CHARACTERIZATIONS OF ULTRADISTRIBUTIONS WITH COMPACT SUPPORT AND DECOMPOSITION BY SUPPORT

SOON-YEONG CHUNG AND DOHAN KIM

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ABSTRACT. In this paper we prove that every nonquasianalytic ultradistribution can be uniformly majorized by the behavior of test functions only on the support and that every ultradistribution with support in the union $K_1 \cup K_2$ of two compact sets can be decomposed as the sum of one with support in K_1 and one with support in K_2 , along the context of Malgrange [17].

1. INTRODUCTION AND PRELIMINARIES

Let $M = (M_p)$ be a sequence of positive numbers satisfying

$$(1.1) \quad \begin{aligned} & \text{(i) } M_0 = 1, \\ & \text{(ii) } M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \dots, \\ & \text{(iii) } \sum_{p=1}^{\infty} M_{p-1}/M_p < +\infty. \end{aligned}$$

If $\Omega \subset \mathbb{R}^n$ is an open set we shall denote by $C_M(\Omega)$ the set of all $\phi \in C^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants $h = h_K > 0$ and $C = C_{K,h} > 0$ such that

$$(1.2) \quad \sup_{x \in K} |D^\alpha \phi(x)| \leq C h^{|\alpha|} M_{|\alpha|}$$

for any multi-index $\alpha \in \mathbb{N}^n$. Here we say that such a ϕ is ultradifferentiable in Ω . Condition (iii) above guarantees that we can construct cutoff functions and partitions of unity from the Denjoy–Carleman theorem. The class is then called nonquasianalytic. In particular when $M_p = (p!)^s$ for $s > 1$, the class C_M is Gevrey class of order s . Let $C_M(K, h)$ be the set of all $\phi \in C^\infty(\Omega)$ with

$$(1.3) \quad |\phi|_{K,h} = \sup_{\substack{x \in K \\ \alpha \in \mathbb{N}^n}} \frac{|D^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty.$$

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Then $C_M(K, h)$ with the norm $|\cdot|_{K, h}$ is a Banach space and

$$C_M(\Omega) = \bigcap_{K \in \Omega} \bigcup_{h > 0} C_M(K, h).$$

The topology on $C_M(\Omega)$ is $\varprojlim_{K \in \Omega} \varinjlim_{h \rightarrow \infty} C_M(K, h)$ where $\varprojlim$ and $\varinjlim$ mean the projective limit topology and the inductive limit topology respectively.

Let $K \subset R^n$ be a compact set. The space $\mathcal{E}'_M(K)$ of ultradistributions with support in K is defined to be the set of all continuous linear functionals on $C_M(R^n)$ such that for every neighborhood Ω of K and for every $h > 0$ we have

$$(1.4) \quad |u(\phi)| \leq C|\phi|_{\Omega, h}, \quad \phi \in C_M(\overline{\Omega}, h)$$

for some constant $C = C_{\Omega, h} > 0$.

In (1.4) if we replace Ω by K then the choice of the constant C may not be uniform. In other words, $u(\phi)$ cannot be uniformly majorized by the behavior of ϕ on K . In this paper, by obtaining an alternative description of $\mathcal{E}'_M(K)$, it will be shown that $u(\phi)$ can be estimated by the behavior of ϕ only on K . The space $\mathcal{E}'_M(K)$ is characterized in several ways in §2.

In §3 we examine whether every $u \in \mathcal{E}'_M(K_1 \cup K_2)$ can be written as $u = u_1 + u_2$, $u_j \in \mathcal{E}'_M(K_j)$, $j = 1, 2$. This is possible positively for hyperfunctions and quasianalytic ultradistributions (see Hörmander [11, 12]) but, in general, not true for the Schwartz distributions and nonquasianalytic ultradistributions (Lojasiewicz [16], Hörmander [11]). Lojasiewicz [16] gave the geometric condition for K_1 and K_2 that such a decomposition is possible. Here we give a necessary and sufficient condition that such a decomposition theorem holds for the nonquasianalytic ultradistributions, which will be verified along the context of Malgrange [17].

2. THE CHARACTERIZATIONS OF $\mathcal{E}'_M(K)$

Let $M = (M_p)$ be a sequence of positive numbers satisfying, in addition to (1.1), that for some constant $H > 1$

$$(2.1) \quad \begin{aligned} & \text{(i) } M_{p+1} \leq H^{p+q} M_p M_q, \quad p, q = 1, 2, \dots, \\ & \text{(ii) } M_{p+1}^p \leq H^p M_p^{p+1}, \quad p = 0, 1, 2, \dots, \\ & \text{(iii) } M_p / p M_{p-1} \text{ is increasing,} \\ & \text{(iv) } \sum_{q=p+1}^{\infty} M_{q-1} / M_q \leq H^p M_p / M_{p+1}, \quad p = 1, 2, \dots \end{aligned}$$

Let K be a compact set in R^n . A jet in K is a multisequence $F = (f^k)$ of continuous functions on K . For a jet, for $x, y \in K$, $m \in N$ we put

$$(R_y^m F)^k(x) = f^k(x) - \sum_{|l| \leq m - |k|} f^{k+l}(y) \frac{(x-y)^l}{l!}.$$

It is clear that if $f \in C_M(R^n)$, the jet $F = (f^k)$, with $f^k = D^k f$, satisfies the conditions

$$(2.2) \quad |f^k(x)| \leq Ch^{|k|} M_{|k|}, \quad k \in N^n, \quad x \in K,$$

$$(2.3) \quad |(R_y^m F)^k(x)| \leq \frac{C|x-y|^{m-|k|+1} h^{m+1} M_{m+1}}{(m-|k|+1)!},$$

$$x, y \in K, \quad m \in N, \quad |k| \leq m,$$

for some constants $C > 0$ and $h > 0$.

Definition 2.1. A jet $F = (f^k)$ on K is called a (Whitney) jet of class M if it satisfies conditions (2.2) and (2.3). We write $\mathcal{E}_M(K)$ for the space of Whitney jets of class M on K .

We need the following lemma later in this paper; (given in Bruna [3]). It is an extension theorem of Whitney type.

Lemma 2.2. For any $F \in \mathcal{E}_M(K)$ there exists $\tilde{f} \in C_M(R^n)$ such that $D^k \tilde{f}(x) = f^k(x)$ for all $x \in K$ and $k \in N^n$.

We denote by $\mathcal{E}_M(K, h)$ the space of jet $F = (f^k)$ of class of M on K such that

$$(2.4) \quad \|F\|_{K,h} = \sup_{\substack{x \in K \\ \alpha \in N^n}} \frac{|f^\alpha(x)|}{h^{|\alpha|} M_{|\alpha|}} + \sup_{\substack{x \neq y \in K \\ m \in N, |k| \leq m}} \frac{(m-|k|+1)! |(R_y^m F)^k(x)|}{|x-y|^{m-|k|+1} h^{m+1} M_{m+1}}$$

is finite. Then $\mathcal{E}_M(K, h)$ is a Banach space with compact imbedding $\mathcal{E}_M(K, h) \hookrightarrow \mathcal{E}_M(K, h')$ when $h < h'$. Then the space $\mathcal{E}_M(K)$ is $\bigcup_{h>0} \mathcal{E}_M(K, h)$ and the topology on $\mathcal{E}_M(K)$ is given by the inductive limit $\varinjlim_{h \rightarrow \infty} \mathcal{E}_M(K, h)$.

It follows from Lemma 2.2 that we may consider the space $\mathcal{E}_M(K)$ as the set of jets $(D^k f)_{k \in N^n}$ where $f \in C_M(R^n)$. From this point of view, we have the following lemma.

Lemma 2.3. Let K be a compact convex set in R^n . Then the topologies on $\mathcal{E}_M(K)$ given by the norm $\|\cdot\|_{K,h}$ and the norm $|\cdot|_{K,h}$ are equivalent.

Proof. First, it follows from the definition that

$$\|F\|_{K,h} \geq |F|_{K,h}.$$

On the other hand, Taylor's formula implies that there is $z \in K$ such that

$$(2.5) \quad |(R_y^m F)^k(x)| = \left| \sum_{|l|=m-|k|+1} f^{k+l}(z) \frac{(y-x)^l}{l!} \right|$$

$$\leq \sum_{|l|=m-|k|+1} |F|_{K,h} h^{m+1} M_{m+1} \frac{|x-y|^{m-|k|+1}}{l!}$$

$$\leq \frac{|F|_{K,h} (nh)^{m+1} M_{m+1} |x-y|^{m-|k|+1}}{(m-|k|+1)!}.$$

Then it follows that

$$(2.6) \quad \|F\|_{K, nh} \leq 2|F|_{K, h}$$

which completes the proof.

Now we define an ideal of $C_M(\Omega)$ as follows:

$$(2.7) \quad J_M(K : \Omega) = \{\phi \in C_M(\Omega) | D^\alpha \phi|_K = 0 \text{ for all } \alpha\}.$$

Lemma 2.4. $J_M(K : \Omega)$ is the closure of the set of all functions in $C_M(\Omega)$ vanishing in a neighborhood of K .

Proof. The proof of the closedness is trivial. Now let ϕ be an element in $J_M(K : \Omega)$ and ε arbitrary. Choose a function $\theta(x)$ in $C_M(R^n)$ such that

$$(2.8) \quad \theta(x) = \begin{cases} 1 & \text{if } x \in R^n \setminus K_\varepsilon, \\ 0 & \text{if } x \in K_{\varepsilon/2}, \end{cases}$$

where $K_\varepsilon = \{x \in R^n | d(x, K) < \varepsilon\}$. Then $\theta\phi$ vanishes in $K_{\varepsilon/2}$. It follows from Taylor's formula and (2.1) that for some C and h ,

$$(2.9) \quad \begin{aligned} |D^\alpha \phi(x)| &\leq Ch^{|\alpha|} M_{|\alpha|} d(x, K), \\ |D^\alpha(\theta\phi)| &\leq Ch^{|\alpha|} M_{|\alpha|} d(x, K), \end{aligned}$$

for all $x \in \Omega$ and $\alpha \in N^n$. Then for any compact set K' in Ω ,

$$\sup_{\substack{x \in K' \\ \alpha \in N^n}} \frac{|D^\alpha(\phi - \theta\phi)|}{h^{|\alpha|} M_{|\alpha|}} \leq 2C\varepsilon,$$

which completes the proof.

Now consider a map $r: C_M(R^n) \rightarrow \mathcal{E}_M(K)$ with $r(\phi) = (D^\alpha \phi|_K)$. Then r is surjective by Lemma 2.2 and we obtain from (2.6) that

$$(2.10) \quad \|r(\phi)\|_{K, nh} \leq 2|\phi|_{\hat{K}, h}$$

where $\hat{K}$ is the convex hull of K . Thus r is a continuous surjection. From this property and Lemma 2.4 we have the following theorem.

Theorem 2.5. The sequence $0 \rightarrow J_M(K : R^n) \hookrightarrow C_M(R^n) \xrightarrow{r} \mathcal{E}_M(K) \rightarrow 0$ is topologically exact.

This theorem implies that $\mathcal{E}_M(K)$ is topologically isomorphic to $C_M(R^n)/J(K : R^n)$. Thus a continuous linear functional u on $\mathcal{E}_M(K)$ is an ultradistribution which is orthogonal to $J_M(K : R^n)$, so that it defines an ultradistribution with support in K . It follows from Lemma 2.4 that $\mathcal{E}'_M(K)$ is the space of all ultradistributions with support in K . Thus the following is an alternative description of $\mathcal{E}'_M(K)$ which was defined in §1.

Corollary 2.6. *We have $u \in \mathcal{E}'_M(K)$ if and only if for any $h > 0$ there exists a constant C such that*

$$(2.11) \quad |u(\phi)| \leq C \left[\sup_{\substack{x \in K \\ \alpha \in N^n}} \frac{|D^\alpha \phi|}{h^{|\alpha|} M_{|\alpha|}} + \sup_{\substack{x \neq y \in K \\ m \in N, |k| \leq m}} \frac{(m - |k| + 1)! |(R_y^m F)^k(x)|}{|x - y|^{m - |k| + 1} h^{m+1} M_{m+1}} \right].$$

Corollary 2.7. *If $u \in \mathcal{E}'_M(K)$ and $D^k \phi(x) = 0$ for all α and $x \in K$ then $u(\phi) = 0$.*

The following example shows that the right-hand side of (2.11) cannot be replaced by $|\phi|_{K,h}$ simply.

Example 2.8. Let (x_j) be a sequence in R^n such that

$$|x_1| > |x_2| > \dots \rightarrow 0.$$

Let K be a compact set consisting of the sequence (x_j) and its limit 0. Define

$$u(\phi) = \sum_j m_j [\phi(x_j) - \phi(0)],$$

where (m_j) is a positive sequence such that

$$\sum_j m_j |x_j| = 1, \quad \sum_j m_j = \infty.$$

Such a sequence always exists since $\lim |x_j| = 0$. Then u defines an ultradistribution with support K ; in fact, it defines a distribution of order 1 with support K .

Now we suppose that the following is true: for every $h > 0$, there is a constant $C > 0$ such that $|u(\phi)| \leq C |\phi|_{K,h}$, $\phi \in C_M(R^n)$.

Choose $\phi \in C_M(R^n)$ equal to 1 in a neighborhood of $\{x_1, \dots, x_j\}$ and 0 near $\{x_{j+1}, x_{j+2}, \dots\} \cup \{0\}$. Then it follows that

$$\sum_{i \leq j} m_i \leq C,$$

which gives a contradiction when $j \rightarrow \infty$.

On the other hand, even if the second term of the right-hand side in (2.11) cannot be omitted, we give an optimal condition for this.

Theorem 2.9. *Let K be a compact set in R^n with finitely many connected components such that any two points x, y in the same component can be joined by a rectifiable curve in K of length $\leq B|x - y|$. If u is an ultradistribution with support in K , it follows that for any $h > 0$ there exists a constant $C > 0$ such that*

$$(2.12) \quad |u(\phi)| \leq C \sup_{\substack{x \in K \\ \alpha \in N^n}} \frac{|D^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}}, \quad \phi \in C_M.$$

Proof. Let $s \rightarrow x(s)$ be a curve in K with $x(0) = y$ and arclength s . Then

$$(2.13) \quad |F_k(s)| \leq \frac{Cs^{m-|k|+1}}{(m-|k|+1)!} \sum_{|l|=m+1} \sup_{x \in K} |D^l \phi(x)|$$

if

$$F_k(s) = D^k \phi(x(s)) - \sum_{|l| \leq m-|k|} D^{k+l} \phi(y) \frac{(x(s) - y)^l}{l!}.$$

This is obvious when $|k| = m$. If $|k| < m$ and (2.13) is already proved for derivatives of higher order, we conclude by induction that

$$\left| \frac{dF_k(s)}{ds} \right| \leq \frac{Cs^{m-|k|}}{(m-|k|)!} \sum_{|l|=m+1} \sup_{x \in K} |D^l \phi|.$$

Since $F_k(0) = 0$ we obtain (2.13) with C replaced by nC . If $d(x, y)$ is the infimum of the curves from x to y in K then (2.13) gives

$$(2.14) \quad \begin{aligned} & \left| D^k \phi(x) - \sum_{|l| \leq m-|k|} D^{k+l} \phi(y) \frac{(y-x)^l}{l!} \right| \\ & \leq \frac{Cd(x, y)^{m-|k|+1}}{(m-|k|+1)!} \sum_{|l| \leq m+1} \sup_{x \in K} |D^l \phi(x)| \\ & \leq \frac{C|x-y|^{m-|k|+1}}{(m-|k|+1)!} (nhB)^{m+1} M_{m+1} |\phi|_{K,h}. \end{aligned}$$

Then it follows that

$$(2.15) \quad \|\phi\|_{K,A} \leq 2|\phi|_{K,h},$$

where $A = (nhB)$, which completes the proof.

The sets satisfying the hypotheses in Theorem 2.9 are sometimes said to be regular in the sense of Whitney.

Corollary 2.10. *Let K be a regular compact in R^n and ϕ a function in $C^\infty(K)$ with $|\phi|_{K,h} < +\infty$. Then there exists an extension $\tilde{\phi} \in \mathcal{E}_M(R^n)$ of ϕ .*

Proof. We use (2.14) combined with Lemma 2.2. Then it is obvious.

3. DECOMPOSITION THEOREM

Let u be an ultradistribution with support in the union K_1 and K_2 of two compact sets K_1, K_2 . In 1985, Hörmander [11] has given an example showing that u cannot always be decomposed as the sum of $u_1 \in \mathcal{E}'_M(K_1)$ and $u_2 \in \mathcal{E}'_M(K_2)$. In this section we give a necessary and sufficient condition for K_1 and K_2 so that such a decomposition can be performed.

Definition 3.1. Let X and Y be compact subsets of R^n . We say that X and Y are M -regularly situated if for any $h > 0$ there exist constants $h' > 0$ and $C > 0$ such that for any $m \in N$,

$$(3.1) \quad d(x, X \cap Y)^p h^p M_p / p! \leq C d(x, Y)^m h'^m M_m / m!$$

for some $p \in N$, for any $x \in X$.

Remark 1. We can use the associated function of (M_p) to write (3.1) as

$$M^*[hd(x, X \cap Y)] \leq CM^*[h'd(x, Y)],$$

where $M^*(t) = \exp[-M(\frac{1}{t})]$ and $M(\rho) = \sup_p \log(\rho^p p! / M_p)$.

Remark 2. Łojasiewicz [16] has proved that $u \in \mathcal{E}'(X \cup Y)$ can be decomposed if and only if either $X \cap Y = \emptyset$ or there are constants $C > 0$ and $\alpha > 0$ such that

$$(3.2) \quad d(x, Y) \geq C d(x, X \cap Y)^\alpha.$$

Thus we come to know that (3.1) is stronger than (3.2).

Now let δ be the 'diagonal mapping'

$$\delta: \mathcal{E}_M(X \cup Y) \rightarrow \mathcal{E}_M(X) \times \mathcal{E}_M(Y)$$

defined by $\delta(F) = (F|_X, F|_Y)$. Then δ is a continuous injection.

Let π be the mapping,

$$\pi: \mathcal{E}_M(X) \times \mathcal{E}_M(Y) \rightarrow \mathcal{E}_M(X \cap Y)$$

defined by $\pi(F, G) = F|_{X \cap Y} - G|_{X \cap Y}$. Then π is a continuous surjection by Lemma 2.2 and $\pi \circ \delta = 0$.

Now we can obtain the dual mappings,

$$\delta': \mathcal{E}'_M(X) \oplus \mathcal{E}'_M(Y) \rightarrow \mathcal{E}'_M(X \cup Y)$$

and

$$\pi': \mathcal{E}'_M(X \cap Y) \rightarrow \mathcal{E}'_M(X) \oplus \mathcal{E}'_M(Y)$$

with $\delta'(T, S) = T + S$ and $\pi'(T) = (T, -T)$.

Theorem 3.2. Under the above hypotheses the following statements are equivalent.

- (i) Either $X \cap Y = \emptyset$ or X and Y are M -regularly situated.
- (ii) The sequence

$$0 \rightarrow \mathcal{E}_M(X \cup Y) \xrightarrow{\delta} \mathcal{E}_M(X) \times \mathcal{E}_M(Y) \xrightarrow{\pi} \mathcal{E}_M(X \cap Y) \rightarrow 0$$

is exact.

- (iii) The sequence

$$0 \rightarrow \mathcal{E}'_M(X \cap Y) \xrightarrow{\pi'} \mathcal{E}'_M(X) \oplus \mathcal{E}'_M(Y) \xrightarrow{\delta'} \mathcal{E}'_M(X \cup Y) \rightarrow 0$$

is exact.

- (iv) Any ultradistribution $u \in \mathcal{E}'_M(X \cup Y)$ can be decomposed as $u_1 + u_2$ where $u_1 \in \mathcal{E}'_M(X)$ and $u_2 \in \mathcal{E}'_M(Y)$.

Proof. The equivalence of (ii), (iii), and (iv) is easily obtained from the theory of duality in DFS-space. Now we will prove the equivalence of (i) and (ii).

The proof of (i) $\Rightarrow$ (ii). We have only to show that $\text{Ker } \pi \subset \text{Im } \delta$.

Let $F = (f^k) \in \mathcal{E}_M(X)$ and $G = (g^k) \in \mathcal{E}_M(Y)$ with $f^k = g^k$ on $X \cap Y$ for all k . Define a jet $H = (H^k)$ by

$$H^k(x) = \begin{cases} f^k(x), & x \in X, \\ g^k(x), & x \in Y. \end{cases}$$

Now we must show that H is a Whitney jet of class M on $X \cup Y$. But H satisfies the derivative condition (2.2). Then to prove the remaining condition (2.3) we show that there are constants $A > 0$ and $C > 0$ such that

$$(3.3) \quad |(R_y^m H)^k(x)| \leq \frac{C|x-y|^{m-|k|+1} A^{m+1} M_{m+1}}{(m-|k|+1)!}, \quad |k| \leq m, \quad m \in N.$$

The case where x and y both belong to X or both belong to Y is clear. Therefore, we may suppose that $x \in X$ and $y \in Y$. But it reduces to the case where $G = 0$. In fact, if we extend G to $\tilde{G} \in \mathcal{E}_M(X \cup Y)$ and replace F by $F - \tilde{G}$ then we obtain it. Thus (3.3) is reduced to

$$(3.4) \quad |f^k(x)| \leq \frac{C|x-y|^{m-|k|+1} A^{m+1} M_{m+1}}{(m-|k|+1)!}, \quad |k| \leq m, \quad m \in N$$

for all $x \in X$ and $y \in Y$.

Since $F = (f^k)$ is a jet of class M on X there exist $h > 0$ and $C > 0$ such that

$$(3.5) \quad |f^k(x)| \leq \frac{C|x-z|^{m'-|k|+1} h^{m'+1} M_{m'+1}}{(m-|k|+1)!}, \quad |k| \leq m', \quad m' \in N$$

for all $x \in X$ and $z \in X \cap Y$.

From (i) and (2.1) we obtain constants $A' > 0$ and $C' > 0$ such that for any $m'' \in N$,

$$(3.6) \quad \frac{|x-z|^p (hH)^p M_p}{p!} \leq \frac{C'|x-y|^{m''} A'^{m''} M_{m''}}{m''!}$$

for some $p \in N$, for some $z \in X \cap Y$, for $x \in X$ and $y \in Y$.

Let m be an arbitrary integer and $|k| \leq m$. If we take $m'' = m - |k| + 1$ and $m' = p + |k| - 1$ in (3.5) and (3.6) respectively, it follows that

$$\begin{aligned} |f^k(x)| &\leq C|x-z|^p h^{p+|k|} M_{p+|k|}/p! \\ &\leq C|x-z|^p (hH)^p M_p (hH)^{|k|} M_{|k|}/p! \\ &\leq CC'|x-y|^{m-|k|+1} A'^{m-|k|+1} M_{m-|k|+1} (hH)^{|k|} M_{|k|}/(m-|k|+1)! \\ &\leq C''|x-y|^{m-|k|+1} A''^{m+1} M_{m+1}/(m-|k|+1)!, \end{aligned}$$

where $C'' = CC'$ and $A'' = \max\{1, A', hH\}$, which is required.

The proof of (ii) $\Rightarrow$ (i). We assume that $\text{Ker } \pi = \text{Im } \delta$. Then $\text{Im } \delta$ is closed, so the open mapping theorem for DFS-space implies that δ is a homomorphism. Thus for any $A > 0$ and $B > 0$ there are constants $A' > 0$ and $C > 0$ such that

$$(3.8) \quad \|F\|_{X \cup Y, A'} \leq C[\|F\|_{X, nA} + \|F\|_{Y, B}], \quad F \in \mathcal{E}_M(X \cup Y).$$

In particular, it is true that for $F = (f^k)$ vanishing on Y ,

$$(3.9) \quad |f^k(x)| \leq C|x-y|^{m-|k|+1} A'^{m+1} M_{m+1} \|F\|_{X, nA} / (m-|k|+1)!$$

for any $m \in N$, $|k| \leq m$, and $x, y \in X \cup Y$. In view of Lemma 2.2, (3.9) is true for all $F = (D^k \phi)$, $\phi \in C_M(R^n)$ whose supports are in $R^n \setminus Y$. Then it follows from (2.6) that (3.9) becomes

$$(3.10) \quad |D^k \phi(x)| \leq 2C|x-y|^{m-|k|+1} A'^{m+1} M_{m+1} |\phi|_{L, A} / (m-|k|+1)!,$$

where L is the convex hull of X .

Let ϕ_0 be an ultradifferentiable function with $\phi(0) = 1$ and $\text{supp } \phi_0 \subset \{x | |x| < 1\}$ and $\phi(x) = \phi_0((x-x_0)/\varepsilon)$ where $x_0 \in X$ and $\varepsilon = d(x_0, X \cap Y)$. Then $\text{supp } \phi$ does not meet Y . Applying (3.10) to $\phi(x)$ we have

$$(3.11) \quad 1 \leq 2C|x_0-y|^{m+1} A'^{m+1} M_{m+1} |\phi|_{L, A} / (m+1)!.$$

On the other hand, since $|\phi|_{L, A}$ is positive it follows that there exists an n -tuple $p \in N^n$ such that

$$(3.12) \quad \sup_{\substack{x \in L \\ k \in N^n}} \frac{|D^k \phi(x)|}{\varepsilon^{|k|} A^{|k|} M_{|k|}} \leq 2 \frac{|D^p \phi(x)|}{\varepsilon^{|p|} A^{|p|} M_{|p|}}.$$

Combining (3.11) and (3.12) it follows that

$$(3.13) \quad d(x_0, X \cap Y)^{|p|} A^{|p|} M_{|p|} / p! \leq C'|x_0-y|^{m+1} A'^{m+1} M_{m+1} / (m+1)!.$$

Since x_0 is arbitrary, this implies (3.1).

On the other hand, if X and Y are disjoint then all things considered above are trivial. Therefore, the theorem is proved.

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DEPARTMENT OF MATHEMATICS, DUKSUNG WOMEN'S UNIVERSITY, SEOUL, KOREA 132-031

DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL, KOREA 151-742