FIXED POINTS OF ORIENTATION REVERSING HOMEOMORPHISMS OF THE PLANE

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ABSTRACT. Let h be an orientation reversing homeomorphism of the plane onto itself. If X is a plane continuum invariant under h , then h has a fixed point in X . Furthermore, if at least one of the bounded complementary domains of X is invariant under h , then h has at least two fixed points in X .

1. INTRODUCTION

In 1951, M. L. Cartwright and J. C. Littlewood (see [6]) proved that if X is a nonseparating, connected, compact, nonempty subset of the plane, invariant under an orientation preserving homeomorphism h of the plane, then h has a fixed point in X . Simpler proofs of the Cartwright-Littlewood theorem were later provided by O. H. Hamilton in [7] and by Morton Brown in [5]. In [2], H. Bell proved the Cartwright-Littlewood theorem for an arbitrary homeomorphism of the plane (see also [1] and [4]). The results of this paper depend strongly on Bell's theorem for orientation reversing homeomorphisms of the plane.

Throughout this paper, h is an orientation reversing homeomorphism of the plane R^2 onto itself, and X is a continuum in the plane (i.e. a nonempty, connected and compact subset of R^2), invariant under h . The components of $R^2 - X$ are called the complementary domains of X .

M. Barge asked whether h has always a fixed point in X , and in some cases, for instance if X has exactly two complementary domains, whether h has two fixed points in X . The following theorem provides an answer to Barge's question:

Theorem. *If at least one of the bounded complementary domains of X is invariant under h , then h has at least two fixed points in X . Otherwise, h has at least one fixed point in X .*

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An easy example of two tangent circles with disjoint interiors and an orientation reversing homeomorphism of R^2 interchanging the circles shows that if no bounded complementary domain of a separating plane continuum X is invariant, then h may have only one fixed point in X .

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2. PRELIMINARY LEMMAS

Let A be an annulus in R^2 and let X be a separating plane continuum contained in A such that the two boundary components of A , C_1 and C_2 , are in different complementary domains of X (equivalently, X separates the annulus A between its boundary components C_1 and C_2). Let H be an orientation reversing homeomorphism of R^2 onto itself. Assume that both A and X are invariant under H .

Let $\tilde{A}$ be the universal covering space of A , and let $p : \tilde{A} \rightarrow A$ be the covering map. A homeomorphism $\tilde{H} : \tilde{A} \rightarrow \tilde{A}$ such that the diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{H}} & \tilde{A} \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{H|_A} & A \end{array}$$

commutes, will be called a *lifting* of $H|_A$. (For elements of the theory of covering spaces see for instance [8].)

Let B be a two-point compactification of $\tilde{A}$. Denote the added points by b_0 and b_1 . Hence $B = \tilde{A} \cup \{b_0, b_1\}$. Notice that B is homeomorphic to a disk and we may assume that B is a subset of R^2 .

Lemma P1. *The set $p^{-1}(X) \cup \{b_0, b_1\}$ is a continuum.*

Proof. We can write $X = \bigcap_{n=1}^{\infty} A_n$, where each A_n is a piecewise linear disk with holes, $A_1 = A$, and $A_{n-1} \supset A_n$ for $n = 1, 2, \dots$. Since X separates A between C_1 and C_2 , so does each A_n . Hence, for each $n = 1, 2, \dots$, the set $p^{-1}(A_n) \cup \{b_0, b_1\}$ is a continuum. Therefore, the set $p^{-1}(X) \cup \{b_0, b_1\}$, which is the intersection of the sets $p^{-1}(A_n) \cup \{b_0, b_1\}$, is a continuum.

Lemma P2. *If X has exactly two complementary domains, then the continuum $p^{-1}(X) \cup \{b_0, b_1\}$ does not separate the plane.*

Proof. We may assume that each of the sets A_n defined in the proof of Lemma P1, is an annulus. In this case, each of the sets $p^{-1}(A_n) \cup \{b_0, b_1\}$ is a disk, and Y is the intersection of a nested sequence of disks. Hence Y is a nonseparating plane continuum.

Lemma P3. *If x_0 is a fixed point of H in A , then there is a lifting of $H|_A$, $\tilde{H}_1 : \tilde{A} \rightarrow \tilde{A}$, having exactly one fixed point in $p^{-1}(x_0)$, and there is a lifting of $H|_A$, $\tilde{H}_2 : \tilde{A} \rightarrow \tilde{A}$, without a fixed point in $p^{-1}(x_0)$.*

Proof. Let $\tilde{H}$ be a lifting of $H|A$. Since H is orientation reversing on A and $H(C_i) = C_i$ for $i = 1, 2$, $\tilde{H}$ is orientation reversing on $\tilde{A}$ and $\tilde{H}(p^{-1}(C_i)) = p^{-1}(C_i)$ for $i = 1, 2$. Let Z be the set of integers. There is an integer n_0 , and there is a one-to-one and onto function $\mu : p^{-1}(x_0) \rightarrow Z$ such that $\mu \circ \tilde{H} \circ \mu^{-1}(n) = -n + n_0$ for every $n \in Z$. Notice that $\tilde{H}$ has a fixed point $\mu^{-1}(n_0/2) \in p^{-1}(x_0)$ iff n_0 is even. If $\tilde{H}$ has a fixed point in $p^{-1}(x_0)$, let $\tilde{H}_1 = \tilde{H}$, and let $\tilde{H}_2$ be such that $\tilde{H}_2(x) = \mu^{-1}(\mu(\tilde{H}(x)) + 1)$ for $x \in p^{-1}(x_0)$. Then $\mu \circ \tilde{H}_2 \circ \mu^{-1}(n) = -n + n_0 + 1$, and $\tilde{H}_2$ has no fixed points in $p^{-1}(x_0)$. If $\tilde{H}$ has no fixed points in $p^{-1}(x_0)$, let $\tilde{H}_2 = \tilde{H}$ and let $\tilde{H}_1$ be such that $\tilde{H}_1(x) = \mu^{-1}(\mu(\tilde{H}(x)) + 1)$ for $x \in p^{-1}(x_0)$.

Lemma P4. *If $\tilde{H}$ is a lifting of $H|A$, then $\tilde{H}$ can be extended to a homeomorphism of the plane R^2 onto itself.*

Proof. Let K be a properly imbedded arc in A intersecting each of the simple closed curves C_i at exactly one point. For $n = 0, \pm 1, \pm 2, \dots$, denote by $\tilde{A}_n$ the closure of a component of $p^{-1}(A - K)$ in such a way that $\tilde{A}_n \cap \tilde{A}_{n+1} \neq \emptyset$. For each n , the union $\bigcup_{k=-n}^n \tilde{A}_k [\bigcup_{k=-n}^n \tilde{H}(\tilde{A}_k)]$ separates B between b_0 and b_1 . For each $n > 0$, there is an m such that $\bigcup_{k=-\infty}^{-n} \tilde{A}_k \supset \bigcup_{k=m}^{\infty} \tilde{H}(\tilde{A}_k)$ and $\bigcup_{k=n}^{\infty} \tilde{A}_k \supset \bigcup_{k=-m}^{-\infty} \tilde{H}(\tilde{A}_k)$. Therefore, if $\{p_n\}$ is a sequence of points in $\tilde{A}$ such that $\lim_{n \rightarrow \infty} p_n = b_0$ [$\lim_{n \rightarrow \infty} p_n = b_1$], then $\lim_{n \rightarrow \infty} \tilde{H}(p_n) = b_1$ [$\lim_{n \rightarrow \infty} \tilde{H}(p_n) = b_0$]. This implies that $\tilde{H}$ can be extended first to a homeomorphism of B onto itself, and then to a homeomorphism of R^2 onto itself.

The proof of the following lemma is similar to the proof of Lemma P3 and it will be omitted.

Lemma P5. *Let $U \subset A$ be a complementary domain of X invariant under H . Then there is a lifting $\tilde{H}$ of $H|A$ such that $\tilde{H}(V) = V$ for some component V of $p^{-1}(U)$.*

3. TWO COMPLEMENTARY DOMAINS

Lemma 1. *If X has exactly two complementary domains, then h has at least two fixed points in X .*

Proof. Let U be the bounded complementary domain of X . There exists a closed disk D_1 in U and a homeomorphism g_1 of R^2 onto itself such that $g_1 \circ h(D_1) = D_1$ and $g_1(x) = x$ outside U . Similarly, assuming that the unbounded complementary domain of X is a neighborhood of the added point of a one-point compactification of R^2 , we can construct a homeomorphism g_2 of the plane such that $g_2(x) = x$ for $x \in X \cup U$, and $g_2 \circ h(D_2) = D_2$ for some closed disk D_2 containing X . Let $H = g_2 \circ g_1 \circ h$. Notice that $H|X = h|X$, H is an orientation reversing homeomorphism of the plane, and there exists an

annulus A , containing X , which is invariant under H , and whose boundary components are in different complementary domains of X .

Let $\tilde{H}$ be a lifting of $H|A$ as defined in §2.

We will show that $\tilde{H}$ has a fixed point in $p^{-1}(X)$. As in the previous section, let B be a two-point compactification of $\tilde{A}$, and let the added points be denoted by b_0 and b_1 . By Lemma P2, the set $Y = p^{-1}(X) \cup \{b_0, b_1\}$ is a nonseparating plane continuum. Let f be a homeomorphism of R^2 onto itself such that $f|p^{-1}(X) = \tilde{H}$ (see Lemma P4). By Bell's theorem, f has a fixed point in Y . Since $f(b_0) = b_1$ and $f(b_1) = b_0$, $\tilde{H}$ has a fixed point in $p^{-1}(X)$.

First, we will show that H (equivalently h) has at least one fixed point in X . If H has no fixed points in X , then $\tilde{H}$ has no invariant fibers $p^{-1}(x)$ with $x \in X$. Hence $\tilde{H}$ has no fixed points in $p^{-1}(X)$, which contradicts the above.

Next, we will show that H has at least two fixed points in X . Suppose to the contrary that $x_0 \in X$ is the only fixed point of H in X . The only possible fixed point of $\tilde{H}|p^{-1}(X)$ is in $p^{-1}(x_0)$. By Lemma P3, there is a lifting $\tilde{H}_2$ of $H|A$ without a fixed point in $p^{-1}(x_0)$ and hence without a fixed point in $p^{-1}(X)$. By Lemma P4, $\tilde{H}_2$ can be extended to a homeomorphism R^2 without fixed points in Y which, again, contradicts Bell's theorem.

4. THE GENERAL CASE

Denote by:

- (1) $[X, h]$ the union of X and the bounded complementary domains of X which contain no fixed points of h ,
- (2) $P(X, h)$ the set of fixed points of h in the bounded complementary domains of X ,
- (3) $LP(X, h)$ the set of the limit points of $P(X, h)$ in X ,
- (4) $Q(X, h)$ the set of the fixed points of h in X .

Proposition 1. *The continuum $[X, h]$ is invariant under h .*

Proposition 2. $Q(X, h) = Q([X, h], h)$.

Proposition 3. *If $LP(X, h) = \emptyset$, then $[X, h]$ has finitely many complementary domains.*

Proposition 4. $LP(X, h) \subset Q(X, h)$.

The following lemma is an immediate consequence of Bell's theorem.

Lemma 2. *If $[X, h]$ does not separate R^2 , then h has a fixed point in X .*

The proofs of Lemmas 3 and 4 are based on a construction similar to that of the proof of Lemma 1.

Lemma 3. *If $LP(X, h)$ consists of exactly one point, then there is an orientation reversing homeomorphism f of R^2 onto itself, and there exists a continuum Y invariant under f such that (1) Y has finitely many complementary domains, and (2) if $Q(X, h)$ contains n (finitely many) points, then $Q(Y, f)$ contains at most $n - 1$ points.*

Proof. Let U_1 be a bounded complementary domain of X invariant under h . Denote by U_2 the unbounded complementary domain of X . Let A be an annulus in R^2 with boundary components C_1 and C_2 contained in U_1 and U_2 respectively. There is an orientation reversing homeomorphism H of R^2 such that A is invariant under H , and $H(x) = h(x)$ for x in an open neighborhood of $R^2 - (U_1 \cup U_2)$. Then $LP(X, h) = LP(X, H)$.

Denote by x_0 the only point of $LP(X, h)$.

Let $\tilde{A}$ be the universal covering space of A , and let $p : \tilde{A} \rightarrow A$ be the covering map. Let $\tilde{H} : \tilde{A} \rightarrow \tilde{A}$ be a lifting of $H|_A$ with no fixed points in the fiber $p^{-1}(x_0)$ (see Lemma P4).

Let $B = \tilde{A} \cup \{b_0, b_1\}$ be a two-point compactification of $\tilde{A}$, embedded in R^2 , and f a homeomorphism of the plane such that $f|_{\tilde{A}} = \tilde{H}$ (f interchanges the points b_0 and b_1). By Lemma P1 the set $p^{-1}(X) \cup \{b_0, b_1\}$ is a continuum. Let $Y = [p^{-1}(X) \cup \{b_0, b_1\}, f]$. By Proposition 1, Y is invariant under f , and by Proposition 2, $Q(Y, f) = Q(p^{-1}(X) \cup \{b_0, b_1\}, f)$. If y is a fixed point of f in $\tilde{A}$, then $y \in p^{-1}(x)$, where x is a fixed point of H in A . Hence $LP(X, H) \cap p(Q(Y, f)) \supset p(LP(Y, f))$ (see Proposition 4) and therefore $LP(Y, f) = \emptyset$. By Proposition 3, Y has finitely many complementary domains. Since for each $x \in X$, the set $p^{-1}(x)$ contains at most one fixed point of $\tilde{H}$, and $\tilde{H}$ has no fixed points in $p^{-1}(x_0)$, the cardinality of $Q(Y, f)$ is not greater than the cardinality of $Q(X, h)$ minus 1.

Lemma 4. *Let $k > 2$ be an integer. If $[X, h]$ has k complementary domains, then there exist an orientation reversing homeomorphism f of R^2 onto itself, a continuum Y invariant under f , and an integer j , $2 \leq j \leq k - 1$, such that (1) Y has j invariant complementary domains, and (2) the cardinality of $Q(Y, f)$ does not exceed the cardinality of $Q(X, h)$.*

Proof. Let $U_1, \dots, U_{k-1}$ be the bounded complementary domains of $[X, h]$, and let U_k be the unbounded complementary domain.

There is an annulus A in R^2 with boundary components C_1 and C_2 in U_{k-1} and U_k respectively, and there is an orientation reversing homeomorphism H of the plane such that $H(x) = h(x)$ for $x \in X$ and $H(A) = A$.

Let $\tilde{A}$ be the universal covering space of A , $p : \tilde{A} \rightarrow A$ the covering map, and let $\tilde{H} : \tilde{A} \rightarrow \tilde{A}$ be a lifting of $H|_A$ such that there is a component V_1 of $p^{-1}(U_1)$ which is invariant under $\tilde{H}$ (see Lemma P5). Let $B = \tilde{A} \cup \{b_0, b_1\}$ be a two-point compactification of $\tilde{A}$, embedded in the plane and let f be a homeomorphism of the plane such that $f|_{\tilde{A}} = \tilde{H}$. By Lemma P1, the set

$\hat{Y} = p^{-1}([X, h]) \cup \{b_0, b_1\}$ is a continuum. Let $Y = [\hat{Y}, f] - V_1$. Notice that Y is a continuum invariant under f with at least two invariant complementary domains.

For each $1 \leq i \leq k-2$, there is at most one component of $p^{-1}(U_i)$ which is invariant under $\tilde{H}$. Therefore, the number j of the complementary domains of Y satisfies the inequality $2 \leq j \leq k-1$.

The only fixed points of f in Y are in the fibers $p^{-1}(x)$, where $x \in X$ is a fixed point of h . Since there is at most one fixed point of f in such a fiber, the cardinality of $Q(Y, f)$ does not exceed the cardinality of $Q(X, h)$.

Theorem. *If at least one of the bounded complementary domains of X is invariant under h , then h has at least two fixed points in X . Otherwise h has at least one fixed point in X .*

Proof. By Lemma 2, if there are no invariant bounded complementary domains, then h has a fixed point in X .

If $LP(X, h)$ contains more than one point, then clearly h has at least two fixed points in X .

By Lemma 3, if $LP(X, h)$ contains exactly one point which is the only point of $Q(X, h)$, then there exists an orientation reversing homeomorphism f of the plane and a continuum Y invariant under f , such that f has no fixed points in Y , and Y has finitely many complementary domains. If $[Y, f]$ does not separate the plane, then f has a fixed point in Y . If $[Y, f]$ separates the plane, then by Lemma 4 applied inductively, f has a fixed point in Y . Hence, $Q(X, h)$ contains at least two points.

Assume now that X has at least one invariant bounded complementary domain U . If $LP(X, h) = \emptyset$, then $[X, h]$ has finitely many complementary domains. The continuum $[X, h] \cup U$ separates the plane. Either $[X, h] \cup U$ has exactly two complementary domains, or by Lemma 4, applied inductively, we obtain an orientation reversing homeomorphism g of the plane having an invariant continuum Z with exactly two complementary domains. By Lemma 1, g has at least two fixed points in Z . Therefore, h has at least two fixed points in X .

Remark. The following question, asked W. Kuperberg, seems to be related to the above theorems:

Let A be an annulus, and let φ be an orientation reversing homeomorphism of the annulus A onto itself which does not interchange the two boundary components of the annulus. Do there always exist two disjoint subcontinua F_1 and F_2 of the annulus, each invariant under φ , such that the set $F_1 \cup F_2$ separates the annulus?

In response to this question, H. Bell [3] constructed two interesting examples of orientation reversing homeomorphisms, satisfying the above conditions, and for which such continua do not exist.

Let A be the annulus defined in polar coordinates by $A = \{(r, \theta) : 1 \leq r \leq 2\}$. Bell proved that:

- (i) There is an orientation reversing homeomorphism $h_1 : A \rightarrow A$ such that
 - (1) if $r = 1$ or $r = 2$, then $h_1(r, \theta) = (r, -\theta)$,
 - (2) if $1 < r < 2$, then the closure of $\{h_1^i(r, \theta) : i = 1, 2, \dots\}$ contains the set $\{(r, \theta) : r = 1\}$,
 - (3) if $1 < r < 2$, then the closure of $\{h_1^{-i}(r, \theta) : i = 1, 2, \dots\}$ contains the set $\{(r, \theta) : r = 2\}$.
- (ii) There is an orientation reversing homeomorphism $h_2 : A \rightarrow A$ such that
 - (1) if $r = 1$ or $r = 2$, then $h_2(r, \theta) = (r, -\theta)$,
 - (2) if $1 < r < 2$, then $\lim_{i \rightarrow \infty} h_2^i(r, \theta) = (1, 0)$,
 - (3) if $1 < r < 2$, then $\lim_{i \rightarrow \infty} h_2^{-i}(r, \theta) = (2, 0)$.

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