SOME TRIGONOMETRIC IDENTITIES RELATED TO EXACT COVERS

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ABSTRACT. Sherman K. Stein proves that if $\sin \pi z = k \prod_{i=1}^n \sin(\pi/d_i)(b_i - z)$ where the b_i are integers, the d_i are positive integers, k is a constant, then $\{(d_i : b_i)\}$ is an exact cover. It is shown here that if $0 \leq b_i < d_i$ then $k = -2^{n-1}$, that the converse is also true, and an analogous formula is conjectured for infinite exact covers. Many well known and lesser known trigonometric and functional identities can be derived from this result and known families of exact covers. A procedure is given for constructing exact covers by induction.

PRELIMINARIES

Let $(d : b)$ be the arithmetic progression (AP) $\{x : x = b + \alpha d, \alpha \in \mathbb{Z}\}$. A finite collection of disjoint AP's $\{(d_i : b_i) : 1 \leq i \leq n\}$ is called an exact cover if each integer belongs to precisely one AP. The numbers d_i are the moduli and the numbers b_i are the offsets. Since $(d_i : b_i) = (d_i : b_i + \gamma d_i)$ it is usual to assume that the offsets have been standardized, $0 \leq b_i < d_i$. A necessary and sufficient condition for a finite collection of AP's to be an exact cover is that $\sum_{i=1}^n 1/d_i = 1$ and $(d_i, d_j) \nmid b_i - b_j$. As a consequence there is only one AP in an exact cover with $b_i = 0 \pmod{d_i}$. If we let $n = \infty$ the collection of AP's is called an infinite exact cover. For an infinite exact cover it is possible for either $\sum_{i=1}^{\infty} 1/d_i < 1$, in which case the cover is called unsaturated, or $\sum_{i=1}^{\infty} 1/d_i = 1$, in which case it is saturated.

In [7] (Corollary 6.5) Stein proves that if

$$(1) \quad \sin \pi z = k \prod_{i=1}^n \frac{\pi}{d_i} (b_i - z)$$

where the b_i are integers, the d_i are positive integers, and k is a constant then $\{(d_i : b_i)\}$ is an exact cover. Trigonometric identities like (1), for example

$$(2) \quad \sin n\theta = 2^{n-1} \sin \theta \sin \left(\theta + \frac{\pi}{n} \right) \sin \left(\theta + \frac{2\pi}{n} \right) \cdots \sin \left(\theta + \frac{(n-1)\pi}{n} \right)$$

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appear frequently in analysis books. (See Hobson [3, p. 119]). Many other identities can be derived from (1) or (2) by taking derivatives of both sides, dividing both sides by $\sin \pi z/d_1$ and taking $\lim(z \rightarrow 0)$, choosing special values for z , etc. (See Hansen [2], particularly formulas 24, 29, and 91.) I show here that if $\{(d_i : b_i) : 1 \leq i \leq n\}$ is an exact cover with standardized offsets then (1) is true with $k = -2^{n-1}$, and derive an analogous formula that I conjecture is true for the case $n = \infty$.

CONSTRUCTION OF EXACT COVERS

I first show how to construct exact covers by induction on the number of AP's in the cover. There is an exact parallel between the method of construction for exact covers given here and the trigonometric proof of an identity like (1). This parallel could be used to write a computer program to create a proof of any such identity. Theorem 1 below says that to verify an identity like (1) it is only necessary to show that $\{(d_i : b_i) : 1 \leq i \leq n\}$ is an exact cover.

THE REDUCTION OF $C \bmod (q : k)$

Let C be an exact cover, and let $(q : k)$ be an AP. Every number in $(q : k)$ belongs to exactly one AP in C , so for each integer α there is a unique index i and integer γ such that

$$(3) \quad k + \alpha q = b_i + \gamma d_i,$$

where we must have $(d_i, q) \mid b_i - k$. For each i with $(d_i, q) \mid b_i - k$ there is a unique particular solution, $\alpha = a_i$ to (3) with

$$(4) \quad 0 \leq a_i < d_i/(d_i, q) \quad \text{and} \quad a_i q/(d_i, q) = (b_i - k)/(d_i, q) + \gamma_i d_i/(d_i, q)$$

and α satisfies (3) if and only if $\alpha = a_i \pmod{d_i/(d_i, q)}$; i.e. $\alpha \in (d_i/(d_i, q) : a_i)$. But every integer α satisfies (3) for exactly one i with $(d_i, q) \mid b_i - k$. This proves that $\{(d_i/(d_i, q) : a_i) : (d_i, q) \mid b_i - k\}$ is an exact cover, called the reduction of $C \bmod (q : k)$. (See Simpson [5].)

Now let q be prime, let $0 \leq k < q$, and let B_k be the reduction of $C \bmod (q : k)$. To indicate that the offset a_i exists for a particular combination of i and k I will write $a_i = a_i(k)$. Then

$$(5) \quad \begin{aligned} B_k &= \{(d_i : a_i(k)) : (d_i, q) = 1\} \cup \{(d_i/q : a_i(k)) : (d_i, q) = q \text{ and } q \mid b_i - k\} \\ &= A_k \cup A'_k \end{aligned}$$

is an exact cover. If $(d_i, q) = q$ for some i , B_k must have fewer AP's than C , because the sum of the reciprocals of its moduli must be 1.

Example 1. Let $C = \{(6 : 0), (6 : 2), (10 : 1), (10 : 5), (10 : 7), (10 : 9), (15 : 13), (30 : 3), (30 : 4), (30 : 10), (30 : 16), (30 : 22), (30 : 23)\} = \{(d_i : b_i) : 1 \leq i \leq 13\}$. Let $q = 2$. Then B_0 and B_1 are calculated from (5) and

the offsets $a_i(0)$ and $a_i(1)$ are calculated from (4).

$$\begin{aligned} B_0 &= A_0 \cup A'_0 \\ &= \{(15 : 14)\} \cup \{(3 : 0), (3 : 1), (15 : 2), (15 : 5), (15 : 8), (15 : 11)\} \\ &= \{(d_7 : a_7(0))\} \cup \{(d_1/2 : a_1(0)), (d_2/2 : a_2(0)), (d_9/2 : a_9(0)), \\ &\quad (d_{10}/2 : a_{10}(0)), (d_{11}/2 : a_{11}(0)), (d_{12}/2 : a_{12}(0))\}. \end{aligned}$$

$$\begin{aligned} B_1 &= A_1 \cup A'_1 \\ &= \{(15 : 6)\} \cup \{(5 : 0), (5 : 2), (5 : 3), (5 : 4), (15 : 1), (15 : 11)\} \\ &= \{(d_7 : a_7(1))\} \cup \{(d_3/2 : a_3(1)), (d_4/2 : a_4(1)), (d_5/2 : a_5(1)), \\ &\quad (d_6/2 : a_6(1)), (d_8/2 : a_8(1)), (d_{13}/2 : a_{13}(1))\}. \end{aligned}$$

Lemma 1. *If $(d_i, q) = 1$ then $a_i(k)$ exists for all k so $(d_i : a_i(k)) \in A_k$ for all k and $\bigcup_{k=0}^{q-1} (d_i q : k + a_i(k)q) = (d_i : b_i)$.*

Proof. If $(d_i, q) = 1$ there is a solution $a_i(k)$ and $\gamma_i(k)$ to (4) for every k . By (4), as k varies over its range, $\gamma_i(k) \pmod{q}$ takes on each integer value in the same range. Thus $\bigcup_{k=0}^{q-1} (d_i q : k + a_i(k)q) = \bigcup_{k=0}^{q-1} (d_i q : b_i + \gamma_i(k)d_i) = (d_i : b_i)$. $\square$

Example 2. In example 1 $(d_7, q) = (15, 2) = 1$ so $(15 : 14) = (d_7 : a_7(0)) \in A_0$ and $(15 : 6) = (d_7 : a_7(1)) \in A_1$ and $(d_7 \cdot 2 : 0 + a_7(0) \cdot 2) \cup (d_7 \cdot 2 : 1 + a_7(1) \cdot 2) = (30 : 28) \cup (30 : 13) = (15 : 13) = (d_7 : b_7)$.

Lemma 2. *If $q \mid d_i$ there is exactly one value of k , $0 \leq k < q$, for which $a_i(k)$ exists, so $(d_i/q : a_i(k)) \in A'_k$ for this value of k and $(d_i : k + a_i(k)q) = (d_i : b_i)$.*

Proof. Clearly $q \mid b_i - k$ for some value of k , so $a_i(k)$ exists for this value of k . If $a_i(r)$ exists, $0 \leq r < q$, then $q \mid b_i - r$ and $q \mid b_i - k$, so $q \mid k - r$. This implies $k = r$, so k is unique. If $q \mid d_i$, $(d_i : b_i)$ must be a subset of some AP $(q : k)$, $0 \leq k < q$. By (3), $k + a_i(k)q = b_i + \gamma d_i$. Thus $(d_i : k + a_i(k)q) = (d_i : b_i)$. $\square$

Example 3. In example 1, $2 \mid d_i$ for $i \neq 7$. Thus $a_i(0)$ exists for $i = 1, 2, 9, 10, 11, 12$ and $a_i(1)$ exists for $i = 3, 4, 5, 6, 8, 13$. For instance, if $k = 1$ and $i = 8$ then $(d_8 : 1 + a_8(1) \cdot 2) = (30 : 1 + 1 \cdot 2) = (30 : 3) = (d_8 : b_8)$. The generalization of the partition $\{7\} \cup \{1, 2, 9, 10, 11, 12\} \cup \{3, 4, 5, 6, 8, 13\}$ to arbitrary primes q and exact covers C has appeared previously in the literature about Znám's conjecture (see Simpson [6] and Znám [8]).

The following lemma describes a class of exact covers called the natural irreducible covers by Korec [4], who also describes constructions 1 and 2 below.

Lemma 3. *If q is a prime, then $Z_q = \{(q : k) : 0 \leq k < q\}$ is an exact cover. Also, $Z_1 = \{(1 : 0)\}$ is an exact cover.*

Construction 1. If $C_1 = \{(d_i : b_i) : 1 \leq i \leq n\}$ and if $C_2 = \{(e_j : c_j) : 1 \leq j \leq m\}$ are exact covers, then $C_3 = \{(d_i : b_i) : i \neq I\} \cup \{(e_j d_I : b_I + c_j d_I) : 1 \leq j \leq m\}$ is an exact cover with $n + m - 1$ AP's.

Korec writes $C_3 = \text{Split}(C_1, b_I, C_2)$. I call this construction covering $(d_I : b_I)$ with C_2 .

Example 4. Let $Z_2 = \{(2 : 0), (2 : 1)\}$. Cover $(2 : 0)$ with B_0 from Example 1, getting the exact cover $\{(30 : 28), (6 : 0), (6 : 2), (30 : 4), (30 : 10), (30 : 16), (30 : 22), (2 : 1)\}$. Cover $(2 : 1)$ in the latter with B_1 , getting

$$C' = \{(30 : 28), (6 : 0), (6 : 2), (30 : 4), (30 : 10), (30 : 16), (30 : 22), \\ (30 : 13), (10 : 1), (10 : 5), (10 : 7), (10 : 9), (30 : 3), (30 : 23)\}.$$

Construction 2. Let $C_1 = \{(d_i : b_i) : 1 \leq i \leq n\}$ be an exact cover. Suppose the union of some subcollection $D = \{(d_{i_j} : b_{i_j}) : 1 \leq j \leq m\}$ is the AP $(d : b)$. Then $C_3 = (C_1 - D) \cup \{(d : b)\}$ is an exact cover with $n - m + 1$ AP's.

I call this construction the substitution of $(d : b)$ for D . Note that $d \mid d_{i_j}$ and $d \mid b_{i_j} - b$, so $C_2 = \{(d_{i_j}/d : (b_{i_j} - b)/d : 1 \leq j \leq m\}$ is an exact cover, the reduction of $C_1 \bmod (d : b)$. If we cover $(d : b)$ with C_2 , we just get C_1 back again, so Constructions 1 and 2 are inverses to each other.

Example 5. Let $D = \{(30 : 28), (30 : 13)\}$. From Example 2 $(30 : 28) \cup (30 : 13) = (15 : 13)$, so substitute $(15 : 13)$ for D in C' from Example 4 getting

$$C'' = \{(15 : 13), (6 : 0), (6 : 2), (30 : 4), (30 : 10), (30 : 16), (30 : 22), \\ (10 : 1), (10 : 5), (10 : 7), (10 : 9), (30 : 3), (30 : 23)\}.$$

Except for the order of the AP's C'' is the same as C from Example 1. The intuitive meaning of the following lemma and its proof is that any exact cover C can be disassembled and reassembled in the same way. I will choose q so that the covers B_k always have fewer AP's than C . Thus it gives a theoretical method to construct an exact cover by induction. Here it is used to prove Theorem 1 below, which shows that there is an exact parallel between the proof of identities like (1) and the construction of exact covers.

Lemma 4. Suppose we have a list of all exact covers with M or fewer AP's. Any exact cover C with $M + 1$ AP's, except Z_{M+1} (if $M + 1$ is prime) can be constructed by first using Construction 1 to cover the AP's $(q : k) \in Z_q$ with q exact covers from the list, where q is a prime that is a proper divisor of some modulus of C . This gives an exact cover C' . If q does not divide every modulus of C , it will then be necessary to apply Construction 2 to some subcollections of C' to get C .

Proof. Let $C = \{(d_i : b_i) : 1 \leq i \leq M + 1\}$ be an exact cover, where C is not the natural irreducible cover Z_{M+1} described in Lemma 3. Then there is a

prime q which is a proper divisor of at least one modulus of C . Let B_k be the reduction of $C \bmod (q : k)$, $0 \leq k < q$. By the remark following (5) the B_k 's have fewer AP's than C and hence are in the list of exact covers with M or fewer AP's. The proof of the lemma will be completed by showing how to reconstruct C from the B_k 's and Z_q . The procedure is illustrated in Examples 4 and 5 above and Example 6 below.

For each k cover the AP $(q : k) \in Z_q$ with B_k . From (5) and Construction 1 we obtain the exact cover

$$C' = \left(\bigcup_{k=0}^{q-1} \{(d_i q : k + a_i(k)q) : (d_i, q) = 1\} \right) \cup \left(\bigcup_{k=0}^{q-1} \{(d_i : k + a_i(k)q) : (d_i, q) = q \text{ and } q \mid b_i - k\} \right).$$

By Lemma 1, $\bigcup_{k=0}^{q-1} (d_i q : k + a_i(k)q) = (d_i : b_i)$ for each i with $(d_i, q) = 1$. Using Construction 2, substitute $(d_i : b_i)$ for $D = \{(d_i q : k + a_i(k)q) : 0 \leq k < q\}$, getting the new exact cover

$$C'' = \{(d_i : b_i) : (d_i, q) = 1\} \cup \bigcup_{k=0}^{q-1} \{(d_i : k + a_i(k)q) : (d_i, q) = q \text{ and } q \mid b_i - k\}.$$

By Lemma 2 if $(d_i, q) = q$ then $(d_i : k + a_i(k)q) = (d_i : b_i)$, so

$$C'' = \{(d_i : b_i) : (d_i, q) = 1\} \cup \{(d_i : b_i) : (d_i, q) = q\} = C.$$

In the preceding proof all of the AP's except perhaps those belonging to C have standardized offsets. Thus, when Constructions 1 and 2 are used to build an exact cover with standardized offsets we can assume that we are applying Constructions 1 and 2 to AP's with standardized offsets. $\square$

Example 6. Let C be as in Example 1, but let $q = 5$. Then

$$B_0 = A_0 \cup A'_0 = \{(6 : 0), (6 : 4)\} \cup \{(2 : 1), (6 : 2)\}$$

$$B_1 = A_1 \cup A'_1 = \{(6 : 1), (6 : 5)\} \cup \{(2 : 0), (6 : 3)\}$$

$$B_2 = A_2 \cup A'_2 = \{(6 : 2), (6 : 0)\} \cup \{(2 : 1), (6 : 4)\}$$

$$B_3 = A_3 \cup A'_3 = \{(6 : 3), (6 : 1)\} \cup \{(3 : 2), (6 : 0), (6 : 4)\}$$

$$B_4 = A_4 \cup A'_4 = \{(6 : 4), (6 : 2)\} \cup \{(2 : 1), (6 : 0)\}$$

Cover the AP $(5 : k) \in Z_5$ with B_k , $0 \leq k < 5$ getting

$$\begin{aligned} C' = & \{(30 : 0), (30 : 20)\} \cup \{(30 : 6), (30 : 26)\} \cup \{(30 : 12), (30 : 2)\} \\ & \cup \{(30 : 18), (30 : 8)\} \cup \{(30 : 24), (30 : 14)\} \\ & \cup \{(10 : 5), (30 : 10)\} \cup \{(10 : 1), (30 : 16)\} \cup \{(10 : 7), (30 : 22)\} \\ & \cup \{(15 : 13), (30 : 3), (30 : 23)\} \cup \{(10 : 9), (30 : 4)\}. \end{aligned}$$

Now $(30 : 0) \cup (30 : 6) \cup (30 : 12) \cup (30 : 18) \cup (30 : 24) = (6 : 0)$ and $(30 : 20) \cup (30 : 26) \cup (30 : 2) \cup (30 : 8) \cup (30 : 14) = (6 : 2)$. Making these substitutions in C' we get

$$\begin{aligned} C'' &= \{(6 : 0), (6 : 2)\} \cup \{(10 : 5), (30 : 10), (10 : 1), (30 : 16), (10 : 7), \\ &\quad (30 : 22), (15 : 13), (30 : 3), (30 : 23), (10 : 9), (30 : 4)\} \\ &= C. \end{aligned}$$

TRIGONOMETRIC IDENTITIES

Theorem 1. *The set of AP's $C = \{(d_i : b_i) : 1 \leq i \leq n\}$, with standardized offsets, is an exact cover if and only if*

$$(6) \quad \sin \pi z = -2^{n-1} \prod_{i=1}^n \sin \frac{\pi}{d_i} (b_i - z).$$

If the offsets are not standardized then write $b_i = b'_i + \eta_i d_i$, where $0 \leq b'_i < d_i$. The product on the right in (6) must then be multiplied by $(-1)^{\sum_{i=1}^n \eta_i}$.

Proof. If (6) holds then C is an exact cover because the zeros of the function on the left must be the same as the zeros of the function on the right and have the same multiplicity.

The first proof of the converse uses the inductive construction of exact covers described in Lemma 4. A different proof is given after Corollary 1 below. Substitute $-\pi z = n\theta$ in (2), getting the identity

$$\sin \pi z = -2^{n-1} \prod_{k=0}^{n-1} \frac{\pi}{n} (k - z).$$

Thus Theorem 1 is true for the covers Z_q described in Lemma 3.

To complete the proof it is necessary to show that if covers C_1 and C_2 satisfy (6), then the cover C_3 obtained from Construction 1 or 2 satisfies (6). If $C = \{(d_i : b_i) : 1 \leq i \leq n\}$ let

$$\prod C = -2^{n-1} \prod_{i=1}^n \sin \frac{\pi}{d_i} (b_i - z).$$

Consider Construction 1 and suppose

$$\begin{aligned} (7) \quad \sin \pi z &= -2^{n-1} \prod_{i=1}^n \sin \frac{\pi}{d_i} (b_i - z) \\ &= -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{e_j} (c_j - z). \end{aligned}$$

Then

$$\begin{aligned}\sin \frac{\pi}{d_I}(z - b_I) &= -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{e_j} \left(c_j - \frac{z - b_I}{d_I} \right) \\ &= -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{e_j d_I} (b_I + c_j d_I - z).\end{aligned}$$

Substituting the latter in (7),

$$\sin \pi z = -2^{m+n-2} \prod_{\substack{i=1 \\ i \neq I}}^n \sin \frac{\pi}{d_i} (b_i - z) \prod_{j=1}^m \sin \frac{\pi}{e_j d_I} (b_I + c_j d_I - z) = \prod C_3.$$

Now consider Construction 2. Since $\bigcup_{j=1}^m (d_{i_j} : b_{i_j}) = (d : b)$ we can show that $d \mid d_{i_j}$ and $d \mid b_{i_j} - b$ and $\{(d_{i_j}/d : (b_{i_j} - b)/d) : 1 \leq j \leq m\}$ is an exact cover, the reduction of $C_1 \bmod (d : b)$. Thus

$$\begin{aligned}\sin \pi z &= -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{d_{i_j}/d} ((b_{i_j} - b)/d - z) \\ &= -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{d_{i_j}} (b_{i_j} - b - dz).\end{aligned}$$

Substituting $(z - b)/d$ for z in the last equation,

$$\sin \frac{\pi}{d}(z - b) = -2^{m-1} \prod_{j=1}^m \sin \frac{\pi}{d_{i_j}} (b_{i_j} - z).$$

Substituting for $\prod_{j=1}^m \sin(\pi/d_{i_j})(b_{i_j} - z)$ in (7),

$$\sin \pi z = -2^{n-m} \sin \frac{\pi}{d}(b - z) \prod_{\substack{i=1 \\ i \neq i_j}}^n \sin \frac{\pi}{d_i} (b_i - z) = \prod C_3.$$

If the offsets are not standardized, let $b_i = b'_i + \eta_i d_i$, where $0 \leq b'_i < d_i$. Then

$$\sin \frac{\pi}{d_i}(b_i - z) = (-1)^{\eta_i} \sin \frac{\pi}{d_i}(b'_i - z),$$

so

$$\sin \pi z = (-1)^{\sum_{i=1}^n \eta_i} (-2^{n-1}) \prod_{i=1}^n \sin \frac{\pi}{d_i}(b_i - z). \quad \square$$

Corollary 1. Suppose $(d_1 : b_1)$ is the unique AP in the exact cover $\{(d_i : b_i) : 1 \leq i \leq n\}$ with $b_1 = 0$ and the offsets are standardized. Then

$$\frac{d_1}{2^{n-1}} = \prod_{i=2}^n \sin \frac{\pi b_i}{d_i}$$

and hence

$$(8) \quad \sin \pi z = d_1 \sin \frac{\pi z}{d_1} \prod_{i=2}^n \frac{\sin(\pi/d_i)(b_i - z)}{\sin(\pi b_i/d_i)}.$$

Proof. Divide both sides of (6) by $\sin(\pi/d_1)(b_1 - z)$ and take $\lim(z \rightarrow 0)$ on both sides. $\square$

I now give a proof of Corollary 1 (and hence Theorem 1) that is quite different from the one just given. This proof should make the conjecture below more plausible.

Proof of Corollary 1. For each real number $\rho > 0$ define

$$f_\rho(z) = \pi z \prod_{k=1}^{\rho} \left(1 - \frac{z}{k}\right) \left(1 + \frac{z}{k}\right) = \pi z \prod_{\substack{k=-\rho \\ k \neq 0}}^{\rho} \left(1 - \frac{z}{k}\right),$$

where the notation $\prod_{k=-\rho}^{\rho}$ means that the product is to be evaluated for integers k , $-\rho \leq k \leq \rho$. Now let $C = \{(d_i : b_i) : 1 \leq i \leq n\}$ be an exact cover with $b_1 = 0$. Then

$$(9) \quad f_\rho(z) = \pi z \prod_{\substack{k=-\rho/d_1 \\ k \neq 0}}^{\rho/d_1} \left(1 - \frac{z}{kd_1}\right) \prod_{i=2}^n \left(\prod_{k=(-\rho-b_i)/d_i}^{(\rho-b_i)/d_i} \left(1 - \frac{z}{b_i + kd_i}\right) \right)$$

From Hobson [3, p. 350] we know that $\lim_{\rho \rightarrow \infty} f_\rho(z) = \sin \pi z$.

But if $b_i \neq 0$ then

$$\begin{aligned} f_\rho \left(\frac{b_i - z}{d_i} \right) &= \pi \left(\frac{b_i - z}{d_i} \right) \prod_{\substack{k=-\rho \\ k \neq 0}}^{\rho} \left(1 - \frac{b_i - z}{kd_i} \right) \\ &= \pi \frac{b_i}{d_i} \left(1 - \frac{z}{b_i} \right) \prod_{\substack{k=-\rho \\ k \neq 0}}^{\rho} \left(\frac{kd_i - b_i}{kd_i} + \frac{z}{kd_i} \right) \\ &= \pi \frac{b_i}{d_i} \left(1 - \frac{z}{b_i} \right) \prod_{\substack{k=-\rho \\ k \neq 0}}^{\rho} \left(1 - \frac{b_i}{kd_i} \right) \left(1 + \frac{z}{kd_i - b_i} \right) \\ &= f_\rho \left(\frac{b_i}{d_i} \right) \prod_{k=-\rho}^{\rho} \left(1 - \frac{z}{b_i + kd_i} \right). \end{aligned}$$

Hence

$$f_{\rho/d_i} \left(\frac{b_i - z}{d_i} \right) = f_{\rho/d_i} \left(\frac{b_i}{d_i} \right) \prod_{k=-\rho/d_i}^{\rho/d_i} \left(1 - \frac{z}{b_i + kd_i} \right).$$

Thus

$$\begin{aligned}
 \sin \pi \left(\frac{b_i - z}{d_i} \right) &= \lim_{\rho \rightarrow \infty} f_{\rho/d_i} \left(\frac{b_i - z}{d_i} \right) \\
 (10) \quad &= \sin \frac{\pi b_i}{d_i} \left(\lim_{\rho \rightarrow \infty} \prod_{k=-\rho/d_i}^{\rho/d_i} \left(1 - \frac{z}{b_i + k d_i} \right) \right) \\
 &= \sin \frac{\pi b_i}{d_i} \left(\lim_{\rho \rightarrow \infty} \prod_{k=(-\rho-b_i)/d_i}^{(\rho-b_i)/d_i} \left(1 - \frac{z}{b_i + k d_i} \right) \right).
 \end{aligned}$$

Also

$$(11) \quad \sin \frac{\pi z}{d_1} = \frac{\pi z}{d_1} \left(\lim_{\rho \rightarrow \infty} \prod_{\substack{k=-\rho/d_1 \\ k \neq 0}}^{\rho/d_1} \left(1 - \frac{z}{k d_1} \right) \right).$$

Taking the limit of both sides of (9) and substituting from (10) and (11) we get

$$\sin \pi z = d_1 \sin \frac{\pi z}{d_1} \prod_{i=2}^n \frac{\sin(\pi/d_i)(b_i - z)}{\sin(\pi b_i/d_i)}.$$

If we could somehow let $n \rightarrow \infty$ as well, we could prove the conjecture below.

I will now show that if $b_1 = 0$, then $d_1/2^{n-1} = \prod_{i=2}^n \sin \pi b_i/d_i$. Let $m = \text{lcm}\{d_1, d_2, \dots, d_n\}$. Since C is an exact cover,

$$x^{2m} - 1 = \prod_{i=1}^n \left(x^{2m/d_i} - e^{2\pi i b_i/d_i} \right).$$

(See the proof of Theorem 5.1 in Stein [7].) Let $x = e^{\pi i z/m}$ in this equation. Then

$$e^{2\pi i z} - 1 = \prod_{i=1}^n (e^{2\pi i z/d_i} - e^{2\pi i b_i/d_i}).$$

Divide both sides by $e^{2\pi i z/d_1} - e^{2\pi i b_1/d_1} = e^{2\pi i z/d_1} - 1$, and take $\lim(z \rightarrow 0)$ of both sides, getting

$$d_1 = \prod_{i=2}^n (1 - e^{2\pi i b_i/d_i}) = \prod_{i=2}^n e^{\pi i b_i/d_i} (-2i \sin \pi b_i/d_i).$$

By Theorem 1 of Fraenkel [1] (with $n=1$), if $0 \leq b_i < d_i$, then

$$\sum_{i=1}^n \frac{b_i}{d_i} = \sum_{i=2}^n \frac{b_i}{d_i} = \frac{n-1}{2}.$$

Thus

$$d_1 = 2^{n-1} e^{\pi i(n-1)/2} e^{-\pi i(n-1)/2} \prod_{i=2}^n \sin \frac{\pi b_i}{d_i} = 2^{n-1} \prod_{i=2}^n \sin \frac{\pi b_i}{d_i}. \quad \square$$

Corollary 2. *If $\{(d_i : b_i) : 1 \leq i \leq n\}$ is an exact cover with standardized offsets then*

- (1) $-2^{n-1} \prod_{i=1}^n \sin(\pi/2d_i)(2b_i - 1) = 1,$
- (2) $\cot \pi z = \sum_{i=1}^n (1/d_i) \cot(\pi/d_i)(z - b_i),$
- (3) $\csc^2 \pi z = \sum_{i=1}^n (1/d_i^2) \csc^2(\pi/d_i)(z - b_i).$

Proof of (1). Let $z = 1/2$ in (6). $\square$

Proof of (2). Take the logarithmic derivative of both sides of (6). $\square$

Proof of (3). Take the derivative of (2) of Corollary 2. $\square$

If $C = \{(d_i : b_i) : 1 \leq i \leq n\}$ is an exact cover, and $q > 0$ is relatively prime to all d_i and k is any integer, then $C' = \{(d_i : k + qb_i \pmod{d_i}) : 1 \leq i \leq n\}$ is an exact cover. More generally, if d is a positive integer and b is any integer, then C'' , the reduction of $C \pmod{d : b}$ is an exact cover.

Corollary 3. *If $\sin \pi z = \prod C$, then $\sin \pi z = \prod C' = \prod C''$.*

Conjecture. *If $\{(d_i : b_i) : 1 \leq i < \infty\}$ is an infinite exact cover (saturated or unsaturated) with $b_1 = 0 \pmod{d_1}$ and standardized offsets then*

$$\sin \pi z = d_1 \sin \frac{\pi z}{d_1} \prod_{i=2}^{\infty} \frac{\sin(\pi/d_i)(b_i - z)}{\sin \pi b_i/d_i}.$$

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