SOME PROPERTIES OF NONCOMMUTATIVE H^1 SPACES

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ABSTRACT. In this paper it is proved that every T in the trace class operators, c_1 , has a unique best approximant from Δ , the set of upper triangular matrices in c_1 ; and that c_1/Δ is weakly sequentially complete.

1. INTRODUCTION

This paper studies some analogies between the H^p spaces and their noncommutative counterpart: the nest algebras in the Schatten p class c_p .

Prior research in this direction includes the work of Macaev [Ma] and Gohberg and Krein [GK] who proved that nest algebras in c_p are complemented in c_p for $1 < p < \infty$ and uncomplemented for $p = 1$ and $p = \infty$, as in the H^p case in L_p ; the work of Fall, Arveson, and Muhly [FAM] who proved the analogue of the result of Sarason [S] that says that $H^\infty + C$ is closed in L_∞ ; and the work of Feeman [F] who proved an analogue of the result of Luecking [Lu] that says that $(H^\infty + C)/H^\infty$ is an M -ideal in L_∞/H^∞ . See Power [Po2] for an excellent survey paper on the subject.

We are going to show that every $T \in c_1$ has a unique best approximant from any nest algebra in c_1 ; i.e., nest algebras in c_1 are Chebychev subspaces of c_1 ; as in the H^1 , L_1 case, (see [K]); and that c_1 modulo a nest algebra in c_1 is weakly sequentially complete, as in L_1/H^1 (see [P] and [G]).

2. PRELIMINARIES

In this paper H denotes a separable Hilbert space and c_p the Schatten p class, i.e., the space of all operators in $B(H)$ for which $|T|_p = (\text{tr}(T^*T)^{p/2})^{1/p}$ is finite. For information on these spaces see [M].

We identify $B(H)$ with $(c_1)^*$ under the trace duality; i.e., for $T \in B(H)$, $\phi_T \in (c_1)^*$ is defined by $\phi_T(A) = \text{tr}(TA)$.

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By a nest of projections in H we mean any linearly ordered set $\mathcal{P}$ of orthogonal projections that is closed in the strong operator topology and contains 0 and I . For $P \in \mathcal{P}$ we denote $P^- = \sup\{Q \in \mathcal{P}: Q < P\}$. Two of the most common nests of projections are the following:

The *discrete nest*. Let (e_i) be an orthonormal basis for H and let P_i be the orthogonal projection onto $\text{span}\{e_j: 1 \leq j \leq i\}$.

The *Volterra nest*. Let $H = L_2[0, 1]$ and for $0 \leq t \leq 1$ let P_t be the orthogonal projection onto the subspace of all functions supported on $[0, t]$.

If $\mathcal{P}$ is a nest of projections on H , the *nest algebra* induced by $\mathcal{P}$ is the set of all operators T in $B(H)$ that leave every element of $\mathcal{P}$ invariant; i.e.,

$$\mathcal{A} = \text{Alg } \mathcal{P} = \{T \in B(H): (I - P)TP = 0 \text{ for every } P \in \mathcal{P}\}.$$

We will also consider

$$\mathcal{A}_0 = \{T \in B(H): (I - P^-)TP = 0 \text{ for every } P \in \mathcal{P}\}.$$

For the discrete nest, $\mathcal{A}$ represents the upper triangular operators T_1 , and $\mathcal{A}_0$ the strictly upper triangular. For the Volterra nest, $\mathcal{A} = \mathcal{A}_0$.

For $1 \leq p < \infty$ we denote the nest algebras in c_p by

$$\mathcal{A}^p = \text{Alg } \mathcal{P} \cap c_p \quad \text{and} \quad \mathcal{A}_0^p = \mathcal{A}_0 \cap c_p.$$

We are ready to make the analogies explicit:

$$\begin{aligned} L_p(T), \quad 1 \leq p < \infty &\leftrightarrow c_p, \quad 1 \leq p < \infty, \\ L_\infty(T) &\leftrightarrow B(H), \\ C(T) &\leftrightarrow K(H), \\ H^p(T), \quad 1 \leq p < \infty &\leftrightarrow \mathcal{A}^p, \quad 1 \leq p < \infty, \\ H^\infty(T) &\leftrightarrow \mathcal{A}. \end{aligned}$$

We need to introduce one more concept. Let (e_i) be a fixed orthonormal basis for H and D a subset of $N \times N$. By the **-diagram* induced by D we mean the set $\mathcal{S}$ of all operators in $B(H)$ or c_1 that have no component outside D ; i.e., $(i, j) \notin D$ implies that $(e_i, Te_j) = 0$. To represent them we write a $*$ at the elements of D and 0 otherwise; e.g.,

$$\mathcal{S} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & * & 0 \end{pmatrix}.$$

It is easy to see that if $\mathcal{S}$ is a **-diagram* then $(\mathcal{S} \cap c_1)^\perp$, the set of all T in $B(H)$ that annihilates $\mathcal{S} \cap c_1$ under the trace duality, is also a **-diagram* satisfying: $(i, j) \in D_{\mathcal{S}} \Leftrightarrow (j, i) \notin D_{\mathcal{S}^\perp}$. This follows from the fact that the elements of $\mathcal{S}^\perp \mathcal{S}$ must have zero diagonal.

Finally, if X is a Banach space we say that a subspace Y of X is Chebychev if for every $x \in X$ there is a unique $y \in Y$ such that

$$\|x - y\| = d(x, Y) = \inf\{\|x - z\| : z \in Y\}.$$

3. UNIQUE BEST APPROXIMATIONS IN NEST ALGEBRAS

In this section we prove the noncommutative analogue of the unique best approximations of H^1 in L_1 .

Theorem 1. *Nest algebras in c_1 are Chebychev subspaces of c_1 .*

The proof of Theorem 1 follows immediately from the following proposition and lemmas. Proposition 1 finds necessary conditions on a w^* -closed subspace of c_1 to have more than one best approximant and Lemmas 2, 3 show that nest algebras do not have that condition. We want to point out that both lemmas are trivial for the discrete nest algebra.

We start with an easy lemma.

Lemma 1. *Let $T \in c_1$ and $U \in B(H)$ be such that $\|U\| \leq 1$ and $\text{tr}(UT) = |T|_1$. Then $UT = |T|$ and U is an isometry on the range of T .*

Proof. Find (ϕ_i) and (ψ_i) orthonormal sequences and $\lambda_i \geq 0$ such that

$$T = \sum_i \lambda_i \phi_i \otimes \psi_i \quad \text{and} \quad |T|_1 = \sum_i \lambda_i.$$

Since $UT = \sum_i \lambda_i \phi_i \otimes U\psi_i$, we have $\text{tr}(UT) = \sum_i \lambda_i (\phi_i, U\psi_i)$. Hence, $\|U\| \leq 1$ implies that for every i , $(\phi_i, U\psi_i) = 1$; i.e., $U\psi_i = \phi_i$. $\square$

Proposition 1. *Let $\mathcal{S}$ be a w^* -closed subspace of c_1 . If $\mathcal{S}$ is not Chebychev then $\mathcal{S}^\perp \mathcal{S}$ contains a nonzero selfadjoint element.*

Proof. The w^* -closedness of $\mathcal{S}$ guarantees the existence of the best approximants. Assume that we do not have uniqueness. Then we can find $T \notin \mathcal{S}$ and $A \in \mathcal{S}$, $A \neq 0$ satisfying:

$$\begin{aligned} |T|_1 &= d(T, \mathcal{S}) = 1, \\ |T \pm A|_1 &= |T|_1. \end{aligned}$$

Since $d(T + A, \mathcal{S}) = |T + A|_1$ we can find $U \in \mathcal{S}^\perp$, $\|U\| \leq 1$, such that $\text{tr}(U(T + A)) = |T + A|_1$. Hence, by Lemma 1, $U(T + A) = |T + A|$.

Since $U \in \mathcal{S}^\perp$ and $A \in \mathcal{S}$ we have that $\text{tr}(U(T + A)) = \text{tr}(UT)$. Therefore, $|T|_1 = |T + A|_1 = \text{tr}(UT)$; and then, by Lemma 1 again, $UT = |T|$.

Therefore,

$$\begin{aligned} UA &= U(T + A) - UT \\ &= |T + A| - |T| \end{aligned}$$

is clearly selfadjoint. It remains to prove that $UA \neq 0$.

If $UA = 0$, $U(T + A) = UT$; and since U is an isometry on the ranges of $T + A$ and T , we have that for every $x \in H$

$$\begin{aligned}\|(T + A)x\| &= \|U(T + A)x\| \\ &= \|UTx\| = \|Tx\|.\end{aligned}$$

Similarly, obtain $\|(T - A)x\| = \|Tx\|$. Then, by the parallelogram law we get the equality

$$\|(T + A)x\|^2 + \|(T - A)x\|^2 = 2[\|Tx\|^2 + \|Ax\|^2];$$

which implies trivially that $A = 0$, contradicting the original assumption. $\square$

The next lemma is an easy consequence of a result from [FAM] that says that $(\mathcal{A}_0^1)^\perp = \mathcal{A}$; and its corollary: If $T \in \mathcal{A}_0^1$ then $\text{tr}(T) = 0$.

Lemma 2. $(\mathcal{A}^1)^\perp = \mathcal{A}_0$.

Proof. It follows from the comment above that $(\mathcal{A}^1)^\perp \subset \mathcal{A}$. Assume that we can find $T \in (\mathcal{A}^1)^\perp$ such that $T \notin \mathcal{A}_0$. Then we can find $P \in \mathcal{P}$ such that

$$(P - P^-)T(P - P^-) \neq 0.$$

Find $x \in (P - P^-)H$ such that $(P - P^-)Tx = y \neq 0$. It is easy to see that $y \otimes x \in \mathcal{A}^1$ and that

$$\begin{aligned}\text{tr}(T(y \otimes x)) &= (y, Tx) \\ &= ((P - P^-)y, Tx) \\ &= (y, y) > 0,\end{aligned}$$

contradicting the fact that $T \in (\mathcal{A}^1)^\perp$.

For the other inclusion let $T \in \mathcal{A}_0$ and $A \in \mathcal{A}^1$. Since $TA \in \mathcal{A}_0^1$ we have that $\text{tr}(TA) = 0$ as mentioned above. Therefore $T \in (\mathcal{A}^1)^\perp$. $\square$

The proof of Theorem 1 will follow from the next lemma because if we had more than one best approximation in $\mathcal{A}^1$ we could find a nonzero selfadjoint element in $(\mathcal{A}^1)^\perp \mathcal{A}^1 = \mathcal{A}_0 \mathcal{A}^1 = \mathcal{A}_0^1$.

Lemma 3. If $T \in \mathcal{A}_0^1$ and T is selfadjoint then $T = 0$.

Proof. Since T is selfadjoint and $T \in \mathcal{A}_0^1$ we have that $T^* \in \mathcal{A}_0^1$ as well. Therefore, $T^*T \in \mathcal{A}_0^1$. By the comment above Lemma 2 we have that $\text{tr}(T^*T) = 0$. Hence $T = 0$. $\square$

Remark. It follows also from the previous lemma and the comment above Lemma 2 that $\mathcal{A}_0^1$ is Chebychev as well.

4. UNIQUE BEST APPROXIMATIONS IN *-DIAGRAMS

In this section we prove that the converse of Proposition 1 holds if the w^* -closed subspace is a $*$ -diagram.

Proposition 2. *Let $\mathcal{S}$ be a $*$ -diagram. If $\mathcal{S}^\perp \mathcal{S}$ contains a nonzero selfadjoint element then $\mathcal{S}$ is not Chebychev.*

The proof of the proposition depends on the following particular case.

Lemma 4. *Let $\mathcal{S} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Then $\mathcal{S}$ is not Chebychev.*

Proof. Let $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $T \in \mathcal{S}^\perp$, $\|T\| = 1$ and

$$\begin{aligned} TT &= |T|, \\ T(T + I) &= |T + I|. \end{aligned}$$

Therefore, $d(T, \mathcal{S}) = |T|_1$ and $d(T + I, \mathcal{S}) = |T + I|_1$. Since $(T + I) - T$ is in $\mathcal{S}$, we have more than one best approximant. $\square$

Remark. The previous lemma is immediately more general. If we pick the rows i and j and the columns k and l in such a way that we have a $*$ at (i, k) and (j, l) and a zero at (i, l) and (j, k) , then we can find U and V unitary (changing rows and columns) such that

$$U\mathcal{S}V = \begin{pmatrix} * & 0 & \cdots \\ 0 & * & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, since $U\mathcal{S}V$ clearly is not Chebychev, $\mathcal{S}$ also is not.

Proof of Proposition 2. If $\mathcal{S}^\perp \mathcal{S}$ contains a nonzero selfadjoint element, we can find $i \neq j$ such that $\mathcal{S}^\perp \mathcal{S}$ contains an element with components at (i, j) and (j, i) .

The only way this can happen is if for some k we have a $*$ at (i, k) in $\mathcal{S}^\perp$ and a $*$ at (k, j) in $\mathcal{S}$. Since the diagonal elements of $\mathcal{S}^\perp \mathcal{S}$ must be zero we have a zero at (k, i) in $\mathcal{S}$. Then we have:

$$\begin{aligned} &* \text{ at } (k, j) \text{ in } \mathcal{S}, \\ &0 \text{ at } (k, i) \text{ in } \mathcal{S}. \end{aligned}$$

Similarly, we can find some l such that we have a $*$ at (j, l) in $\mathcal{S}^\perp$ and at (l, i) in $\mathcal{S}$ and a zero at (l, j) in $\mathcal{S}$; i.e.,

$$\begin{aligned} &* \text{ at } (l, i) \text{ in } \mathcal{S}, \\ &0 \text{ at } (l, j) \text{ in } \mathcal{S}. \end{aligned}$$

Therefore, it follows from the previous remark that $\mathcal{S}$ does not have unique best approximations. $\square$

Remark. The hypothesis of Proposition 2 is very easy to check for $*$ -diagrams. Take the smallest $*$ -diagram containing $\mathcal{S}^\perp \mathcal{S}$ and check that for some i, j it contains a $*$ at (i, j) and at (j, i) . It is clear from the proof of Proposition 2 that this forces $\mathcal{S}^\perp \mathcal{S}$ to have a nonzero selfadjoint element.

Remark. In his dissertation the author asked if Proposition 2 holds for arbitrary w^* -closed subspaces. Recently, V. Mascioni gave a counterexample for it that will appear elsewhere.

5. CONTINUITY OF THE UNIQUE BEST APPROXIMATION

In this section we study some properties of the unique best approximation projection whenever it exists. In particular, we show that it is continuous for nest algebras in c_1 but not uniformly continuous, as in the commutative case (see [K]).

Proposition 3. *Let $\mathcal{S}$ be a w^* -closed subspace of the trace class operators with unique best approximations and $P: c_1 \rightarrow \mathcal{S}$ the unique best approximation projection. Then P is continuous.*

Proof. The projection is not linear but for $A \in \mathcal{S}$ and for $T \in c_1$ we have $P(T + A) = P(T) + A$. This allows us to reduce the proof to the case $|T|_1 = d(T, \mathcal{S})$.

Let $T \in c_1$ such that $|T|_1 = d(T, \mathcal{S}) = 1$. Let $T_n \rightarrow T$ and $\varepsilon > 0$.

For simplicity let $A_n = P(T_n)$. Since $d(\cdot, \mathcal{S})$ is continuous we have that

$$|T - A_n|_1 \rightarrow |T|_1.$$

Find $N \geq 1$ and E an N -dimensional projection such that

$$|T - ETE|_1 \leq \varepsilon.$$

Claim. $\lim_{n \rightarrow \infty} EA_nE = 0$.

We postpone the proof of the claim and finish.

It is clear that we have

$$|T - A_n|_1 \geq \sqrt{|E(T - A_n)E|_1^2 + |E^\perp(T - A_n)E|_1^2},$$

$$|T - A_n|_1 \geq \sqrt{|E(T - A_n)E|_1^2 + |E(T - A_n)E^\perp|_1^2},$$

$$|T - A_n|_1 \geq |E(T - A_n)E|_1 + |E^\perp(T - A_n)E^\perp|_1.$$

Since $EA_nE \rightarrow 0$ and $E^\perp TE$, $ETE^\perp$, and $E^\perp TE^\perp$ are very small, a standard computation gives that for n large enough

$$|E^\perp A_n E|_1 \leq 4\varepsilon,$$

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Therefore, for n large enough

$$|A_n|_1 \leq 15\varepsilon.$$

It remains to prove the claim.

Let B be a w^* -limit point of EA_nE . Since E is finite-dimensional we can find a subsequence $EA_{n_k}E \rightarrow B$. If $B \neq 0$ it is clear that any w^* -limit point C of A_{n_k} is nonzero, giving us the equality

$$\|T - C\|_1 = d(T, \mathcal{S}),$$

and contradicting the uniqueness of the best approximation.

Remark. If $\mathcal{A} = \text{Alg } \mathcal{P}$ is a nest algebra and $P: c_1 \rightarrow \mathcal{A}_0^1$ is the unique best approximation projection, then P is not uniformly continuous. Indeed, since $\mathcal{A}_0^1$ is a dual space (see [Po2]), then by a result of Lindenstrauss [L], we would have a bounded linear projection onto $\mathcal{A}_0^1$; but Macaev [Ma] proved that $\mathcal{A}_0^1$ is uncomplemented in c_1 .

6. $c_1/\mathcal{A}_0^1$ IS WEAKLY SEQUENTIALLY COMPLETE

In this section we point out that by combining results of Feeman [F], Godefroy [G], Li [Li], and the classical result of Dixmier [D] we get:

Proposition 4. $c_1/\mathcal{A}_0^1$ is weakly sequentially complete.

We now list those theorems. The proof of Proposition 4 is an easy consequence of them.

Theorem 2 (Godefroy [G], Li [Li]). *Let F be a Banach space which is an L -summand in its bidual; i.e., $F^{**} = F \oplus_1 F_s$ for some F_s subset of F^{**} ; and $E \subset F$ an L -summand in its bidual. Then F/E is weakly sequentially complete.*

Theorem 3 (Dixmier [D]). $B(H)^* = c_1 \oplus_1 K(H)^\perp$.

Theorem 4 (Feeman [F]). *Suppose that $\phi \in B(H)^*$ and has the decomposition, as in Theorem 3, $\phi = \phi_0 + \phi_T$ where $\phi_0 \in K(H)^\perp$ and ϕ_T is induced by the operator $T \in c_1$. If $\phi \in \mathcal{A}^\perp$, then $\phi_0 \in \mathcal{A}^\perp$ and $\phi_T \in \mathcal{A}^\perp$ as well.*

Remark. Theorem 4 is also true for $\mathcal{A}_0^1$; the proof is basically the same. It follows then that $c_1/\mathcal{A}^1$ is weakly sequentially complete.

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REFERENCES

- [D] J. Dixmier, *Les fonctionnelles linéaires sur l'ensemble des operateurs bornés d'un espace de Hilbert*, Ann. of Math. (2) **51** (1950), 387–408.
- [FAM] T. Fall, W. Arveson, and P. Muhly, *Perturbations of nest algebras*, J. Operator Theory **1** (1979), 137–150.
- [F] T. Feeman, *M -ideals and quasi-triangular algebras*, Illinois J. Math. **31** (1987), 89–98.
- [G] G. Godefroy, *Sous-espaces bien disposés de L^1 et applications*, Trans. Amer. Math. Soc. **286** (1984), 227–249.

- [GK] I. C. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*, Transl. of Math. Monographs, vol. 24, Amer. Math. Soc., Providence, RI, 1970.
- [K] J. P. Kahane, *Best approximations in $L^1(T)$* , Bull. Amer. Math. Soc. **80** (1974), 788–804.
- [Li] D. Li, *Espaces L -facteurs de leurs biduax: Bonne disposition, meilleure approximation et propriété de Radon-Nikodým*, preprint, Equipe d'Analyse Université Paris VI.
- [L] J. Lindenstrauss, *On nonlinear projections in Banach spaces*, Michigan Math. J. **11** (1964), 263–287.
- [Lu] D. Luecking, *The compact Hankel operators form an M -ideal in the space of Hankel operators*, Proc. Amer. Math. Soc. **79** (1980), 222–224.
- [Ma] V. I. Macaev, *Volterra operators produced by perturbations of selfadjoint operators*, Soviet Math. Dokl. **2** (1961), 1013–1016.
- [M] C. A. McCarthy, c_p , Israel J. Math. **5** (1967), 249–271.
- [P] A. Pelczyński, *Banach spaces of analytic functions and absolutely summing operators*, Regional Conference Series in Math., vol. 30, Amer. Math. Soc., Providence, RI, 1977.
- [Po1] S. Power, *Commutators with the triangular projection and Hankel forms on nest algebras*, J. London Math. Soc. **32** (1985), 272–282.
- [Po2] ———, *Analysis in nest algebras*, in Surveys of recent results in operator theory (J. Conway, ed.), Pitman Research Notes in Math., vol. 192, Longman, New York, 1988, pp. 189–234.
- [S] D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–299.

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