(pq)^j + (pq)^{j-1} + \dots + pq + 1],$$

where

(2.8a)

$$f_j(t) = (1 + 4\alpha t)^{-(N/2)p^{j+1}q^j} (1 + 4\alpha qt)^{-(N/2)p^j q^j} (1 + 4\alpha pqt)^{-(N/2)p^j q^{j-1}} \\ \dots (1 + 4\alpha p^j q^{j+1} t)^{-N/2}$$

(2.8b)

$$B_{j+1} = \left(\frac{1}{p+1} \right)^{(pq)^j} \left(\frac{1}{(p+1)(pq+1)} \right) (pq)^{j-1} \\ \dots \left(\frac{1}{(p+1)((pq)^j + (pq)^{j-1} + \dots + 1)} \right)$$

$$(2.8c) \quad A_{j+1} = \left[\left(\frac{1}{(p+1)(q+1)} \right)^{q^{j-1} p^{j-1}} \left(\frac{1}{((p+1)(pq+1)q+1)} \right)^{q^{j-2} p^{j-2}} \right. \\ \left. \dots \left(\frac{1}{((p+1)((pq)^{j-1} + (pq)^{j-2} + \dots + 1)q+1)} \right) \right]^p.$$

Notice that

(2.9)

$$B_{j+1} = \left(\frac{1}{p+1} \right)^{(1-(pq)^{j+1})/(1-pq)} [(1 + pq)^{-(pq)^{j-1}} \dots ((pq)^j + (pq)^{j-1} + \dots + 1)^{-1}] \\ \geq \left(\frac{1}{p+1} \right)^{1/(1-pq)} (1 - pq)^{1/(1-pq)},$$

whereas

$$A_{j+1} = \left(\prod_{k=0}^{j-1} ((p+1)((pq)^k + (pq)^{k-1} + \dots + pq + 1)q + 1)^{-(pq)^{j-1-k}} \right)^p \\ = \left(\prod_{k=0}^{j-1} \left((p+1) \left(\frac{1 - (pq)^{k+1}}{1 - pq} \right) q + 1 \right)^{-(pq)^{j-1-k}} \right)^p \\ \geq \left(\prod_{k=0}^{j-1} \left(\frac{(p+1)q}{1 - pq} + 1 \right)^{-(pq)^{j-1-k}} \right)^p = \left(\prod_{k=0}^{j-1} \left(\frac{q+1}{1 - pq} \right)^{-(pq)^{j-1-k}} \right)^p,$$

so that

$$(2.10) \quad A_{j+1} \geq \left(\frac{1 - pq}{q+1} \right)^{p/(1-pq)}.$$

Finally,

$$\begin{aligned} f_j(t) &= \left(\prod_{k=0}^j (1 + 4\alpha(pq)^k t)^{-(N/2)p(pq)^{j-k}} \right) \left(\prod_{k=0}^j (1 + 4\alpha p^k q^{k+1} t)^{-(N/2)(pq)^{j-k}} \right) \\ &\equiv (f_j^1(t))(f_j^2(t)) \end{aligned}$$

and, since

$$4\alpha t(pq)^j(j+1) = \sum_{k=0}^j (pq)^{j-k} 4\alpha(pq)^k t \geq \sum_{k=0}^j (pq)^{j-k} \log(1 + 4\alpha(pq)^k t),$$

we deduce that

$$(2.11) \quad \lim_{j \rightarrow \infty} f_j^{(i)}(t) \geq 1 \quad \text{for } i = 1, 2.$$

Taking into account (2.8)–(2.11), we let $j \rightarrow \infty$ in (2.7) to obtain (2.5a) under our current assumptions. Estimate (2.5b) can be obtained in an analogous way. The case where p or q are larger than or equal to 1 is similarly dealt with.

As to the general case, we take $\varepsilon > 0$ but otherwise arbitrary, and set $u_\varepsilon(t) \equiv u(t + \varepsilon)$. One then has

$$u_\varepsilon(t) = S(t)u_\varepsilon(0) + \int_0^t S(t-s)v_\varepsilon^p(s)ds,$$

where, by Lemma 1, $u_\varepsilon(0) \geq c \exp(-\alpha|x|^2)$ for some c and α . Therefore $u_\varepsilon(t) \geq c_1 t^\alpha$, and accordingly

$$u(t) = u(\varepsilon + (t - \varepsilon)) \geq c_1(t - \varepsilon)^\alpha,$$

whence the result, since $\varepsilon > 0$ is arbitrary.

As a next step, we show

Lemma 3. Assume that $0 < pq < 1$, and suppose that $(u_0, v_0) \neq (0, 0)$. Then there exists at most one solution of (1.1), (1.2).

Proof. Suppose first that $0 < p < 1$ and $0 < q < 1$. We shall argue by contradiction, thus assuming that for some $(u_0, v_0) \neq (0, 0)$ there exist two different solutions $(u(t), v(t))$ and $(\bar{u}(t), \bar{v}(t))$ defined in some strip S_T . It then follows from (2.2), (2.5), and the mean value theorem that

$$\begin{aligned} (2.12) \quad (u(t) - \bar{u}(t))_+ &\leq \int_0^t S(t-s)(v^p(s) - \bar{v}^p(s))_+ ds \\ &\leq p c_2^{p-1} \int_0^t S(t-s)(v(s) - \bar{v}(s))_+ s^{(p-1)(q+1)/(1-pq)} ds. \end{aligned}$$

In a similar way, we get

$$(2.13) \quad (v(t) - \bar{v}(t))_+ \leq q c_1^{q-1} \int_0^t S(t-s)(u(s) - \bar{u}(s))_+ s^{(q-1)(p+1)/(1-pq)} ds.$$

From (2.12) and (2.13), it follows that, setting $\|f(t, \cdot)\|_\infty = \sup_{x \in \mathbf{R}^N} |f(t, \cdot)|$,

$$\begin{aligned}
 & \|(u - \bar{u})_+(t)\|_\infty \\
 & \leq pq c_1^{q-1} c_2^{p-1} \int_0^t s^{(q+1)(p-1)/(1-pq)} \left(\int_0^s \tau^{(p+1)(q-1)/(1-pq)} \|(u - \bar{u})_+(\tau)\|_\infty d\tau \right) ds \\
 & = pq(1-pq)^{-2}(p+1)(q+1) \\
 & \quad \times \int_0^t s^{(q+1)(p-1)/(1-pq)} \left(\int_0^s \tau^{(p+1)(q-1)/(1-pq)} \|(u - \bar{u})_+(\tau)\| d\tau \right) ds.
 \end{aligned}
 \tag{2.14}$$

We next show that the integrand above is indeed locally integrable. To this end, we first notice that, since p and q are less than one,

$$\|(u - \bar{u})_+(t)\| \leq \int_0^t \|(v - \bar{v})_+(s)\|_\infty^p ds \leq \int_0^t \left(\int_0^s \|(u - \bar{u})_+(\tau)\|_\infty^q d\tau \right)^p ds,
 \tag{2.15}$$

so that

$$\|(u - \bar{u})_+(t)\|_\infty \leq (p+1)^{-(1/(1-pq))} t^{(p+1)/(1-pq)}.
 \tag{2.16}$$

This implies that the right-hand side in (2.14) is convergent. Moreover, substituting (2.16) in (2.14) yields

$$\|(u - \bar{u})_+(t)\|_\infty \leq pq(p+1)^{-(1/(1-pq))} t^{(p+1)/(1-pq)}.
 \tag{2.17}$$

We may now use (2.17) to obtain a new bound for $\|(u - \bar{u})_+(t)\|_\infty$ via (2.14). Iterating this procedure k times, we obtain

$$\|(u - \bar{u})_+(t)\|_\infty \leq (pq)^k (p+1)^{-(1/(1-pq))} t^{(p+1)/(1-pq)}.
 \tag{2.18}$$

Now letting $k \rightarrow \infty$, it follows that $u \equiv \bar{u}$, whence $v \equiv \bar{v}$.

It remains to consider yet the situation where one of the exponents p, q is larger than or equal to one. Assume for instance that $q > 1$, and set $f(s) = s^p$, $g(v) = S(s)v_0 + \int_0^s S(s-\tau)v^q(\tau)d\tau$. Using (2.2) as well as the mean value theorem for $(f \circ g)$, it follows that for some $w = \theta u + (1-\theta)\bar{u}$ with $\theta \equiv \theta(\tau)$ and $0 < \theta < 1$,

$$\begin{aligned}
 (2.19) \quad u(t) - \bar{u}(t) & \leq pq \int_0^t S(t-s) \left(\int_0^s S(s-\tau)w^q(\tau) d\tau \right)^{p-1} \\
 & \quad \times \left(\int_0^s S(s-\tau)w^{q-1}(\tau)(u - \bar{u})(\tau) d\tau \right) ds.
 \end{aligned}$$

On the other hand, by the Hölder inequality,

$$\begin{aligned}
 & \int_0^s S(s-\tau)w^{q-1}(\tau)(u - \bar{u})(\tau) d\tau \\
 & \leq \left(\int_0^s S(s-\tau)|w(\tau)|^q d\tau \right)^{(q-1)/q} \left(\int_0^s S(s-\tau)|(u - \bar{u})(\tau)|^q d\tau \right)^{1/q},
 \end{aligned}$$

and therefore

$$\begin{aligned} \|(u - \bar{u})_+(t)\|_\infty &\leq pq \int_0^t \left(\int_0^s S(s - \tau) |w(\tau)|^q d\tau \right)^{p-1+(q-1)/q} \\ &\quad \times \left(\int_0^s S(s - \tau) |(u - \bar{u})(\tau)|^q d\tau \right)^{1/q} ds. \end{aligned}$$

Taking (2.5) into account again, we obtain

$$(2.20) \quad \|(u - \bar{u})_+(t)\| \leq c \int_0^t s^{-(q+1)/q} \left(\int_0^s \|(u - \bar{u})_+(\tau)\|_\infty^q d\tau \right)^{1/q} ds,$$

where $c = pq(1 - pq)^{-(q+1)/q}(q + 1)^{1/q}(p + 1)$. We now claim that

$$(2.21) \quad v^p(t) - \bar{v}^p(t) \leq \left(\int_0^t S(t - s)(u - \bar{u})_+^q(s) ds \right)^p.$$

Assuming this inequality for the moment, the proof is complete, since (2.12) implies that

$$\|(u - \bar{u})_+(t)\|_\infty \leq \int_0^t \|v^p(s) - \bar{v}^p(s)\|_\infty ds \leq \int_0^t \left(\int_0^s \|(u - \bar{u})_+(\tau)\|_\infty^q d\tau \right)^p ds,$$

so that (2.16) holds, and the result follows at once from (2.20) and (2.16), as in the previous case. To show (2.21), let us write

$$K_t(x) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

For arbitrary $\alpha \in (0, 1)$, one then has

$$\begin{aligned} v(t) &= S(t)v_0 + \int_0^t S(t - s)u^q(s) ds = S(t)v_0 + \int_0^t \int_{\mathbb{R}^N} K_{t-s}(x - y)u^q(s, y) dy ds \\ &= S(t)v_0 + \int_0^t \int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x - y)u^{q-\alpha}(s, y)K_{t-s}^{\alpha/q}(x - y)u^\alpha(s, y) dy ds. \end{aligned}$$

Since $u = \bar{u} + (u - \bar{u}) \leq \bar{u} + (u - \bar{u})_+$ and $u^\alpha \leq \bar{u}^\alpha + (u - \bar{u})_+^\alpha$, we have

$$\begin{aligned} v(t) &\leq S(t)v_0 + \int_0^t \int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x - y)u^{q-\alpha}(s, y) \cdot K_{t-s}^{\alpha/q}(x - y)\bar{u}^\alpha(s, y) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^N} K_{t-s}^{(q-\alpha)/q}(x - y)u^{q-\alpha}(s, y)K_{t-s}^{\alpha/q}(x - y)(u - \bar{u})_+^\alpha(s, y) dy ds. \end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
v(t) &\leq S(t)v_0 + \int_0^t \left(\int_{\mathbb{R}^N} K_{t-s}(x-y) u^q(s, y) dy \right)^{(q-\alpha)/q} \\
&\quad \times \left(\int_{\mathbb{R}^N} K_{t-s}(x-y) \bar{u}^q(s, y) dy \right)^{\alpha/q} ds \\
&\quad + \int_0^t \left(\int_{\mathbb{R}^N} K_{t-s}(x-y) u^q(s, y) dy \right)^{(q-\alpha)/q} \\
&\quad \times \left(\int_{\mathbb{R}^N} K_{t-s}(x-y) (u - \bar{u})_+^q(s, y) dy \right)^{\alpha/q} ds \\
&\leq S(t)v_0 + \left(\int_0^t S(t-s) u^q(s) ds \right)^{(q-\alpha)/q} \\
&\quad \times \left(\int_0^t S(t-s) \bar{u}^q(s) ds \right)^{\alpha/q} \\
&\quad + \left(\int_0^t S(t-s) u^q(s) ds \right)^{(q-\alpha)/q} \left(\int_0^t S(t-s) (u - \bar{u})_+^q(s) ds \right)^{\alpha/q},
\end{aligned}$$

so that, using the fact that for any nonnegative a , b , and c and any $\theta \in (0, 1)$,

$$a + b^{1-\theta} c^\theta \leq (a + b)^{1-\theta} (a + c)^\theta,$$

one arrives at

$$v(t) \leq v(t)^{(q-\alpha)/q} \bar{v}(t)^{\alpha/q} + v(t)^{(q-\alpha)/q} \left(\int_0^t S(t-s) (u - \bar{u})_+^q(s) ds \right)^{\alpha/q}$$

and, setting $\alpha = pq$, (2.21) follows.

Concerning the case where $u_0 = v_0 = 0$, we have the following lemma:

Lemma 4. *Assume that $0 < pq < 1$ and $u_0 = v_0 = 0$. The set of nontrivial nonnegative solutions of (1.1), (1.2) is then given by the family*

$$u(t; s) = c_1 (t-s)_+^{(p+1)/(1-pq)}, \quad v(t; s) = c_2 (t-s)_+^{(q+1)/(1-pq)},$$

where $0 \leq s < t$ and c_1, c_2 are the constants in (1.6).

Proof. The proof consists of a slight modification of the corresponding result for the scalar equation (1.4) (cf. [1]). Let (u, v) be a nontrivial solution of (1.1), (1.2) under our current hypotheses. Then, by (2.2)

$$\|u(t)\|_\infty \leq \int_0^t \left(\int_0^s \|u(\tau)\|_\infty^q d\tau \right)^p ds, \quad \|v(t)\|_\infty \leq \int_0^t \left(\int_0^s \|v(\tau)\|_\infty^p d\tau \right)^q ds,$$

so that

$$(2.22) \quad \|u(t)\|_\infty \leq c_1 t^{(p+1)/(1-pq)}, \quad \|v(t)\|_\infty \leq c_2 t^{(q+1)/(1-pq)},$$

with c_1, c_2 as in (1.6). By hypothesis, there exist $t > 0$ and $x \in \mathbb{R}^N$ such that $u(t, x) > 0$ or $v(t, x) > 0$. Assume, for definiteness, that $u(t, x) > 0$, and define τ as follows

$$\tau = \inf\{t > 0 : u(t, x) > 0\}.$$

By standard results, $u(t, x) > 0$ and $v(t, x) > 0$ for any $x \in \mathbb{R}^N$ and $t > \tau$. Now take $\bar{t} > \tau$ and set

$$\bar{u}(t, x) = u(t + \bar{t}, x), \quad \bar{v}(t, x) = v(t + \bar{t}, x).$$

Then $(\bar{u}, \bar{v})$ solves (1.1) and $\bar{u}(0, x) > 0$, $\bar{v}(0, x) > 0$. Therefore, by Lemma 2,

$$u(t + \bar{t}, x) \geq c_1 t^{(p+1)/(1-pq)}, \quad v(t + \bar{t}, x) \geq c_2 t^{(q+1)/(1-pq)}$$

for any $t \geq 0$. This implies that

$$(2.23) \quad u(t, x) \geq c_1 (t - \tau)_+^{(p+1)/(1-pq)}, \quad v(t, x) \geq c_2 (t - \tau)_+^{(q+1)/(1-pq)}$$

for $x \in \mathbb{R}^N$, $t \geq 0$. Now choose $\underline{t} < \tau$ and define

$$\underline{u}(t, x) = u(t + \underline{t}, x), \quad \underline{v}(t, x) = v(t + \underline{t}, x).$$

By our choice of τ , $(\underline{u}, \underline{v})$ solves (1.1), (1.2) with $\underline{u}(0) = \underline{v}(0) = 0$. Therefore, by (2.22)

$$u(t + \underline{t}, x) \leq c_1 t^{p+1/1-pq}, \quad v(t + \underline{t}, x) \leq c_2 t^{(q+1)/(1-pq)}$$

for any $t \geq 0$ and $x \in \mathbb{R}^N$, and this implies

$$(2.23) \quad u(t, x) \leq c_1 (t - \tau)_+^{(p+1)/(1-pq)}, \quad v(t, x) \leq c_2 (t - \tau)_+^{(q+1)/(1-pq)}$$

for $x \in \mathbb{R}^N$, $t \geq 0$, and the conclusion follows from (2.22) and (2.23).

Our last step consists of this lemma:

Lemma 5. *Let $p > 0$ and $q > 0$ be such that $pq \geq 1$. Then, if $(u(t), v(t))$ and $(\bar{u}(t), \bar{v}(t))$ are solutions of (1.1), (1.2) in some strip S_T , it follows that $u(t) = \bar{u}(t)$ and $v(t) = \bar{v}(t)$ in S_T .*

Proof. The result is rather classical if $p \geq 1$ and $q \geq 1$. Assume for instance that $0 < p < 1 \leq q$. As in the proof of Lemma 3, we set $f(s) = s^p$, $g(v)(s) = S(s)v_0 + \int_0^s S(s-\tau)v^q(\tau)d\tau$ and use the Mean Value Theorem to get (2.19). We now set

$$F(t) = \sup_{0 \leq s \leq t} \|(u - \bar{u})(s)\|_\infty, \quad 0 < t < T.$$

Clearly

$$\begin{aligned}
 & \int_0^s S(s-\tau)w^{q-1}(\tau)(u-\bar{u})(\tau) d\tau \\
 & \leq F(t) \int_0^s S(s-\tau)w^{q-1}(\tau) d\tau \\
 & \leq F(t) \left(\int_0^s S(s-\tau)w^q(\tau) d\tau \right)^{(q-1)/q} \\
 & \leq F(t) \left\| \int_0^s S(s-\tau)w^q(\tau) d\tau \right\|_\infty^{(q-1)/q+p-1} \cdot \left(\int_0^s S(s-\tau)w^q(\tau) d\tau \right)^{1-p},
 \end{aligned}$$

and substituting this into (2.19) yields

$$\|(u-\bar{u})(t)\|_\infty \leq pqF(t) \int_0^t S(t-s) \left\| \int_0^s S(s-\tau)w^q(\tau) d\tau \right\|_\infty^{p-1/q} ds.$$

As $pq \geq 1$, one is thus led to

$$(2.25) \quad F(t) \leq KtF(t) \quad \text{for } t \text{ small enough,}$$

for some constant K depending on p, q, T and the bounds on u and $\bar{u}$ in $\mathbb{R}^N \times [0, t]$. Since (2.25) implies $F(t) \equiv 0$ for t small enough, the result follows by a suitable iteration of the previous argument. The case $0 < q < 1 \leq p$ is similar.

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DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DEL PAIS VASCO, APARTADO 644, 48080 BILBAO, SPAIN

DEPARTAMENTO DE MATEMÁTICA APLICADA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE, 28040 MADRID, SPAIN