PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 1, May 1991

# SOME REMARKS ON THE HOMOLOGY OF MODULI SPACE OF INSTANTONS WITH INSTANTON NUMBER 2

YASUHIKO KAMIYAMA

(Communicated by Frederick R. Cohen)

*Dedicated to Professor Akio Hattori on his sixtieth birthday*

ABSTRACT. Let $M_2$ be the framed moduli space of SU(2) instantons with instanton number 2. By combining the results of Boyer and Mann and the results of Hattori, we determine the structure of $H^*(M_2; \mathbf{Z}_2)$.

## 1. INTRODUCTION

We shall denote by $M_k$ the framed moduli space of $SU(2)$ instantons with instanton number $c_2 = -k$. Recently Boyer and Mann [1] constructed homology operations on $M_k$ for all $k$ and thus constructed new homology classes in $H_*(M_k; \mathbf{Z}_p)$. In the case $k = p = 2$, the result is as follows.

**Theorem 1** [1]. *The elements of $H_*(M_2; \mathbf{Z}_2)$ constructed by Boyer and Mann are given by the following table*:

| $q$ | $H_q(M_2; \mathbf{Z}_2)$ |
|---|---|
| 1 | $z_1 * [1]$ |
| 2 | $z_1^2 \quad z_2 * [1]$ |
| 3 | $Q_1(z_1) \quad z_2 * z_1 \quad z_3 * [1]$ |
| 4 | $Q_2(z_1) \quad z_2^2 \quad z_3 * z_1$ |
| 5 | $Q_1(z_2) \quad z_3 * z_2 \quad Q_3(z_1)$ |
| 6 | $z_3^2 \quad Q_2(z_2)$ |
| 7 | $Q_3(z_2) \quad Q_1(z_3)$ |
| 8 | $Q_2(z_3)$ |
| 9 | $Q_3(z_3)$ |

In another direction Hattori [4] completely determined the homotopy type of $M_2$ and as a result computed $H^*(M_2; \mathbf{Z})$ and $H^*(M_2; \mathbf{Z}_2)$. The results are as follows.

Received by the editors December 5, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58E15; Secondary 55S12.
*Key words and phrases.* Instantons, loop run, Dyer–Lashof operations.

**Theorem 2** [4]. *The cohomology groups of* $M_2$ *with* $\mathbf{Z}$ *coefficients are given by the following table*:

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $H^q(M_2;\mathbf{Z})$ | 0 | $\mathbf{Z}_2$ | $\mathbf{Z}_2$ | $\mathbf{Z}_3\oplus\mathbf{Z}_4$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2$ | $\mathbf{Z}\oplus\mathbf{Z}_2$ | 0 | $\mathbf{Z}_2$ |
| *generators* | | $\beta$ | $\gamma$ | $p^*z \quad \delta$ | $\beta\gamma$ | $\beta\delta$ | $\xi \quad \gamma\delta$ | | $\beta\gamma\delta$ |

**Theorem 3** [4]. *The cohomology groups of* $M_2$ *with* $\mathbf{Z}_2$ *coefficients are given by the following table*:

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $H^q(M_2;\mathbf{Z}_2)$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2\oplus\mathbf{Z}_2$ | $\mathbf{Z}_2$ | $\mathbf{Z}_2$ |
| *generators* | $u$ | $u^2 \quad v$ | $u^3 \quad uv$ | $w \quad u^2v$ | $uw \quad u^3v$ | $u^2w \quad vw$ | $u^3w \quad uvw$ | $u^2vw$ | $u^3vw$ |

The choice of the elements $v$ and $w$ will be specified later.

In this paper we combine these results to obtain further homological information about $M_2$.

## 2. Main results

We first study the following problem. Do the elements of Theorem 1 generate $H_*(M_2;\mathbf{Z}_2)$?

**Proposition 1.** *The elements of Theorem* 1 *generate* $H_*(M_2;\mathbf{Z}_2)$ *and the following relations hold*:

(1) $Q_1(z_1)+z_2*z_1+z_3*[1]=0$.
(2) $Q_2(z_1)=z_3*z_1$.
(3) $Q_1(z_2)+z_3*z_2+Q_3(z_1)=0$.

*Proof.* Let $\mathscr{B}_2$ be the orbit space of $\mathrm{SU}(2)$ connections with instanton number 2 by the action of the based gauge group and let $i: M_2\to\mathscr{B}_2$ be the inclusion.

Direct computations show that each element of Theorem 1 is nontrivial in $H_*(\mathscr{B}_2;\mathbf{Z}_2)$ and differs in $H_*(\mathscr{B}_2;\mathbf{Z}_2)$ except for

$$i_*Q_2(z_2)=i_*z_3*z_1.$$

Therefore by using Theorem 3 we see that the elements of Theorem 1 generate $H_*(M_2;\mathbf{Z}_2)$ and there must be one relation for $q=3,4,5$.

But [1, Proposition 9.10] shows that there are the following relations.

(i) $i_*(Q_1(z_1)+z_2*z_1+z_3*[1])=0$.
(ii) $i_*(Q_2(z_1)+z_3*z_1)=0$.

Using Cartan formula and Adem relation [2] we also see the following relation.

(iii) $i_*(Q_1(z_2) + z_3 * z_2 + Q_3(z_1)) = 0$.

Now by using Theorem 3 we see that the relations (i)–(iii) imply the relations (1)–(3) in Proposition 1.

Next we shall study the Kronecker products of elements of Theorems 1 and 3. On account of Proposition 1 we can take a basis of $H_q(M_2; \mathbf{Z}_2)$ for $q = 3, 4, 5$ as follows:

$$\begin{array}{lll} q = 3 & Q_1(z_1) & z_2 * z_1 \\ q = 4 & z_2^2 & z_3 * z_1 \\ q = 5 & Q_1(z_2) & z_3 * z_2. \end{array}$$

**Theorem 4.** *The Kronecker products of elements of Theorems* 1 *and* 3 *are given by the following table*:

| $q$ | 1 | 2 | |
|---|---|---|---|
| Kronecker products | $\langle u, z_1 * [1]\rangle = 1$ | $\langle u^2, z_1^2\rangle = 0$ | $\langle v, z_1^2\rangle = 1$ |
| | | $\langle u^2, z_2 * [1]\rangle = 1$ | $\langle v, z_2 * [1]\rangle = 0$ |

| 3 | | 4 | |
|---|---|---|---|
| $\langle u^3, Q_1(z_1)\rangle = 0$ | $\langle uv, Q_1(z_1)\rangle = 1$ | $\langle w, z_2^2\rangle = 1$ | $\langle u^2v, z_2^2\rangle = 0$ |
| $\langle u^3, z_2 * z_1\rangle = 1$ | $\langle uv, z_2 * z_1\rangle = 1$ | $\langle w, z_3 * z_1\rangle = 0$ | $\langle u^2v, z_3 * z_1\rangle = 1$ |

| 5 | | 6 | |
|---|---|---|---|
| $\langle uw, Q_1(z_2)\rangle = 1$ | $\langle u^3v, Q_1(z_2)\rangle = 0$ | $\langle u^2w, z_3^2\rangle = 0$ | $\langle vw, z_3^2\rangle = 1$ |
| $\langle uw, z_3 * z_2\rangle = 1$ | $\langle u^3v, z_3 * z_2\rangle = 1$ | $\langle u^2w, Q_2(z_2)\rangle = 1$ | $\langle vw, Q_2(z_2)\rangle = 0$ |

| 7 | | 8 |
|---|---|---|
| $\langle u^3w, Q_3(z_2)\rangle = 1$ | $\langle uvw, Q_3(z_2)\rangle = 0$ | $\langle u^2vw, Q_2(z_3)\rangle = 1$ |
| $\langle u^3w, Q_1(z_3)\rangle = 0$ | $\langle uvw, Q_1(z_3)\rangle = 1$ | |

| 9 |
|---|
| $\langle u^3vw, Q_3(z_3)\rangle = 1$ |

In the above table we define $v$ by

$$\langle v, z_1^2\rangle = 1, \qquad \langle v, z_2 * [1]\rangle = 0.$$

Note that

$$\langle u^2, z_1^2\rangle = 0, \qquad \langle u^2, z_2 * [1]\rangle = 1.$$

We define $w$ by

$$\langle w, z_2^2\rangle = 1, \qquad \langle w, z_3 * z_1\rangle = 0.$$

Note that

$$\langle u^2v, z_2^2\rangle = 0, \qquad \langle u^2v, z_3 * z_1\rangle = 1.$$

*Proof*. Let $\Delta : M_k \to M_k \times M_k$ be the diagonal. Then we can easily show the following relations.

$$\Delta_* z_1 = z_1 \otimes [1] + [1] \otimes z_1 ,$$
$$\Delta_* z_2 = z_2 \otimes [1] + z_1 \otimes z_1 + [1] \otimes z_2 ,$$
$$\Delta_* z_3 = z_3 \otimes [1] + z_2 \otimes z_1 + z_1 \otimes z_2 + [1] \otimes z_3 .$$

The following relation is known in [2].

$$\Delta_* Q_j(a) = \sum_{r,s} Q_{j-r}(a'_s) \otimes Q_r(a''_s) ,$$

where $\Delta_* a = \sum_s a'_s \otimes a''_s$. Theorem 4 easily follows from these results.

Next we shall study the integral classes. On account of Theorem 2 there exists an element $\sigma$ that generates $\mathbf{Z}_4$ in $H_3(M_2 ; \mathbf{Z})$ and there exists an element $\tau$ that generates $\mathbf{Z}$ in $H_7(M_2 ; \mathbf{Z})$. Let

$$j_* : H_*(M_2 ; \mathbf{Z}) \to H_*(M_2 ; \mathbf{Z}_2)$$

be mod 2 reduction.

We shall study $j_* \sigma$ and $j_* \tau$.

**Theorem 5.** *The following relations hold.*

$$j_* \sigma = z_3 * [1] ,$$
$$j_* \tau = Q_3(z_2) .$$

*Proof*. Let $\{E^r_*\}$ be the mod 2 homology Bockstein spectral sequence of $M_2$. The following Nishida relation is known in [2].

$$\beta Q^j(a) = (j-1) Q^{j-1}(a) ,$$

where $\beta$ is the Bockstein operation.

By using the Nishida relation we compute $E^2_*$ as follows.

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $E^2_q$ | 0 | 0 | $z_3 * [1]$ | $z_2^2$ | 0 | 0 | $Q_3(z_2)$ | 0 | 0 |

From this table Theorem 5 follows.

Next as an application of Proposition 1 and Theorem 4, we prove the following theorem.

**Theorem 6.** *The elements of Theorem* 2 *satisfy the following relations*:

(1) $\beta^2 = 2\delta$,
(2) $\delta^2 = 0$,
(3) $\gamma^2 = \beta\delta$.

Note that Theorem 6 completely determines the ring structure of $H^*(M_2 ; \mathbf{Z})$.

*Proof*. (1) is shown in [4].

As $H^8(M_2; \mathbf{Z}) = 0$ holds, (2) follows.

We shall prove (3). Let

$$j_* : H^*(M_2; \mathbf{Z}) \to H^*(M_2; \mathbf{Z}_2)$$

be mod 2 reduction.

All we have to show to prove (3) is $j_*\gamma^2 \neq 0$. Let $u, v, w$ be elements in Theorem 3. Either $j_*\gamma = u^3$ or $uv$ or $u^3 + uv$ holds. We shall show that $j_*\gamma = u^3$ cannot occur. Assertion 1. The following relations hold.

$$u^4 = 0, \qquad v^2 = w.$$

In fact, in the same way as the proof of Theorem 4, we see the following Kronecker products.

$$\langle u^4, z_2^2\rangle = 0, \qquad \langle u^4, z_3 * z_1\rangle = 0,$$
$$\langle v^2, z_2^2\rangle = 1, \qquad \langle v^2, z_3 * z_1\rangle = 0.$$

Assertion 2. The following relation holds.

$$j_*\beta = u^2.$$

In fact, the following holds.

$$j_*\beta = Sq^1 u = u^2.$$

Now suppose $j_*\gamma = u^3$. The table in Theorem 2 shows that

$$j_*(\beta\gamma) \neq 0.$$

But from Assertions 1 and 2 we have

$$j_*(\beta\gamma) = (j_*\beta)(j_*\gamma) = u^2 u^3 = 0.$$

This is a contradiction. Therefore either $j_*\gamma = uv$ or $u^3 + uv$ holds. Anyway

$$(j_*\gamma)^2 = u^2 v^2 = u^2 w \neq 0.$$

This completes the proof of (3).

*Remark*. In [4], whether $\gamma^2 = 0$ or not is left unknown.

Now by using the above results, we can completely determine $H^*(M_2; \mathbf{Z}_2)$.

**Theorem 7.** *$H^*(M_2; \mathbf{Z}_2) = \mathbf{Z}_2[u, v]/(u^4, v^4)$ and $Sq^1 v = uv$ hold. Note that the $\mathscr{A}(2)$-module structure of $H^*(M_2; \mathbf{Z}_2)$ is completely determined.*

*Proof*. The ring structure follows from Theorem 3 and Assertion 1 in Theorem 6. By using Theorem 4 and the following Kronecker products we can easily prove $Sq^1 v = uv$.

$$\langle Sq^1 v, Q_1(z_1)\rangle = 1, \qquad \langle Sq^1 v, z_2 * z_1\rangle = 1.$$

## 3. Appendix

The proof of Proposition 9.5 seems incomplete in [1]. By using Theorem 3, we shall give an explicit proof of this proposition.

**Proposition 9.5** [1]. *$z_i * [1] = Q_i[1]$ for $i = 1, 2, 3$.*

*Proof*. The proof of $z_1 * [1] = Q_1[1]$ is given in [1].

(i) Proof of $z_2 * [1] = Q_2[1]$. Let $i : M_2 \to \Omega_2^3 S^3$ be the inclusion. Clearly $H_2(\Omega_2^3 S^3; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the basis of $Q_1[1]^2 * [-2]$ and $Q_2[1]$. By Theorem 3, $H_2(M_2; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Note that $z_1^2$, $Q_2[1]$, $z_2 * [1]$ are elements of $H_2(M_2; \mathbf{Z}_2)$. But

$$\text{(A)} \qquad \begin{cases} i_* z_1 = Q_1[1] * [-1], \\ i_* z_2 = Q_2[1] * [-1] \end{cases}$$

are given in [1, Theorem 8.6]. Hence $i_* z_1^2 = (Q_1[1] * [-1])^2 = Q_1[1]^2 * [-2]$ and $i_* Q_2[1] = Q_2[1]$. Therefore $i_* : H_2(M_2; \mathbf{Z}_2) \to H_2(\Omega_2^3 S^3; \mathbf{Z}_2)$ is an isomorphism. But $i_*(z_2 * [1]) = (Q_2[1] * [-1]) * [1] = Q_2[1]$ by (A). Therefore $z *_2 [1] = Q_2[1]$ holds.

(ii) *Proof of* $z *_* 3[1] = Q_3[1]$. Let $f : \mathrm{SO}(3) \to M_2$ be the composite of $\mathrm{SO}(3) \to M_1 \times 1 \to M_1 \times M_1 \xrightarrow{*} M_2$ and let $g : \mathrm{SO}(3) \to M_2$ be the composite of

$$\mathrm{SO}(3) \to S^3 \times_{\mathbf{Z}_2} 1 \times 1 \to S^3 \times_{\mathbf{Z}_2} M_1 \times M_1 \xrightarrow{\theta} M_2 .$$

Clearly $f_* z_i = z_i * [1]$ and $g_* z_i = Q_i[1]$ hold for $i = 1, 2, 3$. But we have shown the following.

$$f_* z_1 = g_* z_1, \qquad f_* z_2 = g_* z_2 .$$

By Theorem 3, all we need is to prove the following equalities.

$$\langle u^3, f_* z_3 \rangle = \langle u^3, g_* z_3 \rangle, \qquad \langle uv, f_* z_3 \rangle = \langle uv, g_* z_3 \rangle .$$

Let $\Delta$ be the diagonal; then we easily see the following.

$$\Delta_* z_3 = z_3 \otimes 1 + z_2 \otimes z_1 + z_1 \otimes z_2 + 1 \otimes z_3 .$$

Then $\langle u^3, f_* z_3 \rangle = \langle u^2, f_* z_2 \rangle \langle u, f_* z_1 \rangle = \langle u^2, g_* z_2 \rangle \langle u, g_* z_1 \rangle = \langle u^3, g_* z_3 \rangle$.

$\langle uv, f_* z_3 \rangle = \langle uv, g_* z_3 \rangle$ is similarly proved.


## Acknowledgment

The author is grateful to A. Kono and M. Tezuka for many useful comments.


## References


1. C. Boyer and B. Mann, *Homology operations on instantons*, J. Differential Geom. **28** (1988), 423–465.
2. F. Cohen, J. Lada, and P. May, *The homology of iterated loop spaces*, Lecture Notes in Math., vol. 533, Springer-Verlag, Berlin and New York, 1976.
3. A. Hattori, *Topology of the moduli space of* SU(2)*-instantons with instanton number* 2, J. Fac. Sci. Univ. Tokyo **34** (1987), 741–761.
4. ——, Corrections to *Topology of the moduli space of* SU(2)*-instantons with instanton number* 2, J. Fac. Sci. Univ. Tokyo **36** (1989), 387–388.


Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo 113 Japan