A CERTAIN CLASS OF TRIANGULAR ALGEBRAS IN TYPE II_1 HYPERFINITE FACTORS

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ABSTRACT. Let $\mathcal{S}$ be the standard triangular UHF algebra in a UHF algebra $\mathcal{A}$, where the rank of $\mathcal{A}$ is a strictly increasing sequence of positive integers. Let $\mathcal{M}$ be the type II_1 hyperfinite factor defined as the weak closure of $\mathcal{A}$ in the tracial representation of $\mathcal{A}$. Define $\mathcal{T}$ to be the weak closure of $\mathcal{S}$ in this representation. Then $\mathcal{T}$ is a reflexive, maximal weakly closed triangular algebra in $\mathcal{M}$. Moreover, $\mathcal{T}$ is irreducible relative to $\mathcal{M}$. We exhibit a strongly closed sublattice $\mathcal{L}$ of $\text{lat } \mathcal{T}$ such that $\mathcal{T} = \text{alg } \mathcal{L}$.

1. INTRODUCTION

Let $\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_n}^{\|\cdot\|}$ UHF be a C^* -algebra of rank (p_n) , where (p_n) is a strictly increasing sequence of positive integers. Assume that the embeddings of the $\mathcal{M}_n$ are standard. Define $\mathcal{S} = \overline{\bigcup_{n=1}^{\infty} \mathcal{T}_n}^{\|\cdot\|}$ to be the *standard* triangular UHF algebra in $\mathcal{A}$, where $\mathcal{T}_n$ is the full algebra of upper triangular matrices in $\mathcal{M}_n$. Let $\{\pi_\tau, \mathcal{H}_\tau\}$ be the tracial representation of $\mathcal{A}$, and let $\mathcal{M}$ be the type II_1 hyperfinite factor defined as the weak closure of $\mathcal{A}$ in $B(\mathcal{H}_\tau)$. In this paper we study the algebra $\mathcal{T} = \overline{\mathcal{S}}^w$, the weak closure of $\mathcal{S}$ in $B(\mathcal{H}_\tau)$. We show that $\mathcal{T}$ is a reflexive, maximal weakly closed triangular algebra in $\mathcal{M}$. Moreover, we prove that $\mathcal{T}$ is *irreducible relative to* $\mathcal{M}$; i.e., $\mathcal{M} \cap (\text{lat } \mathcal{T}) = \{0, 1\}$, where $\text{lat } \mathcal{T}$ is the lattice of all $\mathcal{T}$ -invariant projections in $B(\mathcal{H}_\tau)$. We exhibit a strongly closed sublattice $\mathcal{L}$ of $\text{lat } \mathcal{T}$ with the property that $\mathcal{T} = \text{alg } \mathcal{L}$.

The following two theorems are the main results of this paper:

Theorem I. Let $\mathcal{D}_n$ be the diagonal of the algebra $\mathcal{M}_n$, and $\mathcal{C} = \overline{\bigcup_{n=1}^{\infty} \mathcal{D}_n}^{\|\cdot\|}$. Then $\mathcal{T}$ is a maximal weakly closed triangular algebra in $\mathcal{M}$ with diagonal $\mathcal{D} = \overline{\mathcal{C}}^w$. Moreover, $\mathcal{T}$ is irreducible relative to $\mathcal{M}$. Finally, $\mathcal{T}$ is a reflexive algebra in $B(\mathcal{H}_\tau)$.

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Theorem II. For each left ideal $\mathcal{I}$ of $\mathcal{S}$, let $P_{\mathcal{I}}$ be the projection in $B(\mathcal{H}_{\tau})$ determined by the subspace $M_{\mathcal{I}}$, where

$$M_{\mathcal{I}} = \bigvee \{\xi_x | x \in \mathcal{I}\}.$$

Here ξ_x is the vector in $\mathcal{H}_{\tau}$ corresponding to the element $x \in \mathcal{A}$. Then $P_{\mathcal{I}}$ is $\mathcal{I}$ -invariant. Moreover, let $\mathcal{P}$ be defined as

$$\mathcal{P} = \{P_{\mathcal{I}} | \mathcal{I} \text{ is a left ideal of } \mathcal{S}\},$$

and define $\mathcal{L}$ to be the strongly closed subspace lattice in $B(\mathcal{H}_{\tau})$ generated by $\mathcal{P} \cup \text{lat } \mathcal{M}$. Then $\mathcal{L} \subseteq \text{lat } \mathcal{S}$ and $\mathcal{S} = \text{alg } \mathcal{L}$.

2. TRIANGULARITY AND MAXIMAL PROPERTIES OF $\mathcal{S}$

In this section we prove the first part of Theorem I, namely that the algebra $\mathcal{S}$ is a maximal weakly closed triangular algebra in $\mathcal{M}$ with diagonal $\mathcal{D}$. The main tool developed in this section is presented in Theorem 2.6. In that theorem we construct a family of functions $\varphi_n: \mathcal{M} \rightarrow \mathcal{M}_n \otimes \mathcal{D}^{(n)}$, where $\mathcal{D}^{(n)} = \overline{\bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{D}_k}^w$ and $\mathcal{M}_n^c$ is the commutant of $\mathcal{M}_n$ in $\mathcal{M}$. These functions will allow us to approximate elements of $\mathcal{M}$ by elements of the subalgebras $\mathcal{M}_n \otimes \mathcal{D}^{(n)}$. Our construction of the functions φ_n is a direct adaptation of the construction found in [P, Lemma 1.2]. We will need the following three lemmas concerning the factor $\mathcal{M}$:

Convention. Henceforth, if $\mathcal{X}$ is a subset of $\mathcal{M}$, then $\overline{\mathcal{X}}^{\|\cdot\|_2}$ will denote the $\|\cdot\|_2$ -closure of $\mathcal{X}$ in $\mathcal{M}$.

Lemma 2.1 ([MVN, 1.6.I]). Let $\mathcal{A}$ be a $*$ -subalgebra of $\mathcal{M}$. Then $\overline{\mathcal{A}}^{\|\cdot\|_2} = \overline{\mathcal{A}}^w$. Moreover, for each $x \in \mathcal{A}$, there exists a sequence (x_n) in $\mathcal{A}$ such that $\sup_n \|x_n\| < \infty$ and $x_n = \|\cdot\|_2\text{-}\lim_n x_n$.

Lemma 2.2 ([T, III.5.3]). The σ -strong topology on the unit ball of $\mathcal{M}$ is metrized by the trace norm $\|\cdot\|_2$. Moreover, this metric is complete.

Lemma 2.3 ([MVN, 1.3.2]). Let $x \in \mathcal{M}$, and let (x_n) be a sequence in $\mathcal{M}$. Then $x = \text{s-lim}_n x_n$ if and only if $x = \|\cdot\|_2\text{-}\lim_n x_n$ and $\sup_n \|x_n\| < \infty$.

The next result is modeled on [SV, Lemma I.1.6].

Theorem 2.4. Let n be a positive integer, and let $\mathcal{M}_n^c$ denote the commutant of $\mathcal{M}_n$ in $\mathcal{M}$. Then we have

$$\mathcal{M}_n^c = \overline{\bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k}^w.$$

Proof. For each n , let $\{e_{ij}^{(n)} | 1 \leq i, j \leq p_n\}$ be the standard system of matrix units for $\mathcal{M}_n$. By Lemma 2.1, it suffices to prove that

$$(*) \quad \mathcal{M}_n^c = \overline{\bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k}^{\|\cdot\|_2}.$$

To prove (*), fix n and define the mapping $Q_n: \mathcal{M} \rightarrow \mathcal{M}$ as follows:

$$Q_n(x) = \sum_{i=1}^{p_n} e_{i1}^{(n)} x e_{1i}^{(n)}, \quad \forall x \in \mathcal{M}.$$

For $k \geq n$, $Q_n(\mathcal{M}_k) \subseteq \mathcal{M}_n^c \cap \mathcal{M}_k$. Let $x \in \mathcal{M}_n^c$. Then $Q_n(x) = x$. By Lemma 2.1, we have $x = \|\cdot\|_2\text{-}\lim_{k \geq n} x_k$, where (x_k) is some sequence in $\mathcal{M}$ such that $x_k \in \mathcal{M}_k$. Then $x = \|\cdot\|_2\text{-}\lim_{k \geq n} Q_n(x_k)$. For $k \geq n$, $Q_n(x_k) \in \mathcal{M}_n^c \cap \mathcal{M}_k$; hence,

$$x \in \overline{\bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k}^{\|\cdot\|_2}.$$

This proves (*). $\square$

Theorem 2.5. For $n \geq 1$, we have

$$\mathcal{M} = \text{span}\{vx | v \in \mathcal{M}_n, x \in \mathcal{M}_n^c\}.$$

Proof. Use Theorem 2.4 to write

$$\mathcal{M}_n^c = \overline{\bigcup_{k \geq n} \mathcal{M}_n^c \cap \mathcal{M}_k}^w,$$

and then imitate the proof of the analogous result in [P, p. 316]. $\square$

We are now ready to present the construction of the functions φ_n .

Theorem 2.6. Let $1 \leq n \leq k$ be positive integers. Let $g_1, \dots, g_p$ be the minimal projections in $\mathcal{M}_n^c \cap \mathcal{D}_k$. Define the map $\varphi_{nk}: \mathcal{M} \rightarrow \mathcal{M}$ by

$$\varphi_{nk}(x) = \sum_{i=1}^p g_i x g_i, \quad x \in \mathcal{M}.$$

Then the following conditions hold.

- (a) For all $x \in \mathcal{M}$, $\|\varphi_{nk}(x)\| \leq \|x\|$ and $\|\varphi_{nk}(x)\|_2 \leq \|x\|_2$.
- (b) For all $x \in \mathcal{M}$, the limit $\varphi_n(x) = \|\cdot\|_2\text{-}\lim_{k \geq n} \varphi_{nk}(x)$ exists and may be written as

$$\varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} \in \mathcal{D}^{(n)}.$$

- (c) For all $x \in \mathcal{M}$, $\|\varphi_n(x)\| \leq \|x\|$ and $\varphi_n(x) = \text{s-lim}_{k \geq n} \varphi_{nk}(x)$.
- (d) For all $x \in \mathcal{M}$, $x = \text{s-lim}_n \varphi_n(x)$.

Proof. Fix $x \in \mathcal{M}$, and let $k \geq n$. Because the g_i are orthogonal projections with sum 1, we have $\|\varphi_{nk}(x)\| \leq \|x\|$. Now, if $x \in \bigcup_{m=1} \mathcal{M}_m$, then $\|\varphi_{nk}(x)\|_2 \leq \|x\|_2$. By Lemma 2.1, $\mathcal{M} = \overline{\bigcup_{m=1} \mathcal{M}_m}^{\|\cdot\|_2}$; hence, for arbitrary $x \in \mathcal{M}$, we have $\|\varphi_{nk}(x)\| \leq \|x\|_2$. This proves (a).

To prove (b), use Theorem 2.5 to write $x = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} x_{ij}$, $x_{ij} \in \mathcal{M}_n^c$. Now

use Theorem 2.4 and Lemma 2.1 to write $x_{ij} = \|\cdot\|_2\text{-}\lim_{k \geq n} x_{ij}^{(k)}$, $x_{ij}^{(k)} \in \mathcal{M}_n^c \cap \mathcal{D}_n$, $\sup_{k \geq n} \|x_{ij}^{(k)}\| < \infty$. Then $\varphi_{nk}(x_{ij}^{(k)}) \in \mathcal{M}_n^c \cap \mathcal{D}_k$. By (a), $(\varphi_{nk}(x_{ij}^{(k)}))_{k \geq n}$ is a norm-bounded $\|\cdot\|_2$ -Cauchy sequence; hence Lemma 2.2 implies that there exists $\varphi_n(x_{ij}) \in \mathcal{D}^{(n)}$ such that $\varphi_n(x_{ij}) = \|\cdot\|_2\text{-}\lim_{k \geq n} \varphi_{nk}(x_{ij}^{(k)})$. We then have $\varphi_n(x_{ij}) = \|\cdot\|_2\text{-}\lim_{k \geq n} \varphi_{nk}(x_{ij})$. Define $\varphi_n(x)$ by

$$\varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} = \varphi_n(x_{ij}) \in \mathcal{D}^{(n)}.$$

Then the limit $\|\cdot\|_2\text{-}\lim_{k \geq n} \varphi_k(x)$ exists and is equal to $\varphi_n(x)$. This proves (b).

Condition (c) is a direct consequence of (a), (b), Lemma 2.2, and Lemma 2.3.

To prove (d), note first that, by (b), the sequence $(\varphi_n(x))$ is norm bounded. Write $x = \|\cdot\|_2\text{-}\lim_n x_n$, where $x_n \in \mathcal{M}_n$. Then $\|x - \varphi_n(x)\|_2 \leq 2\|x - x_n\|_2$. Thus $x = \|\cdot\|_2\text{-}\lim_n \varphi_n(x)$. It follows from Lemma 2.3 that $x = \text{s-lim}_n \varphi_n(x)$, $x \in \mathcal{M}$. This proves (d). $\square$

The next lemma is a key fact.

Lemma 2.7. *Let $n \geq 1$, and suppose that $d \in \mathcal{D}^{(n)}$. If $i > j$ and $e_{ij}^{(n)} d \in \mathcal{F}$, then $e_{ij}^{(n)} d = 0$.*

Proof. Observe that $e_{ij}^{(n)} d d^* = (e_{ij}^{(n)} d) d^* \in \mathcal{F}$. Write $e_{ij}^{(n)} d d^* = \text{w-lim}_\alpha x_\alpha$, where (x_α) is some net in $\mathcal{F}$. Then, we have $\text{w-lim}_\alpha (e_{ij}^{(n)} d d^* - x_\alpha) = 0$.

Define $a = e_{ij}^{(n)}$, and let ξ_1, ξ_a be the vectors in $\mathcal{H}_\tau$ corresponding, respectively, to the elements 1 and a . We then have $\lim_\alpha \langle (add^* - x_\alpha) \xi_1, \xi_a \rangle = 0$. Now, for all α , we have $\tau(e_{ji}^{(n)} x_\alpha) = 0$; therefore, $\langle (add^* - x_\alpha) \xi_1, \xi_a \rangle = \|e_{ij}^{(n)} d\|_2^2$. It follows that $\|e_{ij}^{(n)} d\|_2 = 0$, i.e., $e_{ij}^{(n)} d = 0$. $\square$

We have now developed enough machinery to prove the main results of this section:

Theorem 2.8. *$\mathcal{F}$ is a triangular algebra in $\mathcal{M}$ with diagonal $\mathcal{F} \cap \mathcal{F}^* = \mathcal{D}$.*

Proof. We first show that $\mathcal{D}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$. To prove this, let $\mathcal{G}$ be an abelian von Neumann subalgebra of $\mathcal{M}$ such that $\mathcal{D} \subseteq \mathcal{G}$. Let $n \geq 1$, and let $x \in \mathcal{G}$ be arbitrary. By Theorem 2.6, we have $\varphi_n(x) = \text{s-lim}_{k \geq n} \varphi_{nk}(x)$. Therefore, $\varphi_n(x) \in \mathcal{G}$. Therefore, if $y \in \mathcal{G}$, then $\varphi_n(x)$ and y commute. In particular, $e_{ij}^{(n)} \varphi_n(x) = \varphi_n(x) e_{ij}^{(n)}$, for $1 \leq i, j \leq p_n$. By Theorem 2.6, we have

$$(**) \quad \varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} \in \mathcal{D}^{(n)}.$$

Therefore, if $i \neq j$, then $e_{ij}^{(n)} d_{ij} = 0$. Thus $\varphi_n(x) \in \mathcal{D}$. By Theorem 2.6, we have $x = \text{s-lim}_n \varphi_n(x)$, consequently, $x \in \mathcal{D}$. This proves that $\mathcal{D} = \mathcal{G}$, so $\mathcal{D}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$.

To show that $\mathcal{T}$ has diagonal $\mathcal{D}$, let $x \in \mathcal{T} \cap \mathcal{T}^*$, and let $n \geq 1$. Express $\varphi_n(x)$ as in (**). We have $\varphi_n(x) \in \mathcal{T} \cap \mathcal{T}^*$; therefore, $e_{ij}^{(n)} d_{ij} = e_{ii}^{(n)} \varphi_n(x) e_{jj}^{(n)} \in \mathcal{T} \cap \mathcal{T}^*$. Thus, Lemma 2.7 implies that $e_{ij}^{(n)} d_{ij} = 0$ for $i \neq j$. Hence, for all n , $\varphi_n(x) \in \mathcal{D}$. By Lemma 2.6, $x = \text{s-lim}_n \varphi_n(x)$, and hence, $x \in \mathcal{D}$. This proves that $\mathcal{T}$ has diagonal $\mathcal{D}$. $\square$

Theorem 2.9. $\mathcal{T}$ is a maximal weakly closed triangular algebra in $\mathcal{M}$.

Proof. Let $\mathcal{R}$ be a weakly closed triangular algebra in $\mathcal{M}$ such that $\mathcal{T} \subseteq \mathcal{R}$. Then $\mathcal{D} = \mathcal{R} \cap \mathcal{R}^*$. Let $n \geq 1$, and let $x \in \mathcal{R}$. Write $\varphi_n(x)$ as in (**) of Theorem 2.8. By (c) of Theorem 2.6, we have $\varphi_n(x) = \text{s-lim}_{k \geq n} \varphi_{nk}(x)$; hence, because $\varphi_{nk}(x) \in \mathcal{R}$, we have $\varphi_n(x) \in \mathcal{R}$. Let $1 \leq j < i \leq p_n$. Then $e_{ij}^{(n)} d_{ij} = e_{ii}^{(n)} \varphi_n(x) e_{jj}^{(n)} \in \mathcal{R}$. Define $y = e_{ij}^{(n)} d_{ij}$. Then we have $y^* = e_{ji}^{(n)} d_{ij}^* \in \mathcal{T}$; therefore, $y + y^* \in \mathcal{R}$. But then $y + y^* \in \mathcal{R} \cap \mathcal{R}^* = \mathcal{D}$. It follows from Lemma 2.7 that $e_{ij}^{(n)} d_{ij} = 0$. Hence, for all n , $\varphi_n(x) \in \mathcal{T}$. Because $x = \text{s-lim}_n \varphi_n(x)$, we conclude that $x \in \mathcal{T}$. Hence $\mathcal{R} = \mathcal{T}$. $\square$

3. $\mathcal{T}$ IS IRREDUCIBLE RELATIVE TO $\mathcal{M}$

In this section we prove the second part of Theorem I, namely that $\mathcal{T}$ is irreducible relative to $\mathcal{M}$. This proof of this is essentially by reductio ad absurdum.

Lemma 3.1. Let (x_n) be a sequence in $\mathcal{T}$ such that $\sup_n \|x_n\| < \infty$. Let $q \in \mathcal{M} \cap (\text{lat } \mathcal{T})$, and suppose that (q_n) is a sequence of projections in $\mathcal{M}$ such that $q = \|\cdot\|_2\text{-lim}_n q_n$. Then $\lim_n \|q_n^\perp x_n q_n\|_2 = 0$.

Proof. Verify that for all n , $\|q_n^\perp x_n q_n\|_2 \leq 3M\|q - q_n\|_2$, where $M = \sup_n \|x_n\|$. $\square$

Straightforward considerations suffice to prove the following lemma; hence we will omit its proof.

Lemma 3.2. Let $q \in \mathcal{D}$ be a projection. Then there exists a sequence (q_n) of projections in $\mathcal{M}$ such that $q_n \in \mathcal{D}_n$ and $q = \|\cdot\|_2\text{-lim}_n q_n$.

We are now in a position to prove that $\mathcal{T}$ is irreducible relative to $\mathcal{M}$.

Theorem 3.3. The algebra $\mathcal{T}$ is irreducible relative to $\mathcal{M}$.

Proof. Let $q \in \mathcal{M} \cap (\text{lat } \mathcal{T})$. By Lemma 3.2, there exists a sequence (q_n) of projections in $\mathcal{D}_n$ such that $q = \|\cdot\|_2\text{-lim}_n q_n$ and $q_n = \sum_{i=1}^{p_n} q_i^{(n)} e_{ii}^{(n)}$, with $q_i^{(n)} \in \{0, 1\}$. Let (N_n) and (L_n) be sequences of positive integers defined as follows:

$$\begin{aligned} N_n &= |\{i: 1 \leq i \leq p_n \text{ and } q_i^{(n)} = 0\}|, \\ L_n &= p_n - N_n. \end{aligned}$$

Then exactly one of the following conditions holds:

- (a) $\liminf_n N_n/p_n = 0$.
 (b) $\liminf_n N_n/p_n > 0$.

If (a) holds, then we are done, for then $\|1 - q\|_2^2 = \lim N_n/p_n = 0$, i.e., $q = 1$. Therefore we may assume that (b) holds. We claim that $q = 0$. To prove this, assume that $q \neq 0$. Then the sequence (N_n/p_n) is a bounded sequence. Therefore, by (b), there exists an $\alpha > 0$ such that some subsequence of (N_n/p_n) converges to α . Without loss of generality, we may assume that $\lim_n N_n/p_n = \alpha$. We then have $\lim_n L_n/p_n = 1 - \alpha$. We cannot have $\alpha = 1$; thus, $0 < \alpha < 1$. Now, there exists a sequence (X_n) of matrices such that for all n , $X_n \in M_{p_n}$, $\|X_n\| = 1$ and

- (c) $\|X_n\|_2 = \begin{cases} N_n/p_n, & \text{if } N_n \leq L_n; \\ L_n/p_n, & \text{otherwise.} \end{cases}$
 (d) $Q_n^\perp X_n Q_n = X_n$.

Here $Q_n = \text{diag}(q_1^{(n)}, \dots, q_{p_n}^{(n)})$. Note that the matrix of q_n with respect to the $e_{ij}^{(n)}$ is Q_n . Now let $R_n \in M_{p_n}$ be the matrix defined by $R_n = W_n \otimes X_n$. Here, W_n is the $r_{n+1,n} \times r_{n+1,n}$ matrix having 1's on the superdiagonal, while all other entries are 0, and $r_{n+1,n} = p_{n+1}/p_n$. It is clear that $\|R_n\| = 1$. Define $x_n \in \mathcal{M}$ to be the element of $\mathcal{M}$ whose matrix with respect to the $e_{ij}^{(n+1)}$ is R_n . Let $r = r_{n+1,n}$; then, by (d), the matrix of $q_n^\perp x_n q_n$ with respect to the $e_{ij}^{(n+1)}$ is given by $(1_r \otimes Q_n^\perp) R_n (1_r \otimes Q_n) = R_n$. Hence we have $q_n^\perp x_n q_n = x_n$. It follows that $\|q_n^\perp x_n q_n\|_2 = \|x_n\|_2 = \|R_n\|_2$. A simple calculation shows that $\|R_n\|_2^2 = (1 - 1/r_{n+1,n}) \|X_n\|_2^2$. Because $p_n < p_{n+1}$, we have $r_{n+1,n} \geq 2$, and hence $(1 - 1/r_{n+1,n}) \geq 1/2$. Consequently for all n , $\|q_n^\perp x_n q_n\|_2^2 = \|R_n\|_2^2 \geq (1/2) \cdot \|X_n\|_2^2$. Therefore

- (e) $\liminf_n \|q_n^\perp x_n q_n\|_2 \geq (1/\sqrt{2}) \cdot \liminf_n \|X_n\|_2$.

Now, $\lim_n N_n/p_n = \alpha > 0$ and $\lim_n L_n/p_n = 1 - \alpha > 0$. Therefore condition (c) implies that $\liminf_n \|X_n\|_2 > 0$. Hence (e) implies that

- (f) $\liminf_n \|q_n^\perp x_n q_n\|_2 > 0$.

Because the matrix R_n is upper triangular, we see that for all n , $x_n \in \mathcal{F}$. Because $\|x_n\| = \|X_n\| = 1$, it follows from Lemma 4.1 that $\lim_n \|q_n^\perp x_n q_n\|_2 = 0$. But this contradicts (e); hence we conclude that $q = 0$. This completes the proof of the theorem. $\square$

4. REFLEXIVITY OF $\mathcal{T}$

In this final section we prove the last part of Theorem I, namely that $\mathcal{T}$ is a reflexive subalgebra of $B(\mathcal{H}_\tau)$. We conclude the section with a proof of Theorem II.

Theorem 4.1 ([L, 4.22]). *Every weakly closed subalgebra of a finite von Neumann algebra containing m.a.s.a. is reflexive.*

Theorem 4.2. *The algebra $\mathcal{T}$ is reflexive.*

Proof. Immediate by Theorems 2.8 and 4.1.

Theorem 4.2. *Let $n \geq 1$, and suppose that $d \in \mathcal{D}^{(n)}$. If $i > j$ and $e_{ij}^{(n)} d \in \mathcal{R}$, then $e_{ij}^{(n)} d = 0$, where $\mathcal{R} = \overline{\bigcup_{n=1} \mathcal{T}_n}^{\|\cdot\|_2}$. It follows that $\mathcal{R} \subseteq \mathcal{T}$.*

Proof. Observe that $|\langle (add^* - x)\xi_1, \xi_a \rangle|^2 \leq \|e_{ii}^{(n)}\|_2^2 \cdot \|e_{ij}^{(n)} dd^* - x\|_2^2$, for all $i > j$, $d, x \in \mathcal{A}$, and $a = e_{ij}^{(n)}$. Use this inequality to proceed as in the proof of Theorem 2.7. Now use Theorem 2.6(b) to check that $\mathcal{R} \subseteq \mathcal{T}$. $\square$

Theorem 4.3. $\mathcal{T} = \text{alg } \mathcal{L}$.

Proof. First, observe that 2.1, 2.6, and 2.7 imply that $\mathcal{L} \subseteq \text{lat } \mathcal{T}$. Hence, to complete the proof, it suffices to show that $\text{alg } \mathcal{L} \subseteq \mathcal{T}$. Let $x \in \text{alg } \mathcal{L}$. Then $x \in \text{alg lat } \mathcal{M} = \mathcal{M}$. We claim that for all $y \in \mathcal{S}$, $xy \in \mathcal{T}$; hence, with $y = 1$, we get $x \in \mathcal{T}$. To prove the claim, write $x = \|\cdot\|_2\text{-}\lim_n x_n$, where $x_n \in \mathcal{M}_n$ and $\sup_n \|x_n\| < \infty$. Let $y \in \mathcal{S}$. Then $x\xi_y \in M_\mathcal{S}$; therefore, there exists a sequence (y_n) in $\mathcal{S}$ such that $x\xi_y = \|\cdot\|\text{-}\lim_n \xi_{y_n}$. We then have $\|x_n \xi_y - \xi_{y_n}\| \leq \|x_n \xi_y - x \xi_y\| + \|x \xi_y - \xi_{y_n}\|$. Because $x = \text{s-lim}_n x_n$, it follows that $\|x_n y - y_n\|_2^2 = \|\xi_{(x_n y - y_n)}\|^2 = \|x_n \xi_y - \xi_{y_n}\|^2 \rightarrow 0$. Hence we have $\lim_n \|x y - y_n\|_2 = 0$. Therefore, $x y = \|\cdot\|_2\text{-}\lim_n y_n$. Theorem 4.2 then gives $x y \in \mathcal{R}$, i.e., $x y \in \mathcal{T}$. $\square$

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