SELF-SIMILAR SETS 5. INTEGER MATRICES AND FRACTAL TILINGS OF $\mathbb{R}^n$

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ABSTRACT. Tilings with self-similar tiles and “just-touching fractals” are constructed from matrices of integers, extending work by Levy, Mandelbrot, Dekking, Bedford, and others.

1. DEFINITIONS AND RESULT

A closed set A_1 in $\mathbb{R}^n$ with nonempty interior is called an *m-rep tile* if there are sets $A_2, \dots, A_m$ congruent to A_1 such that $\text{int } A_i \cap \text{int } A_j = \emptyset$ for $i \neq j$ and

$$A_1 \cup \dots \cup A_m = g(A_1),$$

where g is a similarity mapping. By iterated composition, so-called similarity tilings are obtained from *m-rep tiles* (see Grünbaum and Shephard [13, §10.1]). We allow g to be a linear expansive map: $g(x) = Mx$, where all eigenvalues of the matrix M have modulus > 1 .

We show that *m-rep tiles* can be constructed from any map g which transforms the lattice $L = \mathbb{Z}^n$ of integer vectors to itself. In other words, we assume that M consists of integers. Let $m = |\det M|$. A family $\{y_1, \dots, y_m\} \subseteq L$ is called a *residue system* for g if $y_1 = 0$ and $L = \bigcup \{y_i + g(L) \mid i = 1, \dots, m\}$. ($g(L)$ is a subgroup of L , and we choose one point from each coset.)

Theorem 1. *Let g be a linear expansive map on $\mathbb{R}^n$ with integer matrix and $\{y_1, \dots, y_m\}$ a residue system of g . Then there is a unique *m-rep tile* A_1 such that*

$$g(A_1) = A_1 \cup \dots \cup A_m \quad \text{with } A_i = y_i + A_1.$$

2. EXAMPLES AND COMMENTS

Theorem 1 forms a framework for intuitive constructions of fractal tiles by Mandelbrot [17, Chapter 7], Giles [12], and others. For $\mathbb{R}^2$ and particular choices of y_i . Bedford [5, Theorem 1] proved our theorem using Dekking's

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FIGURE 1. The tame twindragon.

approach [6]. His proof contains a little gap; it does not work for $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, where A_1 turns out to be the well-known twindragon [17, 6, 11, 12]. For $\det \mathbf{M} = 2$, y_2 influences the size and position of A_1 , but not the shape.

Another twindragon, tame and unknown, but implicit in the work of Gilbert [9, 10], is shown in Figure 1. g has matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ with respect to the basis $b_1 = (1, 0)'$, $b_2 = (1/2, -\sqrt{7}/2)'$, and the lattice $\mathbf{L}$ is generated by this basis. Since the theorem is based on the linear structure of $\mathbb{R}^n$, the choice of basis does not really matter, but our basis makes g a similarity mapping with respect to the Euclidean metric, which seems more appealing. For $n = 2$, we can choose a basis such that g is a similitude iff $\mathbf{M}$ either has a pair of complex eigenvalues, or two real eigenvalues with equal modulus and independent eigenvectors.

Now consider the hexagonal lattice $\mathbf{L}$ generated by the basis $b_1 = (1, 0)'$, $b_2 = (1/2, \sqrt{3}/2)'$. With $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $y_2 = (1, 0)'$, $y_3 = (0, 1)'$ we get the terdragon [6, 17], see Figure 2a. An orientation-reversing similitude is induced by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, which gives the "terdragon of second kind" in Figure 2b. Gosper's flake [8; 17, p. 47, pp. 70–71] is obtained from $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$, with $y_1, \dots, y_6$ forming a hexagon around y_7 . The matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ yields "Gosper's flake of the second kind" (Figure 3, p. 552) using the same y_i and Figure 4 (see p. 553) when y_7 is replaced by an equivalent residue $y_7 + g(x)$ outside the hexagon.

FIGURE 2a. Terdragon.

FIGURE 2b. Second-kind terdragon.

We also get unexpected results even with the simplest mappings. Let $g(x) = 2x$. If $y_1, \dots, y_4$ are the vertices of any parallelogram, the uniqueness of A_1 shows that A_1 is that parallelogram. The tiles corresponding to $y_2 = (1, 0)'$, $y_3 = (0, 1)'$, $y_4 = (-1, -1)'$ and $y_2 = (-1, 0)'$, $y_3 = (-2, 1)'$, $y_4 = (1, 1)'$ are given in Figures 5 and 6 (in Figure 5, $b_1 = (\sqrt{3}/2, -1/2)$); see page 553. It seems that such tiles have not been studied, except for an example by Mandelbrot [17, p. 141].

FIGURE 3a. Gosper flake.

FIGURE 3b. Second-kind Gosper flake.

FIGURE 4

FIGURE 5

FIGURE 6

3. SELF-SIMILAR SETS

Let $f_1, \dots, f_m$ be contractions on a complete metric space (X, d) ; that is, $d(f_i(x), f_i(y)) \leq r_i \cdot d(x, y)$ for all x, y , where the factor r_i is < 1 . Hutchinson [14] called a compact set $A \neq \emptyset$ *self-similar* with respect to $f_1, \dots, f_m$ if

$$A = f_1(A) \cup \dots \cup f_m(A).$$

He proved that $F(\mathbf{B}) = f_1(\mathbf{B}) \cup \dots \cup f_m(\mathbf{B})$ is a contradiction on the space $\mathcal{C}$ of nonempty compact sets of X with Hausdorff metric. The fixed point theorem implies the following:

Theorem [14]. *Given $f_1, \dots, f_m$, there is a unique self-similar set A , and for each compact $\mathbf{B}_0 \neq \emptyset$ the sequence $\mathbf{B}_k = F(\mathbf{B}_{k-1})$, $k = 1, 2, \dots$, converges to A in $\mathcal{C}$.*

If the f_i fulfill the following weak contractivity condition, some F^p becomes a contraction, and the statement remains true:

(0) there are constants $C > 0$ and $r < 1$ such that

$$d(f_{i_1} \cdots f_{i_p}(x), f_{i_1} \cdots f_{i_p}(y)) \leq C \cdot r^p \cdot d(x, y)$$

for all $p \in \mathbb{N}$, $i_1, \dots, i_p \in \{1, \dots, m\}$, and all $x, y \in \mathbb{R}^n$.

4. PROOF OF THEOREM 1

Let $\max\{1/\lambda \mid \lambda \text{ eigenvalue of } g\} < r < 1$. There is a $C > 0$ with $|g^{-p}(x)| \leq C \cdot r^p \cdot |x|$ for $p \in \mathbb{N}$, $x \in \mathbb{R}^n$ [6, Lemma 2.3]. Since $|f_{i_1} \cdots f_{i_p}(x) - f_{i_1} \cdots f_{i_p}(y)| = |g^{-p}(x - y)|$, condition (0) is fulfilled for $f_i(x) = y_i + g^{-1}(x)$, $i = 1, \dots, m$, and for Euclidean distance. Now $A = g(A_1)$ is the self-similar set with respect to $f_1, \dots, f_m$. We construct A starting with $\mathbf{B}_0 = \{0\}$. Thus $\mathbf{B}_1 = \{y_1, \dots, y_m\}$, and

$$\mathbf{B}_k = \{y_{i_k} + g^{-1}(y_{i_{k-1}}) + \dots + g^{-k+1}(y_{i_1}) \mid i_1, \dots, i_k \in \{1, \dots, m\}\}.$$

This sequence is increasing; hence, $A = \text{cl}(\bigcup \mathbf{B}_k)$. Since A is compact, $A \subseteq U_c = \{y \mid |y| \leq c\}$ for some c . We shall prove that

(i) there are finitely many points $x_1, \dots, x_q$ in L such that

$$\{g(x_1), \dots, g(x_q)\} + A \supseteq U_c.$$

By the Baire category theorem, this implies $\text{int } A \neq \emptyset$, so the Lebesgue measure $\lambda(A)$ is positive. Since each f_i has determinant $1/m$, we have $\lambda(A_i) = \lambda(A)/m$ for $A_i = f_i(A)$. For any i, j with $i \neq j$, $A = \bigcup A_i$ implies $\lambda(A_i \cap A_j) + \lambda(A) \leq \sum \lambda(A_i)$. Thus $\lambda(A_i \cap A_j) = 0$ and $\text{int } A_i \cap \text{int } A_j = \emptyset$, which means that A is in fact an m -rep tile.

It remains to prove (i). Choose a positive number δ such that each u in $\mathbb{R}^n$ has distance $\leq \delta$ from L . Given $\varepsilon > 0$, there is a k such that $\delta \cdot C \cdot r^{k-1} < \varepsilon$,

and hence each u has distance $\leq \varepsilon$ from the lattice $g^{-k+1}(\mathbf{L})$. Since the y_i form a residue system, each z in $\mathbf{L}$ can be written, for any k , as

$$z = g^k(x) + g^{k-1}(y_{i_k}) + \cdots + g(y_{i_2}) + y_{i_1},$$

with certain residues $y_{i_k}, \dots, y_{i_1}$ and some x in $\mathbf{L}$. Hence

$$g^{-k+1}(z) = g(x) + y_{i_k} + g^{-1}(y_{i_{k-1}}) + \cdots + g^{-k+1}(y_{i_1}),$$

which means that

$$g^{-k+1}(\mathbf{L}) = \bigcup \{g(x) | x \in \mathbf{L}\} + \mathbf{B}_k.$$

The definition of c now implies that

$$g^{-k+1}(\mathbf{L}) \cap \mathbf{U}_c \subseteq \{g(x_1), \dots, g(x_q)\} + \mathbf{B}_k,$$

where $x_1, \dots, x_q$ are those points of $\mathbf{L}$ for which $|g(x)| \leq 3c$. Taking the closure of the union for $k = 1, 2, \dots$ we obtain (i).

5. JUST-TOUCHING FRACTALS

Although Hutchinson's construction yields self-similar sets for any choice of $f_1, \dots, f_m$, it is not so easy to obtain nice pictures: if the factors r_i are small, $\mathbf{A}$ will be a Cantor set, and if the r_i are large, there will be too much overlap of the parts $\mathbf{A}_i$. To identify the interesting cases Barnsley [4] termed *just-touching fractals*, Hutchinson [14] introduced the *open set condition* (OSC): there exists an open set $\mathbf{V}$ with $f_i(\mathbf{V}) \subseteq \mathbf{V}$ and $f_i(\mathbf{V}) \cap f_j(\mathbf{V}) = \emptyset$ for $i, j \in \{1, \dots, m\}$, $i \neq j$. This is an adequate condition, but hard to control.

The relevance of our theorem lies in the fact that it provides the first systematic approach to the construction of self-similar sets with OSC where the $\mathbf{A}_i$ touch each other. Of course it is possible to construct such figures by computer experiment [4]. If we want to develop analysis on such fractal spaces, however (determine shortest connections, define a Laplacian, etc. [3, 16]), we must know their structure exactly.

Our tiles can be made into fractal "gaskets" or "carpets" by just dropping some of the f_i . Since the OSC was proved for $\{f_1, \dots, f_m\}$ with $\mathbf{V} = \text{int } \mathbf{A}$, it will be true for any subset of the f_i . Figures 7 and 8 (see p. 556) are obtained by deleting f_7 and f_3, f_5 from Figure 3. In the same way we can produce subshift constructions [1, 2, 18] with the OSC. For example, if we do not allow the combination $f_2 \cdot f_2$ in Figure 2b, we obtain Figure 9 (p. 557).

Another remark on the OSC: The open set $\mathbf{V}$ can always be chosen to be bounded, and one might expect that one can require other properties (e.g., simple connectedness). Our examples show that this is not true (Figures 4, 5). From $\mathbf{A} \subseteq \text{cl}(\mathbf{V})$ [14], it follows that for our tiles $\mathbf{V}$ must be equal to $\text{int } \mathbf{A}$ minus a nowhere-dense set.

Finally, a one-dimensional example: $f_1(x) = x/3$ and $f_2(x) = (x+2)/3$ generate the middle-third set in $\mathbb{R}$. Take another similitude with the same factor $1/3$ and with fixed point c . The self-similar set $\mathbf{A}$ with respect to f_1 ,

FIGURE 7

FIGURE 8

f_2, f_3 will be disconnected unless $c = 1/2$, and from projection theorems it follows that A has Lebesgue measure zero for almost all c (cf. [7]). Our theorem shows that for a dense set of c —namely $c = (3n + 1)/(3m + 2)$ or $c = (3n + 2)/(3m + 1)$ with $n, m \in \mathbb{Z}$ —the set A is a 3-rep tile, hence regular-closed and not a Cantor set. (Take $g(x) = 3x$, $b = 2/(3 \cdot (3m + 2))$, $y_2 = (3m + 2) \cdot b$, $y_3 = (3n + 1) \cdot b$).

FIGURE 9

6. NOTE ON "NUMBER SYSTEMS"

Gilbert [9, 10, 11] considered complex numbers b (base) and $a_1 = 0, a_2, \dots, a_m$ ("digits", $m = |b|^2$) such that each $z \in \mathbb{C}$ has a representation $z = \sum a_{i_k} \cdot b^k$, where k runs from $-\infty$ to some integer $q(z)$. All z with $q(z) < 0$ form an m -rep tile which could be called the "unit interval" with respect to b and the a_i . The "integers" z ($a_{i_k} = 0$ for $k < 0$) form a lattice L containing $a_1, \dots, a_m$, and to each point in L there corresponds one tile. Our mapping g is $g(z) = b \cdot z$.

Gilbert obtains the twindragon for $b = -1 + i$, $a_2 = 1$, but he does not consider $b = 1 + i$ an admissible base. Nevertheless, $b = 1 + i$, $a_2 = 1$ (our first example, §2) also leads to a twindragon, but with an area five times smaller.

The point is that we also construct A_1 as a "unit interval" with respect to representations $x = \sum g^k(y_{i_k})$, but not all points of $\mathbb{R}^n$ need to be representable. For x_0 in L , the long division $x_0 = g(x_1) + y_{i_1}$, $x_1 = g(x_2) + y_{i_2}, \dots$ need not end with $x_q = 0$, but can lead to another periodic orbit, so that x becomes a "periodic integer with infinitely many digits". Our proof implies that the number of periodic orbits is finite, but we did not study their number, length, and structure.

7. SYMMETRIES WHICH ARE NOT TRANSLATIONS

So far the congruence between parts of our set A was given by translations. Allowing for rotations and reflections, we get more exotic pictures, which can hardly be recognized as tiles. A finite group S of linear mappings with integer matrices and with determinant ± 1 will be called a *symmetry group* of g if $gS = Sg$. We still assume $g(x) = Mx$ and $m = \det M$.

FIGURE 10. Levy's curve.

Theorem 2. Let g be a linear expansive map on $\mathbb{R}^n$ with integer matrix, let $\{s_1, \dots, s_m\}$ be contained in a symmetry group of g , and let

$$L = \bigcup \{s_i^{-1}(y_i + g(L)) \mid i = 1, \dots, m\}.$$

Then the set A self-similar with respect to $f_i(x) = s_i \cdot g^{-1}(x) + y_i$, $i = 1, \dots, m$, has a nonempty interior.

If g is a similitude and the s are Euclidean isometries, the s_i will be in a symmetry group, and A will be an m -rep tile. Simple examples are the right-angled triangles with angles 45, 45 ($m = 2$) or 30, 60 ($m = 3$ or 4; cf. [19]). The "curve" in Figure 10 had already been shown to be a tile by Levy [15] in 1938. In this case, g has matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $s_1 = \text{id}$, $s_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ +y \end{pmatrix}$, and $y_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. With $s_2 = \text{id}$ we had the twindragon, $s_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$ yields the Heighway dragon [6, p. 89; 17, p. 66], and $s_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ gives Figure 1 in [1]. With $s_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$ and $s_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$, we obtain Figures 11 and 12. Changing y_2 , we can generate other shapes or the same shapes in different positions.

For Figures 13–15 (see p. 560), we take $M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ and $y_2 = (1, 0)'$ with respect to the basis of the hexagonal lattice (see §2). In Figure 13 $y_3 = (0, 1)'$, and $s_2 = s_3$ is a rotation about 60° . In Figures 14 and 15 $s_2 = \text{id}$, $y_3 = y_2$, and s_3 is a rotation about -60° resp. $+60^\circ$. Since $y_1 = 0$ and $s_1 = \text{id}$, the mappings $h_i(x) = s_i(x) + y_i$ transform A_1 onto A_i .

In Figure 16 (on p. 561), $g(x) = 2x$, and h_2 , h_3 , and h_4 are reflections in the lines $x = -3/2$, $y = 1/2$, and $y = x + 1$, respectively. Such pictures could be considered generalizations of the "foldable figures" studied by Hoffman and Whithers [19].

FIGURE 11

FIGURE 12

FIGURE 13

FIGURE 14

FIGURE 15. Another terdragon.

FIGURE 16

8. PROOF OF THEOREM 2

We make a few changes in the proof of Theorem 1 in §4. Let

$$K = \max\{\|s\| \mid s \in \mathbf{S}\}.$$

Since $\mathbf{S}$ is a group, and $g^{-1}sg \in \mathbf{S}$ for $s \in \mathbf{S}$, it follows from $s_{i_1}, \dots, s_{i_k} \in \mathbf{S}$ that $t = s_{i_k}g^{-1}s_{i_{k-1}}g^{-1} \dots s_{i_1}g^{-1} \cdot g^k$ is in $\mathbf{S}$. Consequently, $|s_{i_k}g^{-1} \dots s_{i_1}g^{-1}(x)| \leq \|t\| \cdot |g^{-k}(x)| \leq K \cdot |g^{-k}(x)|$, so that (0) is fulfilled with $C' = C \cdot K$ for $f_i(x) = s_i \cdot g^{-1}(x) + y_i$, $i = 1, \dots, m$. We construct the self-similar set $\mathbf{A}$ starting with $\mathbf{B}_0 = \{0\}$:

$$\mathbf{B}_k = \{y_{i_k} + s_{i_k}g^{-1}(y_{i_{k-1}}) + \dots + s_{i_k}g^{-1} \dots s_{i_3}g^{-1}s_{i_2}g^{-1}(y_{i_1}) \mid i_1, \dots, i_k \in \{1, \dots, m\}\}.$$

Instead of (i) we prove that there are finitely many copies $s(g(x) + \mathbf{A})$ of $\mathbf{A}$ which cover $\mathbf{U}_c$. The last assumption of Theorem 2 says that for each z in $\mathbf{L}$ there are $i_1, i_2, \dots \in \{1, \dots, m\}$ such that $z = s_{i_1}^{-1}g(u) + s_{i_1}^{-1}(y_{i_1})$, $u \in \mathbf{L}$, $u = s_{i_2}^{-1}g(v) + s_{i_2}^{-1}(y_{i_2}) \dots$, and

$$\begin{aligned} z &= s_{i_1}^{-1}gs_{i_2}^{-1}g \dots s_{i_k}^{-1}g(x) + s_{i_1}^{-1}g \dots s_{i_k}^{-1}(y_{i_k}) \\ &\quad + s_{i_1}^{-1}g \dots s_{i_{k-1}}^{-1}(y_{i_{k-1}}) + \dots + s_{i_1}^{-1}(y_{i_1}), \end{aligned}$$

with x in $\mathbf{L}$, for an arbitrary k . For t from above we have

$$t^{-1} = g^{-k+1}s_{i_1}^{-1}gs_{i_2}^{-1}g \dots s_{i_k}^{-1}$$

which implies that

$$g^{-k+1}(z) = t^{-1}g(x) + t^{-1}(b), \text{ with } b \text{ in } \mathbf{B}_k.$$

Thus $g^{-k+1}(\mathbf{L}) \subseteq \bigcup \{t^{-1}(g(x) + \mathbf{B}_k) | x \in \mathbf{L}, t \in \mathbf{S}\}$. For $g^{-k+1}(\mathbf{L}) \cap \mathbf{U}_c$, we need only consider the x with $|g(x)| \leq c \cdot (K+2)$, because otherwise $t^{-1}(g(x) + \mathbf{U}_c) \cap \mathbf{U}_c = \emptyset$.

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