INTERPRETATIONS INTO p -ALGEBRAS

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ABSTRACT. We prove that there is only one interpretation from the variety of p -algebras into itself.

1. INTRODUCTION AND PRELIMINARIES

The concept of interpretability of one variety into another was introduced by Neumann [4] in the study of Malt'sev conditions; it was later developed by García and Taylor [3]. Roughly speaking, a variety V is interpretable in a variety W if the operations of V can be defined in terms of the operations of W in such a way that the (underlying sets of the) algebras in W with these new operations are algebras in V . For precise definitions the reader is referred to the above mentioned [3].

In what follows $\mathbf{D}_{01}$ stands for the variety of distributive lattices with greatest and least elements. $\mathbf{B}_\omega$ stands for the variety of distributive pseudocomplemented algebras, or p -algebras. For background on these, check for instance [1].

A p -algebra is a universal algebra $\langle A; +, \cdot, *, 0, 1 \rangle$ such that

- (1) $\langle A; +, \cdot, 0, 1 \rangle \in \mathbf{D}_{01}$,
- (2) $x(xy)^* = xy^*$,
- (3) $0^{**} = 0$,
- (4) $0^* = 1$.

Remark. For any $a \in A$, a^* is called the pseudocomplement of a . We can prove that a^* is the greatest element x such that $a \cdot x = 0$.

Some identities verified in $\mathbf{B}_\omega$ that will be used in the sequel are the following:

Theorem. In $\mathbf{B}_\omega$ the following identities hold:

- (i) $x \cdot x^* = 0$,
- (ii) $x \leq x^{**}$, $x^* = x^{***}$,

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- (iii) $(x + y)^* = x^* y^*$,
- (iv) $(xy)^* \geq x^* + y^*$,
- (v) $x^{**} y^{**} = (xy)^{**}$.

In what follows we need to determine the free p -algebras in one and two generators, $\mathbf{F}_{\mathbf{B}_\omega}(\mathbf{x})$ and $\mathbf{F}_{\mathbf{B}_\omega}(\mathbf{x}, \mathbf{y})$. Using a result by Berman and Dwinger [2, p. 240], after some tedious computations, we can prove that $\mathbf{F}_{\mathbf{B}_\omega}(\mathbf{x})$ is the algebra whose elements are $0, x, x^*, x^{**}, x + x^*, x^* + x^{**}$ and 1 , and that $\mathbf{F}_{\mathbf{B}_\omega}(\mathbf{x}, \mathbf{y})$ is the algebra whose elements are the finite sums of the following set S of terms

$$S = \{0, 1, x, x^*, x^{**}, y, y^*, y^{**}, xy, xy^*, xy^{**}, x^*y, x^*y^*, x^*y^{**}, x^{**}y, x^{**}y^*, x^{**}y^{**}, (xy)^*, (xy^*)^*, (x^*y)^*, (x^*y^*)^*, (x^*y^{**})^*, (xy)^*(x^*y^*)^*, (xy^*)^*(x^*y)^*\}.$$

2. INTERPRETING $\mathbf{D}_{01}$ INTO $\mathbf{B}_\omega$

As stated in the introduction, we must define the operations $\vee, \wedge, \mathbf{0}$ and $\mathbf{1}$ of $\mathbf{D}_{01}$ in terms of the operations of $\mathbf{B}_\omega$, that is, we need binary terms $\sigma(x, y)$ and $\pi(x, y)$, and constant terms c_0 and c_1 in the language of p -algebras such that given any p -algebra A , if we define

$$\begin{aligned} x \vee y &:= \sigma(x, y), \\ x \wedge y &:= \pi(x, y), \\ \mathbf{0} &:= c_0, \\ \mathbf{1} &:= c_1, \end{aligned}$$

$\langle A; \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle \in \mathbf{D}_{01}$.

It is immediate that the “trivial” and the “dual” interpretations will work, that is, define

$$\begin{array}{ll} x \vee y := x + y, & x \vee y := xy, \\ x \wedge y := xy, & \text{or} \quad x \wedge y := x + y, \\ \mathbf{0} := 0, & \mathbf{0} := 1, \\ \mathbf{1} := 1, & \mathbf{1} := 0. \end{array}$$

In what follows we will prove that no other definition is possible.

Let us assume then that

$$(1) \quad x \vee y = \sum t_i(x, y)$$

for a finite set of terms $t_i \in S$.

Since $x \vee y$ is an idempotent operation, we must have that

$$x = x \vee x = \sum t_i(x, x).$$

This implies that the terms 1 , $(x^*y)^*$, $(xy^*)^*$, and $(x^*y)^*(xy^*)^*$ cannot appear in the right-hand side of (1) or else

$$x = \sum t_i(x, x) = 1.$$

In the same way the terms x^* , y^* , x^*y^* , and $(xy)^*$ cannot be summands of σ or else

$$0 = 0 \vee 0 = 1.$$

Similarly, if the terms $x^{**}y^{**}$, $(x^*y^*)^*$, or y^{**} appear as summands

$$x = x \vee x = \sum t_i(x, x) \geq x^{**}$$

which is not possible.

Notice now that the constants **0** and **1** must be defined in terms of the constants 0 and 1 so either **0** = 0 and **1** = 1 or **0** = 1 and **1** = 0.

In any case

$$x \vee 0 = 0 \vee x \in \{x, 0\}$$

and

$$x \vee 1 = 1 \vee x \in \{x, 1\}.$$

Evaluating $x^{**}y^*$, x^*y^{**} , $(xy)^*(x^*y^*)^*$ in $(x, 0)$ or $(0, x)$ yields x^{**} so if these terms appear in σ , we would have $x \vee 0 = 0 \vee x \geq x^{**}$ which is not possible by the remark above.

Similarly, evaluating $x^{**}y$, xy^{**} , x^*y , xy^* in $(x, 1)$ or $(1, x)$ yields x^{**} or x^* , so if these terms appear in σ

$$1 \vee x = x \vee 1 \geq x^* \quad \text{or} \quad x^{**},$$

so

$$x \vee 1 = 1 \vee x = 1.$$

It is not difficult to check that, if this is the case, then both x and y must appear in σ so the terms here considered are absorbed.

We are left with the terms x , y and xy . Obviously if x appears in σ , so does y and if they both appear in σ , the third is absorbed. One of them must be present or else

$$x \vee y = 0$$

which is not possible, so we have only two possibilities, either

$$x \vee y = x + y \quad \text{or} \quad x \vee y = xy.$$

It is easy to check that in these cases

$$x \wedge y = xy, \quad \mathbf{0} = 0 \quad \text{and} \quad \mathbf{1} = 1$$

and

$$x \wedge y = x + y, \quad \mathbf{0} = 1 \quad \text{and} \quad \mathbf{1} = 0$$

respectively.

We have thus proven our main result.

Theorem. *There are only two interpretations from $\mathbf{D}_{01}$ into $\mathbf{B}_\omega$.*

3. INTERPRETING $\mathbf{B}_\omega$ IN $\mathbf{B}_\omega$

All we have to do is to define a new pseudocomplement

$$x' := \tau(x) \in \mathbf{F}_{\mathbf{B}_\omega}(\mathbf{x}),$$

in such a way that for any p -algebra $\langle A; +, \cdot, *, 0, 1 \rangle$, defining $\vee, \wedge, \mathbf{0}$ and $\mathbf{1}$ as in the previous section, $\langle A; \vee, \wedge, ', \mathbf{0}, \mathbf{1} \rangle$ is also a p -algebra.

It is clear that x' is not constant, so recalling that

$$x \wedge x' = \mathbf{0}$$

a quick check at the following table

x'	$x + x'$	$x \cdot x'$
x	x	x
x^*	$x + x^*$	$\mathbf{0}$
x^{**}	x^{**}	x
$x + x^*$	$x + x^*$	x
$x^* + x^{**}$	$x^* + x^{**}$	x

shows the following:

Theorem. *The only interpretation of $\mathbf{B}_\omega$ into itself is the trivial interpretation.*

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