A SYMBOLIC CALCULUS FOR LAYER POTENTIALS ON C^1 CURVES AND C^1 CURVILINEAR POLYGONS

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(Communicated by Barbara L. Keyfitz)

ABSTRACT. A symbolic calculus for some algebras of Mellin operators on the finite interval $J \equiv [0, 1]$ is developed. The algebras are ample enough to include singular integral operators and analytic double layer potentials and their adjoints on C^1 curves and piecewise C^1 curves with corners. Fredholmness and the index of the operators on $L^p(J)$ are completely determined by the principal symbol on $L^p(J)$, $\text{Smb}^{1/p}$.

This note describes some algebras of pseudodifferential operators of Mellin type, or *Mellin operators*, on $L^p(J)$, $J \equiv [0, 1]$, which are useful for double layer potential operators on piecewise C^1 curves and C^1 curvilinear polygons. The resulting symbolic calculus gives necessary and sufficient conditions for the Fredholmness of the operators acting on $L^p(J)$. This paper is a companion to Lewis [L]. The first step is to enlarge the algebras of Mellin operators described in Lewis–Parenti [LP], J. Elschner [E], and [L] to include multiplication by continuous functions in $C(J)$. In §1 we carefully describe the structure of the Mellin operators in $\text{Op}\Sigma_{\alpha,\beta}(J)$ of [L] and then enlarge the operator class to $\text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ and extend the symbol map, $\text{Smb}^{1/p}$. In §2, we show that analytic double layer kernel operators on C^1 curves are in $\text{Op}\Sigma_{0,1} \otimes C(J)$ and show how to calculate their symbols. Theorem 2 shows that the important operators are the finite Hilbert transform and Hardy kernel operators at the endpoints and corners of the curves. The essential step (Theorem 3) is to use the Commutator Theorem of A. P. Calderón and C. P. Calderón [CCFJR] to show that the remainders are compact operators on $L^p(J)$.

The pioneering work on potentials on C^1 domains was done by E. B. Fabes, M. Jodeit, and N. M. Rivire [FJR]. J. Cohen and J. Gosselin [CG] treated multiple layer potentials for the biharmonic operator on C^1 domains. Our proof of Theorem 3 is in the spirit of [FJR] and [CG].

Received by the editors September 21, 1989 and, in revised form, March 1, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35S15, 45E05.

Key words and phrases. Layer potentials, Mellin operators, nonsmooth domains, pseudodifferential operators, symbolic calculus.

Research partially supported by the Italian CNR.

The author thanks A. Bove for several discussions regarding this work. The referee provided clarifications of the statements and proofs.

1. ALGEBRAS OF MELLIN OPERATORS ON A FINITE INTERVAL

For $J \equiv [0, 1]$ the algebra of Mellin operators $\text{Op}\Sigma_{\alpha, \beta}(J)$ on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$, is described in [L, §1]. We follow closely the notions of [E] since Ełschner develops an extension to meromorphic symbols which arise in constructing parametrices. For $0 \leq \alpha < \beta \leq 1$, define the strip $\Gamma_{\alpha, \beta} = \{z \in \mathbb{C} : \alpha < \text{Re } z < \beta\}$, and let Γ_γ be the oriented contour $\{z = \gamma + i\xi : -\infty \leq \xi \leq +\infty\}$.

If $f \in L^p(J)$ let Rf be the reflection

$$(1-1) \quad Rf(t) = f(1-t).$$

We recall some function spaces [LP, E, and L]:

Definition 1.1. For $0 \leq \alpha < \beta \leq 1$, let $\mathcal{O}_{\alpha, \beta}^m$ be the Fréchet space of functions $f(z) \in \mathcal{O}(\Gamma_{\alpha, \beta})$ such that for every ℓ , for every $[a, b] \subset (\alpha, \beta)$,

$$(1-2) \quad |f|_{\ell, [a, b]} = \sup_{\Gamma_{a, b}} \left| (1 + |z|)^{\ell-m} \frac{d^\ell}{dz^\ell} f(z) \right| < \infty.$$

For $0 < \alpha \leq 1$, let $\mathcal{F}^\alpha(J)$ be the class of functions $a(t) \in C^\infty((0, 1)) \cap C(J)$ such that for every ℓ , for every $\alpha' < \alpha$,

$$(1) \quad \left(t \frac{d}{dt} \right)^\ell (a(t) - a(0)) = O(t^{\alpha'}).$$

$$(2) \quad \text{With } Ra(t) = a(1-t),$$

$$\left(t \frac{d}{dt} \right)^\ell (Ra(t) - Ra(0)) = O(t^{\alpha'}).$$

Let H denote the *finite Hilbert transform*:

$$(1-3) \quad Hf(t) = \text{p.v.} \frac{1}{\pi} \int_0^1 \frac{1}{t-s} f(s) ds.$$

Let $\tilde{k}(z) \in \mathcal{O}_{\alpha, \beta}^{-1}$. We shall call the operator

$$(1-4)^1 \quad Kf(t) = \int_0^1 k\left(\frac{t}{s}\right) f(s) \frac{ds}{s} = \frac{1}{2\pi i} \int_{\Gamma_{1/p}} t^{-z} \tilde{k}(z) \hat{f}(z) dz$$

a *Hardy kernel operator*.²

¹For $f \in C_0^\infty(\mathbb{R}^+)$ its Mellin transform $\hat{f}$ is $\hat{f}(z) = \int_0^\infty t^{z-1} f(t) dt$, so that $f(t) = \frac{1}{2\pi i} \times \int_{\Gamma_{1/p}} t^{-z} \hat{f}(z) dz$.

²Other choices of the class for $\tilde{k}_i(z)$ are possible, cf. [LP] and [E]; for applications we often have $\tilde{k}_i(z) \in \mathcal{O}_{\alpha, \beta}^{-\infty}$.

Let $\mathcal{C}_{\alpha,\beta}(J)$ be the class of operators which are compact on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$.

We are now ready to describe the algebra of Mellin operators $\text{Op}\Sigma_{\alpha,\beta}(J)$, on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$. The following definition is equivalent to [L, Def. 1.1]. Let $\chi(t)$ be a fixed cutoff function, $\chi(t) \in C_0^\infty([0, \frac{1}{3}))$, $\chi(t) \equiv 1$ on $[0, \frac{1}{4}]$.

Definition 1.2. An operator on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$, is in $\text{Op}\Sigma_{\alpha,\beta}(J)$ iff

$$(1-5) \quad A = c(t)I + d(t)H + \chi K_0 \chi + R \chi K_1 \chi R + C,$$

where

- (1) $c(t), d(t) \in \mathcal{F}^{\beta-\alpha}(J)$,
- (2) K_0 and K_1 are Hardy kernel operators with $\tilde{k}_i(z) \in \mathcal{O}_{\alpha,\beta}^{-1}$,
- (3) $C \in \mathcal{C}_{\alpha,\beta}(J)$.

If A has the structure (1-5), define

$$(1-6) \quad \begin{cases} a_0(z) = c(0) + d(0)(-\cot \pi z) + \tilde{k}_0(z), \\ a_{0\pm}(t) = a_{1\mp}(1-t) = c(t) \pm id(t), \\ a_1(z) = c(1) + d(1)(\cot \pi z) + \tilde{k}_1(z). \end{cases}$$

As an operator acting on $L^p(J)$, the principal symbol of an operator $A \in \text{Op}\Sigma_{\alpha,\beta}(J)$, $\text{Smb}^{1/p} A$, is a continuous function on $\mathcal{R}_J^{1/p}$, the oriented boundary of the rectangle:

$$(1-7) \quad \begin{array}{ccccc} & t=0 & t \in [0, 1] & t=1 & \\ \frac{1}{p} + i\infty & & & & \frac{1}{p} - i\infty \\ \Gamma_{1/p} & \uparrow & \mathcal{R}_J^{1/p} & \downarrow & \Gamma_{1/p} \\ \frac{1}{p} - i\infty & & & & \frac{1}{p} + i\infty \\ & t=0 & t \in [0, 1] & t=1 & \end{array}$$

Definition 1.3. Let $A \in \text{Op}\Sigma_{\alpha,\beta}(J)$ and $\frac{1}{p} \in (\alpha, \beta)$. The principal symbol of A as an operator on $L^p(J)$, $\text{Smb}^{1/p} A$, is the quadruple of functions $a_0(\frac{1}{p} + i\xi)$, $a_{0+}(t) = a_{1-}(1-t)$, $a_1(\frac{1}{p} + i\xi)$, $a_{0-}(t) = a_{1+}(1-t)$, considered as a continuous function on $\mathcal{R}_J^{1/p}$:

$$(1-8) \quad \begin{array}{ccccc} & t=0 & a_{0+}(t) = a_{1-}(1-t) & t=1 & \\ \frac{1}{p} + i\infty & & & & \frac{1}{p} - i\infty \\ a_0(\frac{1}{p} + i\xi) & \uparrow & \mathcal{R}_J^{1/p} & \downarrow & a_1(\frac{1}{p} + i\xi) \\ \frac{1}{p} - i\infty & & & & \frac{1}{p} + i\infty \\ & t=0 & a_{0-}(t) = a_{1+}(1-t) & t=1 & \end{array}$$

Remark. To verify that Definition 1.2 is equivalent to [L, Def. 1.1], it is enough to verify that an operator A with the structure (1-5) has the correct symbol.

The enlargement of the algebra $\text{Op}\Sigma_{\alpha,\beta}(J)$ to include multiplication by functions in $C(J)$ may be done in several ways.

Definition 1.4. An operator on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$, is in $\text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ iff

$$(1-9) \quad A = c(t)I + d(t)H + \chi K_0 \chi + R\chi K_1 \chi R + C,$$

where

- (1) $c(t), d(t) \in C(J)$,
- (2) K_0 and K_1 are Hardy kernel operators with $\tilde{k}_i(z) \in \mathcal{O}_{\alpha,\beta}^{-1}$,
- (3) $C \in \mathcal{C}_{\alpha,\beta}(J)$.

If $A \in \text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ has the structure (1-9), then $\text{Smb}^{1/p} A$ is defined using (1-6) and (1-8).

To extend the symbolic calculus from $\text{Op}\Sigma_{\alpha,\beta}(J)$ to $\text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$, first topologize the space of symbols. For a given $\text{Smb}^{1/p} A$ let $a_0(z)$, $a_{0\pm}(t) = a_{1\mp}(1-t)$, $a_1(z)$ and $c(t)$, $d(t)$, $\tilde{k}_0(z)$, $\tilde{k}_1(z)$ be related by (1-6). We shall say that a sequence of symbols $\text{Smb}^{1/p}(A^j)$ converges in $\Sigma_{\alpha,\beta} \otimes C(J)$ iff

$$(1-10) \quad \begin{cases} c^j(t) \text{ and } d^j(t) \text{ converge in } C(J), \\ \tilde{k}_0^j(z) \text{ and } \tilde{k}_1^j(z) \text{ converge in } \mathcal{O}_{\alpha,\beta}^{-1}. \end{cases}$$

If $\text{Smb}^{1/p}(A^j)$ is a sequence of symbols of operators in $\text{Op}\Sigma_{\alpha,\beta}(J)$ converging in $\Sigma_{\alpha,\beta} \otimes C(J)$, there is a representative sequence of operators $\{A^j\}$, $A^j \in \text{Op}\Sigma_{\alpha,\beta}(J)$, converging in operator norm on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$; moreover, if $\text{Smb}^{1/p}(A^j)$ converges to 0 in $\Sigma_{\alpha,\beta} \otimes C(J)$, and A^j converges to A in operator norm on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$, then A is a compact operator. Thus an alternate definition of the class $\text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ is:

Definition 1.4'. The algebra $\text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ consists of those operators A such that there is a sequence of operators $\{A^j\} \subset \text{Op}\Sigma_{\alpha,\beta}(J)$ such that

- (1) $\text{Smb}^{1/p}(A^j)$ converges in $\Sigma_{\alpha,\beta} \otimes C(J)$.
- (2) A^j converges to A in operator norm on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$,

If $A \in \text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$, its principal symbol as an operator on $L^p(J)$, $\text{Smb}^{1/p} A$, is

$$(1-11) \quad \text{Smb}^{1/p} A = \lim_{j \rightarrow \infty} \text{Smb}^{1/p}(A^j).$$

With Definition 1.4', it is easy to extend the symbolic calculus of [LP, Theorems 1.2 and 1.3]:

- (1) If $A, B \in \text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$, then $AB \in \text{Op}\Sigma_{\alpha,\beta} \otimes C(J)$ and

$$(1-12) \quad \text{Smb}^{1/p} AB = (\text{Smb}^{1/p} A)(\text{Smb}^{1/p} B),$$

(2) If $A \in \text{Op} \Sigma_{\alpha, \beta} \otimes C(J)$, then $A^* \in \text{Op} \Sigma_{1-\beta, 1-\alpha} \otimes C(J)$, and

$$(1-13) \quad [\text{Smb}^{1/q} A^*](t, z) = \overline{[\text{Smb}^{1/p} A](t, 1 - \bar{z})}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The following estimate also shows that $\text{Smb}^{1/p} A$ is well defined.

Lemma 1. Let $A \in \text{Op} \Sigma_{\alpha, \beta} \otimes C(J)$. For $\frac{1}{p} \in (\alpha, \beta)$,

$$(1-14) \quad |\text{Smb}^{1/p} A| \leq \inf_{C \in \mathcal{E}_{\alpha, \beta}(J)} \|A + C\|_p.$$

Proof. The argument is similar to [L, pp. 502–504].

Fix $f \in C_0^\infty((0, 1))$. If $\frac{1}{p} \in (\alpha, \beta)$ and $C \in \mathcal{E}_{\alpha, \beta}(J)$, then $\lim_{\lambda \rightarrow \pm\infty} C(t^{i\lambda} f) = 0$ in $L^p(J)$. Thus in $L^p(J)$

$$\lim_{\lambda \rightarrow \pm\infty} t^{-i\lambda} (A + C)(t^{i\lambda} f) = a_{0\mp}(t) f(t),$$

so that $|a_{0\mp}(t)| \leq \|A + C\|_p$.

Now fix $f \in C_0^\infty((0, \frac{1}{4}))$. Let

$$(1-15) \quad A_0 f(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/p}} t^{-z} a_0(z) \tilde{f}(z) dz.$$

Let $T_\epsilon^{1/p}$ be the dilation $T_\epsilon^{1/p} f(t) = \epsilon^{-1/p} f(t/\epsilon)$. Again $\lim_{\epsilon \rightarrow 0^+} C(T_\epsilon^{1/p} f) = 0$ in $L^p(J)$. Thus

$$\lim_{\epsilon \rightarrow 0^+} T_{1/\epsilon}^{1/p} (A + C)(T_\epsilon^{1/p} f) = A_0 f(t),$$

so that $|a_0(\frac{1}{p} + i\xi)| \leq \|A_0\|_{L^p(\mathbb{R}^+)} \leq \|A + C\|_p$. Intertwining A with the reflection R shows the estimate for $|a_1(\frac{1}{p} + i\xi)|$. $\square$

We next recall the notion of *ellipticity* of a system of Mellin operators on $L^p(J)$.

Definition 1.5. Let $A = (A_{ij})$ be an $N \times N$ matrix of operators in $\text{Op} \Sigma_{\alpha, \beta}(J)$ or $\text{Op} \Sigma_{\alpha, \beta} \otimes C(J)$. The system A is *elliptic* on $[L^p(J)]^N$ iff $\text{Smb}^{1/p} A$ is a nonsingular matrix on $\mathcal{R}_J^{1/p}$. A number $z_0 \in \Gamma_{\alpha, \beta}$ is a *singularity type* for A at $t = 0$ [$t = 1$] if

$$(1-16) \quad \det(\text{Smb}^{1/p} A(0, z_0)) = 0 \quad [\det(\text{Smb}^{1/p} A(1, z_0)) = 0].$$

We can now extend the Index Theorem of [LP, Theorems 4.1, 4.2; E, Theorems 4.4, 4.6; L, Theorem 1].

Theorem 1. Let $A = (A_{ij})$ be an $N \times N$ matrix of operators in $\text{Op} \Sigma_{\alpha, \beta}(J)$ or $\text{Op} \Sigma_{\alpha, \beta} \otimes C(J)$. Then

- (1) A is a Fredholm operator on $[L^p(J)]^N$ iff A is elliptic on $[L^p(J)]^N$.
- (2) If A is elliptic on $[L^p(J)]^N$, define

$$(1-17) \quad \text{ind}_p(A) = \dim((\ker A) \cap [L^p(J)]^N) - \dim\left((\ker A^*) \cap [L^{\frac{p}{p-1}}(J)]^N\right).$$

Then

$$(1-18) \quad \text{ind}_p(A) = \frac{1}{2\pi} \Delta_{\mathcal{R}_J^{1/p}} \{ \arg (\det(\text{Smb}^{1/p} A)) \},$$

where the change in $\arg$ is taken as $\mathcal{R}_J^{1/p}$ is traversed in the clockwise direction.

Proof. For operators in $\text{Op} \Sigma_{\alpha, \beta}(J)$, this is Theorem 1 of [L]. If $A \in \text{Op} \Sigma_{\alpha, \beta} \otimes C(J)$ is elliptic on $[L^p(J)]^N$, $\frac{1}{p} \in (\alpha', \beta')$, construct an operator [parametrix] B in $\text{Op} \Sigma_{\alpha', \beta'} \otimes C(J)$ with symbol $(\text{Smb}^{1/p} A)^{-1}$ to show that A is Fredholm on $[L^p(J)]^N$, $\frac{1}{p} \in (\alpha', \beta')$. The formula (1-18) for $\text{ind}_p(A)$ now follows by approximating A in operator norm by operators in $\text{Op} \Sigma_{\alpha, \beta}(J)$. $\square$

2. DOUBLE LAYER KERNEL OPERATORS ON C^1 CURVES

In this section we show that double layer potential operators with analytic kernels are examples of operators in $\text{Op} \Sigma_{0,1} \otimes C(J)$.

Definition 2.1. A function $k(x, y)$ is an *analytic double layer kernel* if

- (1) k is real analytic in $\mathbf{R}^2 \setminus \{0\}$,
- (2) k is homogeneous of degree -1 and *odd*: for all real $\lambda \neq 0$, $k(\lambda x, \lambda y) = \lambda^{-1} k(x, y)$.

Let $\vec{\gamma}_0(t)$, $\vec{\gamma}_1(t)$, $\vec{\gamma}_2(t)$, $t \in J$, be simple C^1 curves with $d\vec{\gamma}_i/dt \neq \vec{0}$. Assume that $\vec{\gamma}_i(0) = (0, 0)$, and $d\vec{\gamma}_i(0)/dt = \vec{u}_i$, $\vec{u}_1 \neq \vec{u}_2$.

For $k(x, y)$ an analytic double layer kernel, and $b(t) \in C(J)$ define

$$(2-1) \quad K_b^0 f(t) = \text{p.v.} \int_0^1 k(\vec{\gamma}_0(t) - \vec{\gamma}_0(s)) b(s) f(s) \left| \frac{d\vec{\gamma}_0}{ds} \right| ds,$$

and

$$(2-2) \quad K_b^{12} f(t) = \int_0^1 k(\vec{\gamma}_1(t) - \vec{\gamma}_2(s)) b(s) f(s) \left| \frac{d\vec{\gamma}_2}{ds} \right| ds.$$

Theorem 2. If $\vec{\gamma}_i(t)$ are of class $C^1(J)$, then

- (1) $K_b^0 \in \text{Op} \Sigma_{0,1} \otimes C(J)$ and

$$(2-3) \quad \text{Smb}^{1/p} K_b^0 = k(\vec{\gamma}_0'(t)) b(t) \left| \frac{d\vec{\gamma}_0}{dt} \right| \pi \text{Smb}^{1/p}(H).$$

- (2) $K_b^{12} \in \text{Op} \Sigma_{0,1} \otimes C(J)$ and has the same symbol as the Hardy kernel operator with kernel

$$(2-4) \quad k^{12}(t) = |\vec{u}_2| b(0) k(t\vec{u}_1 - \vec{u}_2);$$

the Mellin transform of k^{12} is in $\mathcal{O}_{0,1}^{-\infty}$.

Proof. Since multiplication by $b(t) \in C(J)$ is in $\text{Op}\Sigma_{0,1} \otimes C(J)$, we may assume that $b(t) \equiv 1$.

To prove (2), as in [L, §2, Ex. 5] we write $\vec{\gamma}_i(t) = t\vec{u}_i + \vec{\epsilon}_i(t)$, with $\vec{\epsilon}_i(t) = t \int_0^1 [\vec{\gamma}'_i(\tau t) - \vec{\gamma}'_i(0)] d\tau = o(t)$. For t and s near 0, $|\vec{\gamma}_1(t) - \vec{\gamma}_2(s)| \approx t + s$. Thus $k(\vec{\gamma}_1(t) - \vec{\gamma}_2(s)) = k(t\vec{u}_1 - s\vec{u}_2 + \vec{\epsilon}_1(t) - \vec{\epsilon}_2(s)) = k(t\vec{u}_1 - s\vec{u}_2) + R^{12}(t, s)$. If $\vec{\gamma}_1$ and $\vec{\gamma}_2$ are smooth, then the kernel $R^{12} \partial/\partial t$ can be dominated by a Hardy kernel, so that $f(t) \mapsto \int_0^1 R(t, s)f(s)ds$ is compact. In the C^1 case, approximate $\vec{\gamma}_1, \vec{\gamma}_2$ in $C^1(J)$ by smooth curves $\vec{\gamma}_1^j, \vec{\gamma}_2^j$; with $\vec{\gamma}_i^{jj'}(0) = \vec{u}_i$. The kernels $[k(\vec{\gamma}_1^j(t) - \vec{\gamma}_2^j(s)) - k(\vec{\gamma}_1^j(t) - \vec{\gamma}_2^j(s))] = [R^{12}(t, s) - (R^{12})^j(t, s)]$ are dominated by (small constant) $\times (t+s)^{-1}$. Thus R^{12} is the kernel of a compact operator on $L^p(J)$. The proof also gives an approximation of K_b^{12} by operators in $\text{Op}\Sigma_{0,1} \otimes C(J)$. In (2-4), $\tilde{k}^{12}(z) \in \mathcal{O}_{0,1}^{-\infty}$ since $k^{12} \in C^\infty([0, \infty))$ and $(td/dt)^\ell k^{12}(t) = O(t^{-1})$ as $t \rightarrow \infty$.

To prove (1) and (2-3), we let

$$\Delta \vec{\gamma}_0(t, s) = \frac{\vec{\gamma}_0(t) - \vec{\gamma}_0(s)}{t - s}$$

and write $k(\vec{\gamma}_0(t) - \vec{\gamma}_0(s)) = k(\Delta \vec{\gamma}_0(t, s))(t - s) = k(\vec{\gamma}'_0(t))(t - s) + R(t, s)$, where

$$(2-5) \quad R(t, s) = \frac{1}{t - s} [k(\Delta \vec{\gamma}_0(t, s)) - k(\vec{\gamma}'_0(t))] \left| \frac{d\vec{\gamma}_0}{ds} \right|.$$

That the operator

$$(2-6) \quad f \mapsto \text{p.v.} \int_0^1 R(t, s)f(s)ds$$

is a compact operator on $L^p(J)$ is a result of Theorem 3 stated below. $\square$

We now prove Theorem 3.

Theorem 3. Let $\vec{\gamma}(t)$, $0 \leq t \leq 1$, be a simple C^1 curve and $k(x, y)$ an analytic double layer kernel. Let

$$\Delta \vec{\gamma}(t) = \frac{\vec{\gamma}(t) - \vec{\gamma}(s)}{t - s},$$

and

$$(2-7) \quad R_{\vec{\gamma}}f(t) = \text{p.v.} \int_0^1 \frac{1}{t - s} [k(\Delta \vec{\gamma}(t, s)) - k(\vec{\gamma}'(t))]f(s)ds.$$

Then $R_{\vec{\gamma}}$ is a compact operator on $L^p(J)$, $1 < p < \infty$.

Proof. The argument is similar to the argument of [FJR] which proves the compactness of the double layer potential on C^1 surfaces. Since the problem is local, assume that $\vec{\gamma}(t) = (t, \phi(t))$, with $|\phi'(t)| \leq \delta$, a small constant. For z near 0, express

$$k(1, z) = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{k(1, \zeta)}{\zeta - z} d\zeta.$$

Then

(2-8)

$R_{\tilde{\gamma}} f(t)$

$$= \text{p.v.} \int_0^1 \frac{1}{t-s} \left[\frac{1}{2\pi i} \oint_{|\zeta|=R} k(1, \zeta) \left\{ \frac{\Delta\phi(t, s) - \phi'(t)}{(\zeta - \Delta\phi(t, s))(\zeta - \phi'(t))} \right\} d\zeta \right] f(s) ds.$$

Define

$$(2-9) \quad R_{\zeta} f(t) = \text{p.v.} \int_0^1 \frac{1}{t-s} \left\{ \frac{\Delta\phi(t, s) - \phi'(t)}{(\zeta - \Delta\phi(t, s))(\zeta - \phi'(t))} \right\} f(s) ds.$$

Let $\{\phi^j\}$ be a sequence of smooth functions converging to ϕ in $C^1(J)$. Let

$$(2-10) \quad R_{\zeta}^j f(t) = \text{p.v.} \int_0^1 \frac{1}{t-s} \left\{ \frac{\Delta\phi^j(t, s) - \phi^{j'}(t)}{(\zeta - \Delta\phi(t, s))(\zeta - \phi'(t))} \right\} f(s) ds,$$

The operator R_{ζ}^j has a continuous kernel and is a compact operator on $L^p(J)$. By the Commutator Theorem of A. P. Calderón and C. P. Calderón [CCFJR, Theorem 4], we have that R_{ζ}^j converges to R_{ζ} in norm on $L^p(J)$, uniformly for $|\zeta| = R$, $|\phi'| < \delta$. In fact, with $\psi^j = \phi^j - \phi$, if

$$(2-11) \quad T_{\zeta}^j f(t) = \sup_{\epsilon > 0} \left| \int_{|t-s| > \epsilon} \frac{1}{t-s} \left\{ \frac{\Delta\psi^j(t, s) - \psi^{j'}(t)}{(\zeta - \Delta\phi(t, s))(\zeta - \phi'(t))} \right\} f(s) ds \right|,$$

we have $\|T_{\zeta}^j f\|_p \leq C_{p, R, \delta} \|\psi^{j'}\|_{\infty} \|f\|_p$. Thus R_{ζ} is a compact operator and the map $\zeta \mapsto R_{\zeta}$ is continuous in operator norm. Finally

$$R_{\tilde{\gamma}} = \frac{1}{2\pi i} \oint_{|\zeta|=R} k(1, \zeta) R_{\zeta} d\zeta$$

is a compact operator on $L^p(J)$. $\square$

Remarks. The operator (2-1) is a bounded operator on C^1 and small Lipschitz constant curves as a consequence of the Theorem of A. P. Calderón [Cal]. R. Coifman, A. McIntosh, and Y. Meyer [CMM] proved the boundedness of (2-1) on Lipschitz curves. Theorems III and V of [CMM] may be used for the bound for (2-11). In the general Lipschitz case the operator (2-7) is not compact since true Hardy kernels may appear.

In Theorem 3, if $\tilde{\gamma}(t)$ is a Lyapunov curve of class $C^{1, \lambda}$, we need not require that the kernel $k(x, y)$ be analytic; $k(x, y) \in C^2(\mathbf{R}^2 \setminus \{0\})$ suffices. For a different approach to the symbol map in this case, see M. Costabel [Cos].

Layer potentials on curvilinear polygons whose sides are of class C^1 may now be reduced to a big system of Mellin operators in $\text{Op} \Sigma_{0,1} \otimes C(J)$ as in [L, §3]. In applications, the functions $b(t)$ of Theorem 2 are components of the exterior unit normal and are only piecewise continuous.

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