WEIGHTED DECAY ESTIMATE FOR THE WAVE EQUATION

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ABSTRACT. The work is devoted to the proof of a new L^∞ - L^2 weighted estimate for the solution to the nonhomogeneous wave equation in $(3+1)$ -dimensional space-time. The weighted Sobolev spaces are associated with the generators of the Poincaré group. The estimate obtained is applied to prove the global existence of a solution to a nonlinear system of wave and Klein-Gordon equations with small initial data.

1. INTRODUCTION

Essential results concerning the global existence of solution to the $(3+1)$ -dimensional nonlinear wave equation have been obtained recently by Klainerman [9, 10], Hörmander [6, 7] (see also [1–4]). The use of Sobolev weighted spaces associated with the generators

$$(1.1) \quad \begin{aligned} \partial_t, \partial_j &= \partial/\partial x_j, & j &= 1, 2, 3, \\ \Omega_{jk} &= x_j \partial_k - x_k \partial_j, & 1 \leq j < k \leq 3, \\ \Omega_{0j} &= t \partial_j + x_j \partial_t, & j &= 1, 2, 3, \end{aligned}$$

of the Poincaré group in $\mathbf{R}^4$ plays the key role in the approach developed by Klainerman and Hörmander. Denote the above generators by $\Gamma_1, \dots, \Gamma_{10}$.

One of the sharpest estimates of the L^∞ -norm of the solution to the nonhomogeneous wave equation is obtained by Hörmander in [7]. This estimate improves some previous estimates due to Klainerman [9, 10]. More precisely, if $u(t, x)$ is a solution to the Cauchy problem

$$(1.2) \quad \square u = f, \quad \square = \partial_t^2 - \sum_{j=1}^3 \partial_j^2, \quad u(0, x) = \partial_t u(0, x) = 0$$

and $f(t, x) \in C^\infty(\mathbf{R}_+; \mathcal{S}(\mathbf{R}^3))$, then the estimate derived in the Appendix of

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[7] implies

$$(1.3) \quad (1+t+|x|)|u(t, x)| \leq C \sum_{|\beta|+j \leq 2} \int_0^t \int_{\mathbf{R}^3} |\Gamma^\beta L_0^j f(s, y)| \frac{dy ds}{(1+s+|y|)}.$$

Here $L_0 = t\partial_t + \sum_{j=1}^3 x_j \partial_j$ is the radial vector field.

On the other hand, some important coupled systems of nonlinear wave and Klein-Gordon equations have been investigated in [1–4]; these show that the use of the radial vector field L_0 is not convenient for the proof of a global existence theorem. To overcome this difficulty, in [2, 3] another L^∞ - L^2 weighted estimate is proposed, when the inclusion

$$(1.4) \quad \text{supp } f(s, y) \subseteq \{|y| \leq s + R\}$$

is fulfilled for some constant $R > 0$. The estimate has the form

$$(1.5) \quad (1+t)|u(t, x)| \leq C \sum_{r=0}^{\infty} \sum_{|\beta| \leq 3} \sup_{s \in I_r \cap [0, t]} (1+s)^{3/2} \|\Gamma^\beta f(s)\|.$$

Here I_r is the interval $[2^{r-1}, 2^{r+1}]$ for $r \geq 1$, $I_0 = [0, 2]$ and $\|g(s)\|$ denotes the norm

$$\|g(s)\| = \left(\int_{\mathbf{R}^3} |g(s, y)|^2 dy \right)^{1/2}.$$

The main goal of this work is to obtain an improved estimate of type (1.5) for the solution $u(t, x)$ to the Cauchy problem (1.2), provided assumption (1.4) is not fulfilled.

Such an estimate will be crucial for the proof of the global existence of solution to some important nonlinear systems of wave and Klein-Gordon equations, when the initial data are not compactly supported (see the approach developed in [1–4]).

2. MAIN RESULT

Given any function $f(s, y) \in C^\infty(\mathbf{R}_+, \mathcal{S}(\mathbf{R}^3))$ consider the weighted L^2 -norm

$$(2.1) \quad \|f(s)\|_k = \left(\int_{\mathbf{R}^3} (1+s+|y|)^{2k} |f(s, y)|^2 dy \right)^{1/2};$$

and, analogously, for $f(y) \in \mathcal{S}(\mathbf{R}^3)$, denote

$$\|f\|_k = \left(\int_{\mathbf{R}^3} (1+|y|)^{2k} |f(y)|^2 dy \right)^{1/2}.$$

If $k = 0$, then we omit the index k and write simply $\|f(s)\|$ and $\|f\|$.

Theorem 1. *Let $f(s, y) \in C^\infty(\mathbf{R}_+; \mathcal{S}(\mathbf{R}^3))$ and let $u(t, x)$ be a solution to the Cauchy problem (1.2). Then we have*

$$(1+t+|x|)|u(t, x)| \leq C \sum_{r=0}^{\infty} \sum_{|\beta| \leq 3} \sup_{s \in I_r \cap [0, t]} \|\Gamma^\beta f(s)\|_{3/2}.$$

Taking advantage of the fact that the generators Γ_j commute with the operator $\square$, we obtain the following.

Corollary 2. *Under the assumptions of Theorem 1 we have*

$$(1+t+|x|)|\Gamma^\alpha u(t, x)| \leq C \sum_{r=0}^{\infty} \sum_{|\beta| \leq 3+|\alpha|} \sup_{s \in I_r \cap [0, t]} \|\Gamma^\beta f(s)\|_{3/2}.$$

Combining the above estimates with the estimates due to W. von Wahl [11], one can treat also the case when the initial data of the solution to (1.2) are nonzero. Namely, assuming $u(0, x) = \varepsilon f(x)$ and $\partial_t u(0, x) = \varepsilon g(x)$, $f, g \in \mathcal{S}(\mathbf{R}^3)$, for any positive real number $\delta > 0$ we have

$$(2.2) \quad (1+t+|x|)|\Gamma^\alpha u(t, x)| \leq C\varepsilon + C \sum_{|\beta| \leq 3+|\alpha|} \sup_{s \in [0, t]} \|\Gamma^\beta f(s)\|_{3/2+\delta}.$$

3. PRELIMINARIES

The main goal of this section is to obtain some estimates of the Radon transform

$$(3.1) \quad \mathcal{R}(f)(\rho, \omega) = \int_{x\omega=\rho} f(x) dS_x, \quad \rho \in \mathbf{R}, \omega \in \mathbf{S}^2$$

of a function $f(x) \in \mathcal{S}(\mathbf{R}^3)$.

Lemma 3.1. *Suppose $f(x) \in C_0^\infty(\mathbf{R}^3)$ and $\text{supp } f \subseteq \{|x| \leq R\}$. Then we have the estimate*

$$\sup_{\rho \in \mathbf{R}, \omega \in \mathbf{S}^2} |\mathcal{R}(f)(\rho, \omega)| \leq C(1+R)^{1/2} \sum_{|\beta| \leq 2} \|\Omega^\beta f\|,$$

where $\Omega^\beta = \Omega_1^{\beta_1} \Omega_2^{\beta_2} \Omega_3^{\beta_3}$ and $\Omega_1, \Omega_2, \Omega_3$ are the generators $\Omega_{jk} = x_j \partial_k - x_k \partial_j$, $1 \leq j < k \leq 3$, of the group of rotations on $\mathbf{R}^3$.

Proof. The assumption $\text{supp } f \subseteq \{|x| \leq R\}$ implies that

$$\text{supp}_\rho \mathcal{R}(f)(\rho, \omega) \subseteq \{|\rho| \leq R\}$$

and we get the inequality

$$|\mathcal{R}(f)(\rho, \omega)| \leq C \int_{-R}^R |\partial_\rho \mathcal{R}(f)(\rho, \omega)| d\rho.$$

Applying the Cauchy-Schwartz inequality we get

$$|\mathcal{R}(f)(\rho, \omega)| \leq C(1+R)^{1/2} \left(\int_{-R}^R |\partial_\rho \mathcal{R}(f)(\rho, \omega)|^2 d\rho \right)^{1/2}.$$

The Sobolev inequality on $\mathbf{S}^2$ gives

$$|g(\omega)|^2 \leq C \sum_{|\beta| \leq 2} \int_{\mathbf{S}^2} |\Omega^\beta g(\theta)|^2 d\theta.$$

Combining the above estimates with the Plancherel identity (see [5])

$$(3.2) \quad \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} |\partial_{\rho} \mathcal{R}(f)(\rho, \omega)|^2 d\omega d\rho = c \|f\|^2,$$

we get the needed estimate.

This completes the proof. $\square$

Corollary 3.2. Suppose $f(x) \in C_0^{\infty}(\mathbf{R}^3)$ and

$$\text{supp } f(x) \subseteq \{c_1 R \leq |y| \leq c_2 R\}$$

for some positive constants c_1, c_2 and $R \geq 1$. Then we have

$$|\mathcal{R}(f)(\rho, \omega)| \leq C \sum_{|\beta| \leq 2} \|\Omega^{\beta} f\|_{1/2}.$$

The next step in this section is the analysis of the case when the function $f(x)$ is not compactly supported.

Lemma 3.3. Let $f(x) \in \mathcal{S}(\mathbf{R}^3)$. Given any $R > 0$, we have the estimate

$$|\mathcal{R}(f)(\rho, \omega)| \leq C(1+R)^{-1/2} \sum_{|\beta| \leq 2} \|\Omega^{\beta} f\|_1 + C(1+R)^{1/2} \sum_{|\beta| \leq 2} \|\Omega^{\beta} f\|.$$

Proof. First, consider the case $|\rho| \geq 1+R$. Then the relation

$$\mathcal{R}(f)(\rho, \omega) = \rho^{-1} \int_0^{\rho} \partial_{\tau}(\tau \mathcal{R}(f)(\tau, \omega)) d\tau$$

and the Cauchy-Schwartz inequality yield

$$(3.3) \quad |\mathcal{R}(f)(\rho, \omega)| \leq \rho^{-1/2} \left(\int_0^{\rho} |\partial_{\tau}(\tau \mathcal{R}(f)(\tau, \omega))|^2 d\tau \right)^{1/2}.$$

Applying the Sobolev inequality on $\mathbb{S}^2$, we find

$$|\mathcal{R}(f)(\rho, \omega)| \leq C(1+R)^{-1/2} \sum_{|\beta| \leq 2} \left(\int_{\mathbb{S}^2} \int_0^{\rho} |\partial_{\tau}(\tau \mathcal{R}(\Omega^{\beta} f)(\tau, \theta))|^2 d\tau d\theta \right)^{1/2}.$$

Since

$$\tau \mathcal{R}(g)(\tau, \theta) = \tau \int_{y \cdot \theta = \tau} g(y) dS_y = \sum_{j=1}^3 \theta_j \mathcal{R}(y_j g)(\tau, \theta),$$

we conclude that

$$|\tau \partial_{\tau}(\mathcal{R}(g)(\tau, \theta))| \leq C \sum_{j=1}^3 |\partial_{\tau} \mathcal{R}(y_j g)(\tau, \theta)|,$$

and via the Plancherel identity (3.2) we get

$$(3.4) \quad |\mathcal{R}(f)(\rho, \omega)| \leq C(1+R)^{-1/2} \sum_{|\beta| \leq 2} \|\Omega^{\beta} f\|_1.$$

The identity (3.1) shows that $\mathcal{R}(f)(\rho, \omega)$ is an odd function. Therefore, to finish the proof it is sufficient to consider only the case $0 \leq \rho \leq R+1$. If this is the case, then we apply the estimate

$$|\mathcal{R}(f)(\rho, \omega) - \mathcal{R}(f)(1+R, \omega)| \leq \int_0^{1+R} |\partial_\tau \mathcal{R}(f)(\tau, \omega)| d\tau.$$

Applying the arguments given in the proof of Lemma 3.1, we obtain

$$\int_0^R |\partial_\tau \mathcal{R}(f)(\tau, \omega)| d\tau \leq C(1+R)^{1/2} \sum_{|\beta| \leq 2} \|\Omega^\beta f\|.$$

This estimate and (3.4) complete the proof. $\square$

4. PROOF OF THEOREM 1

We lose no generality assuming that f is a real-valued function.

First, consider the case $|x| \geq t/2$. Then we can use the following representation of the solution to the wave equation obtained in [2] (see Lemma 3.1 in [2]),

$$(4.1) \quad \begin{aligned} |x|u(t, x) = & -2\pi \sum_{\sigma=\pm 1} \sigma \operatorname{Re} \int_0^\infty \hat{f}(\rho, \sigma\rho\omega, t) e^{i\rho(t+\sigma|x|)} d\rho \\ & + \sum_{1 \leq r < k \leq 3} \operatorname{Re} \int_{\mathbf{R}^3} \frac{c_{kr}(x, \xi)}{|\omega \times \xi|} e^{i(t|\xi|+x \cdot \xi)} \hat{f}_{kr}(|\xi|, \xi, t) \frac{d\xi}{|\xi|}, \end{aligned}$$

where $\omega = x/|x|$, $\hat{f}_{kr}(\tau, \xi, t) = \xi_k \partial_{\xi_r} \hat{f} - \xi_r \partial_{\xi_k} \hat{f}$, $\hat{f}(\tau, \xi, t)$ is the Fourier transform of the function $f(s, y)\chi(0 \leq s \leq t)$, i.e.

$$(4.2) \quad \hat{f}(\tau, \xi, t) = \int_0^t \int_{\mathbf{R}^3} f(s, y) e^{-i(\tau s + \xi \cdot y)} dy ds.$$

Moreover, $c_{kr}(x, \xi)$ are bounded functions, homogeneous of degree 0 with respect to x, ξ and $a \times b$ denotes the vector product in $\mathbf{R}^3$.

If $|x| \leq t/2$, then we use a similar representation derived in [2] (see Lemma 3.3 in [2])

$$(4.3) \quad \begin{aligned} u(t, x) = & \operatorname{Re} \sum_{|\beta| \leq 1} \int_0^\infty \int_{S^2} (t+x \cdot \omega)^{-1} c_\beta(\omega) (\Gamma^\beta f)^\wedge(\rho, \rho\omega, t) e^{i\rho(t+x \cdot \omega)} d\omega d\rho \\ & - \operatorname{Re} \sum_{j=1}^3 \int_0^\infty \int_{S^2} (t+x \cdot \omega)^{-1} \omega_j (\tilde{f}_j(t, \rho\omega) - e^{i\rho t} \tilde{f}_j(0, \rho\omega)) e^{i\rho x \cdot \omega} d\omega d\rho. \end{aligned}$$

Here $(G)^\wedge(\tau, \xi, t)$ denotes the Fourier transform of the function $G(t, y)$ determined by (4.2), $\tilde{f}_j$ is the partial Fourier transform of the function $y_j f(t, y)$ with respect to the spatial variables, i.e.

$$\tilde{f}_j(t, \xi) = \int_{\mathbf{R}^3} y_j f(t, y) e^{-iy \cdot \xi} dy.$$

In the sequel, we shall use also the following simple relation between the Fourier transform of a real-valued function and its Radon transform (see [5], [2])

$$(4.4) \quad 2 \operatorname{Re} \int_0^\infty \tilde{F}(t, \rho\omega) e^{i\rho\tau} d\rho = \mathcal{R}(F)(\tau, \omega, t),$$

where

$$\mathcal{R}(F)(\tau, \omega, t) = \int_{y, \omega=\tau} F(t, y) dS_y$$

is the Radon transform with respect to the spatial variables. The above relation yields

$$(4.5) \quad 2 \operatorname{Re} \int_0^\infty \hat{F}(\rho, \rho\omega, t) e^{i\rho\tau} d\rho = \int_0^t \mathcal{R}(F)(\tau - s, \omega, s) ds$$

for any real-valued function $F(s, y) \in C^\infty(\mathbf{R}_+; \mathcal{S}(\mathbf{R}^3))$.

In the case $|x| \geq t/2$ we use the identities (4.1), (4.4) and derive the estimate

$$(1 + t + |x|)|u(t, x)| \leq C \sup_{(\tau, \omega) \in \mathbf{R} \times \mathbf{S}^2} \sum_{|\beta| \leq 1} \int_0^t |\mathcal{R}(\Gamma^\beta f)(\tau - s, \omega, s)| ds.$$

The application of Lemma 3.3 with $R = s$ gives

$$\begin{aligned} (1 + t + |x|)|u(t, x)| &\leq C \int_0^t \sum_{|\beta| \leq 3} \|\Gamma^\beta f(s)\|_{1/2} ds \\ &\quad + C \int_0^t (1 + s)^{-1/2} \sum_{|\beta| \leq 3} \|\Gamma^\beta f(s)\|_1 ds. \end{aligned}$$

Taking advantage of the simple estimate

$$\int_0^t \phi(s) ds \leq C \sum_{r=0}^\infty \sup_{s \in I_r \cap [0, t]} (1 + s)|\phi(s)|,$$

we obtain the desired estimate.

Finally, let us turn to the case $|x| \leq t/2$. Now we use the relations (4.3), (4.4), (4.5), together with Lemma 3.3, and derive

$$\begin{aligned} (1 + t)|u(t, x)| &\leq C \int_0^t (1 + s)^{-1/2} \sum_{|\beta| \leq 3} \|\Gamma^\beta f(s)\|_1 ds \\ &\quad + C \sum_{j=1}^3 \sup_{(\tau, \omega) \in \mathbf{R} \times \mathbf{S}^2} |\mathcal{R}(f_j)(\tau - t, \omega, t)|. \end{aligned}$$

The finite speed of propagation of the wave guarantees that without loss of generality we can assume that

$$\operatorname{supp}_y f(s, y) \subseteq \{|y| \leq 2t\}$$

for $0 \leq s \leq t$. Now the application of Lemma 3.1 yields

$$|\mathcal{R}(f_j)(\tau - t, \omega, t)| \leq C(1 + t)^{3/2} \sum_{|\beta| \leq 2} \|\Gamma^\beta f(t)\|,$$

and we obtain the desired inequality.

This completes the proof.

5. APPLICATION TO A SYSTEM OF WAVE AND KLEIN-GORDON EQUATIONS

Consider the Cauchy problem

$$(5.1) \quad \begin{aligned} \square u_1 &= F_1(u, \nabla u), & (\square + 1)u_2 &= F_2(u, \nabla u), & \nabla &= (\partial_t, \nabla_x), \\ u(0, x) &= \varepsilon f(x), & \partial_t u(0, x) &= \varepsilon g(x), & x &\in \mathbf{R}^3, & \varepsilon > 0, \end{aligned}$$

where $u = (u_1, u_2)$, and analogously for f and g . A similar system was considered in [2, 3], f and g assumed compactly supported. In contrast to this, here we assume $f, g \in \mathcal{S}(\mathbf{R}^3; \mathbf{R}^2)$. The nonlinearities $F_j(u, w)$ are supposed to be smooth in u, w ; and they have the form $F_j(u, \nabla u) = Q_j(\nabla u) + C_j(u, \nabla u)$, $j = 1, 2$, where the derivatives of $C_j(u, w)$ up to order 2 are zero for $(u, w) = (0, 0)$ while

$$Q_j(\nabla u) = \sum_{r,s=0}^3 \sum_{m,k=1}^2 q_j^{mkr s} \partial_r u_m \partial_s u_k$$

satisfy the strong null condition used in [2], that is

$$\sum_{r,s=0}^3 q_j^{mkr s} \xi_r \xi_s = 0 \quad \text{for any } \xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbf{R}^4, \quad m, k, j = 1, 2.$$

The above condition simply means that

$$(5.2) \quad q_j^{mkr s} = q_j^{mksr}.$$

Examples of quadratic nonlinearities satisfying the strong null condition are discussed in [2]. In order to prove the global existence of a solution of (5.1) for ε small enough, we introduce the following norms

$$\begin{aligned} E_m(u(t)) &= \sum_{|\alpha| \leq m} \sum_{|\beta| \leq 1} \sup_{0 \leq \tau \leq t} \|\partial^\beta \Gamma^\alpha u(\tau)\|, \\ H_n(u(t)) &= \sum_{|\alpha| \leq n} \sup_{0 \leq \tau \leq t} \|(1 + \tau + |y|)\Gamma^\alpha u(\tau, y)\|_{L^\infty(\mathbf{R}_y^3)}. \end{aligned}$$

Denote for the sake of brevity $F_j(t, x) \equiv F_j(u(t, x), \nabla u(t, x))$ and similarly for Q_j and C_j . To estimate the L^2 norm of u_1 we apply the conformal estimate for the linear wave equation due to S. Klainerman [11] (see also Chapter VII in [8]),

$$\|\Gamma^\alpha u_1(t)\| \leq C\varepsilon + C \int_0^t \sum_{|\beta| \leq m-1} \|\Gamma^\beta F_1(\tau)\|_1 d\tau,$$

where $|\alpha| \leq m$. Further, the energy estimate for the wave equation gives

$$\|\nabla \Gamma^\alpha u_1(t)\| \leq C\varepsilon + C \int_0^t \sum_{|\beta| \leq m} \|\Gamma^\beta F_1(\tau)\| d\tau.$$

Uniting the above two estimates we get

$$E_m(u_1(t)) \leq C\varepsilon + C \sum_{|\beta| \leq m-1} \sum_{|\gamma| \leq 1} \int_0^t \|\partial^\gamma \Gamma^\beta F_1(\tau)\|_1 d\tau,$$

since

$$\Gamma_j = \sum_{r,s=0}^3 c_j^{rs} x_r \partial_s, \quad c_j^{rs} \text{-constants}, \quad x_0 = t.$$

Similarly, from the energy estimate for the Klein-Gordon equation it follows that

$$E_m(u_2(t)) \leq C\varepsilon + C \sum_{|\beta| \leq m-1} \sum_{|\gamma| \leq 1} \int_0^t \|\partial^\gamma \Gamma^\beta F_2(\tau)\|_1 d\tau.$$

Since $F_j = Q_j + C_j$, we need estimates of

$$\|\partial^\gamma \Gamma^\beta Q_j(\tau)\|_1 \quad \text{and} \quad \|\partial^\gamma \Gamma^\beta C_j(\tau)\|_1$$

for $|\beta| \leq m-1$, $|\gamma| \leq 1$. From the definition of the norms E_m and H_m it follows that

$$(5.3) \quad \|\partial^\gamma \Gamma^\beta C_j(\tau)\|_1 \leq C(1+\tau)^{-1} E_m(u(\tau)) H_n(u(\tau))$$

provided $n = [m/2]$ (the greatest integer in $m/2$) and $H_n(u(\tau))$ is bounded. To estimate the quadratic nonlinearity, we need the following.

Lemma 5.1. *If the bilinear form*

$$Q_j(\nabla u, \nabla w) = \sum_{r,s=0}^3 \sum_{m,k=1}^2 q_j^{mkrs} \partial_r u_m \partial_s w_k$$

satisfies the strong null condition, then we have

$$|Q_j(\nabla u, \nabla w)(t, x)| \leq C(1+t+|x|)^{-1} |\Gamma u(t, x)| |\Gamma w(t, x)|,$$

where $|\Gamma u(t, x)| = \sum_{|\alpha| \leq 1} |\Gamma^\alpha u(t, x)|$.

Proof. If $|x| \leq t+1$, then the desired inequality is obtained in Lemma 2.1 in [2]. Suppose $|x| \geq t+1$. Without loss of generality we can assume $x = (x_1, 0, 0)$. From

$$\Omega_{1k} = x_1 \partial_k - x_k \partial_1, \quad k = 1, 2, 3,$$

we express ∂_k

$$\partial_k = (x_k/x_1) \partial_1 + (1/x_1) \Omega_{1k}.$$

Analogously, $\Omega_{01} = t \partial_1 + x_1 \partial_t$ implies $\partial_t = (-t/x_1) \partial_1 + (1/x_1) \Omega_{01}$. Then setting $\xi_0 = -t/x_1$, $\xi_k = x_k/x_1$, $k = 1, 2, 3$, we have the inequality

$$\left| Q_j(\nabla u, \nabla w)(t, x) - \sum_{r,s=0}^3 \sum_{m,k=1}^2 q_j^{mkrs} \xi_r \xi_s \partial_1 u_m \partial_1 w_k \right| \leq C |\Gamma u(t, x)| |\Gamma w(t, x)| / |x_1|.$$

The strong null condition guarantees that the left-hand side of this inequality coincides with $|Q_j(\nabla u, \nabla w)(t, x)|$. Keeping in mind that $|x_1| = |x| \geq t + 1$, we complete the proof. $\square$

By virtue of this lemma we obtain

$$(5.4) \quad \|\partial^\gamma \Gamma^\beta Q_j(\tau)\|_1 \leq C(1 + \tau)^{-1} E_m(u(\tau)) H_n(u(\tau)).$$

From (5.3) and (5.4) we arrive at

$$(5.5) \quad E_m(u(t)) \leq C\varepsilon + C \int_0^t E_m(u(\tau)) H_n(u(\tau)) (1 + \tau)^{-1} d\tau.$$

In order to evaluate $H_n(u_1(t))$ we use the estimate (2.2) for the linear wave equation. Thus, for any $\delta > 0$ we have

$$H_n(u_1(t)) \leq C\varepsilon + C \sum_{|\beta| \leq n+3} \sup_{0 \leq \tau \leq t} \|\Gamma^\beta F_2(\tau)\|_{3/2+\delta}.$$

In the same way, from the estimate for the linear Klein-Gordon equation due to T. Sideris [12] we conclude that

$$H_n(u_2(t)) \leq C\varepsilon + C \sum_{|\beta| \leq n+2} \sup_{0 \leq \tau \leq t} \|\Gamma^\beta F_2(\tau)\|_{3/2+\delta}.$$

The quantity $\|\Gamma^\beta F(\tau)\|_{3/2+\delta}$ can be estimated by means of Lemma 5.1; this gives the inequality

$$\|\Gamma^\beta F(\tau)\|_{3/2+\delta} \leq C(1 + \tau + |x|)^{-1/2+\delta} E_{n+3}(u(\tau)) H_n(u(\tau)),$$

provided $E_{n+3}(u(t))$ is bounded. If $n + 3 = [m/2] + 3 \leq m$, i.e. $m \geq 5$, then we derive

$$\|\Gamma^\beta F(\tau)\|_{3/2+\delta} \leq C(1 + \tau + |x|)^{-1/2+\delta} E_m(u(\tau)) H_n(u(\tau));$$

so

$$H_n(u(t)) \leq C\varepsilon + C \sup_{0 \leq \tau \leq t} (1 + \tau + |x|)^{-1/2+\delta} E_m(u(\tau)) H_n(u(\tau)).$$

From this estimate and (5.5) following the argument given by S. Klainerman [11] (see also §5 in [2]), one obtains a global bound for $H_n(u(t)) + (1 + t)^{-\delta} E_m(u(t))$. This fact guarantees the existence of a global solution of the system (5.1) assuming ε small enough.

Another application of our main estimate obtained in Theorem 1 is connected with the Maxwell-Dirac system. This gives us an idea of how to extend the approach of [4] for the case when all the initial data of this nonlinear system are not compactly supported.

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