GOOD APPROXIMATIONS AND CONTINUED FRACTIONS

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(Communicated by William Adams)

ABSTRACT. Let $(q_n)_n$ be the sequence of best approximation denominators of an irrational number α . The set of real numbers x for which $q_n x \rightarrow 0 \pmod{1}$ is studied. It is shown that a number x belongs to $\alpha\mathbb{Z} \pmod{1}$ if and only if a simple condition on the speed of the convergence related to an arithmetic property of α is satisfied. This set is uncountable whenever α has unbounded partial quotients.

1. INTRODUCTION

Let α be a real irrational number with regular continued fraction expansion $[a_0; a_1, \dots, a_n, \dots]$. Let $(q_n(\alpha))_{n \geq -1}$ be the sequence of denominators of the regular convergents $p_n(\alpha)/q_n(\alpha)$ of α , given by

$$q_{-1}(\alpha) := 0, \quad q_0(\alpha) := 1, \quad q_n(\alpha) := a_n q_{n-1}(\alpha) + q_{n-2}(\alpha), \quad n \geq 1.$$

In the sequel we will write, for short, q_n and p_n/q_n instead of $q_n(\alpha)$ and $p_n(\alpha)/q_n(\alpha)$, respectively.

We are interested in the set $E(\alpha)$ of real numbers β for which the sequence $(q_n \beta)_{n \geq 1}$ converges, modulo 1, to 0. Clearly $E(\alpha)$ is an additive subgroup of $\mathbb{R}$ which contains $\alpha\mathbb{Z} + \mathbb{Z}$. From a result of Eggleston [E], if α has bounded partial quotients, then $E(\alpha)$ is at most enumerable. In this particular case a result of Larcher [L] states that $E(\alpha) = \alpha\mathbb{Z} + \mathbb{Z}$. We will generalize this result. Let $\|\xi\|$ be the distance of ξ from the nearest integer and put $\theta_n = \theta_n(\alpha) := q_n |q_n \alpha - p_n|$, $n \geq 0$. Let β be a real number. We will show then, among other things, that for any irrational number α we have $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$ if and only if $\|q_n \beta\| \in o(\theta_n)$, $(n \rightarrow \infty)$.

2. SOME NOTATIONS

Put

$$A_0 := \begin{bmatrix} 1 & a_0 \\ 0 & 1 \end{bmatrix}, \quad A_n := \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}, \quad n \geq 1,$$

Received by the editors October 1, 1989 and, in revised form, June 29, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 11J70, 11A55.

Key words and phrases. Continued fractions, diophantine approximations.

and

$$M_n := A_0 \cdots A_n, \quad n \geq 0.$$

A short calculation then yields

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}, \quad n \geq 0.$$

We consider M_n as the linear transformation $x \rightarrow (p_{n-1}x + p_n)/(q_{n-1}x + q_n)$, so that

$$\frac{p_n}{q_n} = M_n(0), \quad n \geq 0.$$

We furthermore define

$$T_n = [0; a_{n+1}, a_{n+2}, \dots], \quad n \geq 0,$$

and

$$V_n := \frac{q_{n-1}}{q_n}, \quad n \geq 0.$$

We have

$$(1) \quad V_n = [0; a_n, \dots, a_1], \quad n \geq 1, \quad V_0 = 0,$$

$$(2) \quad \alpha = \frac{p_n + T_n p_{n-1}}{q_n + T_n q_{n-1}}, \quad n \geq 0,$$

and

$$(3) \quad \theta_n = \frac{T_n}{1 + T_n V_n}, \quad n \geq 0.$$

Notice that $\theta_n = q_n \|q_n \alpha\|$. Other useful properties are

$$(4) \quad |q_n \alpha - p_n| < \frac{1}{q_{n+1}} \quad \text{and} \quad q_{n+1} \|q_n \alpha\| + q_n \|q_{n+1} \alpha\| = 1, \quad n \geq 0;$$

see [P, K].

Lemma. Let $A, B \in \mathbb{Z}$ and define the real number β by

$$\beta := \frac{AT_n + B}{q_{n-1}T_n + q_n},$$

where $n \geq 0$. Then $\beta = (\delta_n \Delta_n) \alpha - \delta_n (Ap_n - Bp_{n-1})$, where

$$\delta_n := \det M_n = (-1)^n, \quad n \geq 0, \quad \Delta_n := \det \begin{bmatrix} A & B \\ q_{n-1} & q_n \end{bmatrix}.$$

Proof. From (2) and the definition of β we have $\beta = M_n^* M_n^{-1}(\alpha)$, where

$$M_n^* := \begin{bmatrix} A & B \\ q_{n-1} & q_n \end{bmatrix}.$$

But now the result follows at once, since

$$\begin{aligned} M_n^* M_n^{-1} &= \delta_n \begin{bmatrix} A & B \\ q_{n-1} & q_n \end{bmatrix} \begin{bmatrix} q_n & -p_n \\ -q_{n-1} & p_{n-1} \end{bmatrix} \\ &= \delta_n \begin{bmatrix} \Delta_n & -Ap_n + Bp_{n-1} \\ 0 & \delta_n \end{bmatrix}. \quad \square \end{aligned}$$

Theorem 1. Let α be an irrational number. For each real number β we have

$$\beta \in \alpha\mathbb{Z} + \mathbb{Z} \Leftrightarrow \exists n_0 = n_0(\beta), \quad \forall n \geq n_0: \|q_n \beta\| \leq \frac{1}{4} \theta_n.$$

Proof. Let $\beta = a\alpha + b$, for some $a, b \in \mathbb{Z}$. Then

$$(5) \quad \|q_n \beta\| \leq |a| \|q_n \alpha\| = \frac{|a|}{q_n} q_n \|q_n \alpha\| = \frac{|a|}{q_n} \theta_n$$

and the implication follows directly from the fact that the sequence of denominators $(q_n)_{n \geq 0}$ is a monotonically increasing sequence of positive integers.

Now let $n \in \mathbb{N}$. For $k = 0, 1, \dots, q_n - 1$ we define the closed intervals $I_n^k = I_n^k(\alpha)$ by

$$I_n^k := \left[\frac{k}{q_n} - \frac{\theta_n}{4q_n}, \frac{k}{q_n} + \frac{\theta_n}{4q_n} \right],$$

with $\theta_n = \theta_n(\alpha)$, $q_n = q_n(\alpha)$. We call n the rank of I_n^k . Since $0 < \theta_n < 1$ for all n , it is obvious that two different intervals of the same rank are disjoint. Now suppose that an interval of rank n intersects with two intervals of rank $n+1$. This would give

$$\frac{2\theta_{n+1}}{4q_{n+1}} + \frac{2\theta_n}{4q_n} \geq \frac{1}{q_{n+1}}$$

i.e.,

$$\theta_{n+1} + \frac{q_{n+1}}{q_n} \theta_n \geq 2.$$

A direct consequence of (4) is that $(\theta_n q_{n+1})/q_n < 1$ for all n , hence we are led to a contradiction. The conclusion is that an interval of rank n intersects with at most one interval of rank $n+1$. Now let β be a real number with the property that there exists a positive integer $n_0 = n_0(\beta)$ such that

$$\|q_n \beta\| \leq \frac{1}{4} \theta_n, \quad \text{for all } n \geq n_0.$$

Without any loss of generality we may assume that $0 \leq \beta < 1$. Then for every $n \geq n_0$ there exists a unique $k(n) = k(n, \beta)$, $k(n) \in \{0, \dots, q_n - 1\}$, such that

$$(6) \quad \beta \in I_n^{k(n)}.$$

Next we introduce the number β^* , defined by

$$\beta^* := \frac{k(n_0)T_{n_0+1} + k(n_0 + 1)}{q_{n_0}T_{n_0+1} + q_{n_0+1}}.$$

From the lemma

$$\beta^* = \delta_{n_0+1} \Delta_{n_0+1} \alpha - \delta_{n_0+1} (k(n_0)p_{n_0+1} - k(n_0 + 1)p_{n_0});$$

in particular we have $\beta^* \in \alpha\mathbb{Z} + \mathbb{Z}$. For abbreviation we will write Δ instead of Δ_{n_0+1} . Notice that (5) yields

$$(7) \quad \|q_n \beta^*\| \leq \frac{|\Delta|}{q_n} \theta_n.$$

Next observe that

$$\begin{aligned} \left| \frac{k(n_0)}{q_{n_0}} - \frac{k(n_0+1)}{q_{n_0+1}} \right| &\leq \left| \beta - \frac{k(n_0)}{q_{n_0}} \right| + \left| \beta - \frac{k(n_0+1)}{q_{n_0+1}} \right| \\ &\leq \frac{1}{4} \left(\frac{\theta_{n_0}}{q_{n_0}} + \frac{\theta_{n_0+1}}{q_{n_0+1}} \right). \end{aligned}$$

From this it follows that

$$|\Delta| \leq \frac{1}{4} q_{n_0} q_{n_0+1} \left(\frac{\theta_{n_0}}{q_{n_0}} + \frac{\theta_{n_0+1}}{q_{n_0+1}} \right).$$

Hence

$$\frac{|\Delta|}{q_{n_0+1}} \leq \frac{1}{4} \left(\theta_{n_0} + \frac{q_{n_0}}{q_{n_0+1}} \theta_{n_0+1} \right) = \frac{1}{4} (q_{n_0} \|q_{n_0} \alpha\| + q_{n_0} \|q_{n_0+1} \alpha\|),$$

therefore

$$(8) \quad |\Delta| \leq \frac{1}{4} q_{n_0+1}.$$

This combined with (7) shows that $\|q_n \beta^*\| \leq \frac{1}{4} \theta_n$, for all $n > n_0$. In the same way as we showed (6) we find that there exists a sequence of intervals $I_n^{k^*(n)}$, such that $\beta^* \in I_n^{k^*(n)}$ for all $n > n_0$.

The linear transformation

$$f: x \rightarrow \frac{k(n_0)x + k(n_0+1)}{q_{n_0}x + q_{n_0+1}}$$

is strictly monotonic on $[0, 1]$, hence

$$|\beta - f(0)| \leq |f(1) - f(0)| = \frac{|\Delta|}{q_{n_0+1}(q_{n_0} + q_{n_0+1})} < \frac{1}{4} \frac{1}{q_{n_0+1}},$$

the last inequality in view of (8). This means that $I_n^{k(n)} = I_n^{k^*(n)}$ for all $n > n_0$, i.e. $\beta, \beta^* \in I_n^{k(n)}$, $n \geq n_0$, so that $\beta^* = \beta$. Hence $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$. $\square$

Remark 1. It follows from (5) that $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$ implies $\|q_n \beta\| \in o(\theta_n)$. Reciprocally, if $\|q_n \beta\| \in o(\theta_n)$, then there exists a positive integer n_0 such that $\|q_n \beta\| \leq \frac{1}{4} \theta_n$, for all $n \geq n_0$. From Theorem 1 we get $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$. An equivalent formulation for Theorem 1 is therefore

Theorem 1'. Let α be an irrational number. For each real number β we have

$$\beta \in \alpha\mathbb{Z} + \mathbb{Z} \Leftrightarrow \|q_n \beta\| \in o(\theta_n), \quad (n \rightarrow \infty).$$

Remark 2. Assume now that α is an irrational number with bounded partial quotients, i.e., there exists a positive integer K such that $a_n \leq K$, for $n \geq 1$. From formula (3) we derive $\theta_n \geq 1/(K+2)$. This implies the following result:

Corollary [L]. Let α be an irrational number with bounded partial quotients. Then

$$\beta \in \alpha\mathbb{Z} + \mathbb{Z} \Leftrightarrow \lim_{n \rightarrow \infty} \|q_n \beta\| = 0.$$

In fact, we can replace the limit condition by the following weaker form:

$$\limsup_{n \rightarrow \infty} \|q_n \beta\| < \frac{1}{4} \frac{1}{K+2}.$$

Remark 3. A theorem of Stewart [S, Theorem 3] yields that, in case α has bounded partial quotients, the relation $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$ is equivalent to

$$(*) \quad \limsup_{n \rightarrow \infty} (2\|q_n \beta\| - \theta_n) \leq 0.$$

This last form is stronger than the previous one, however the proof given by Stewart is technically quite involved. Moreover, from the construction given in Theorem 3 we will see that in case α has not bounded partial quotients, the inequality $(*)$ is not equivalent to $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$. Now we improve Larcher's result.

Theorem 2. Let α have bounded partial quotients. Then a real number β belongs to $\alpha\mathbb{Q} + \mathbb{Q}$ if and only if the sequence $(\|q_n \beta\|)_n$ has only finitely many limit points.

Proof. Clearly if $\beta = (p\alpha + r)/q$ ($p, q, r \in \mathbb{Z}$, $q \neq 0$) then the sequence $(\|q_n \beta\|)_n$ has limit points in $\{0, 1/q, \dots, (q-1)/q\}$. Conversely, let S be the set of limit points of the above sequence. By the Kronecker approximation theorem there exists a positive integer m such that

$$\forall s \in S : \|m \cdot s\| < \frac{1}{8(K+2)}.$$

By assumption, for any $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ satisfying

$$(\forall n \in \mathbb{N})(n \geq N(\varepsilon) \Rightarrow \exists s_n \in S : \|q_n \cdot \beta - s_n\| \leq \varepsilon).$$

On the other hand, we have

$$\begin{aligned} \|mn \cdot \beta\| &\leq \min_{r, t \in \mathbb{N}} (|m(q_n \cdot \beta - r - s_n)| + |m \cdot s_n - t|) \\ &\leq m\|q_n \cdot \beta - s_n\| + \|m \cdot s_n\|. \end{aligned}$$

Choose $\varepsilon = 1/16m(K+2)$. Then, for sufficiently large n ,

$$\|mq_n \cdot \beta\| < \frac{1}{4} \frac{1}{K+2}$$

and Remark 2 finishes the proof. $\square$

In [L] it is shown that there exist real numbers α and β , $\beta \notin \alpha\mathbb{Z} + \mathbb{Z}$, such that $\lim_{n \rightarrow \infty} \|q_n \beta\| = 0$. Notice that, in view of the above corollary, the number α has unbounded partial quotients. Here we will show that for each irrational number α with unbounded partial quotients there exist uncountably many real numbers β such that $\lim_{n \rightarrow \infty} \|q_n \beta\| = 0$. In fact if $\lim_{n \rightarrow \infty} q_{n+1}/q_n = +\infty$ then the Hausdorff dimension of $E(\alpha)$ is equal to 1 (see [E]). But for almost all α one has $\liminf_{n \rightarrow \infty} q_{n+1}/q_n = 1$ and $\limsup_{n \rightarrow \infty} q_{n+1}/q_n = +\infty$.

Theorem 3. Let α be an irrational number with unbounded partial quotients. Then the additive subgroup $E(\alpha)$ of $\mathbb{R}$ defined by

$$E(\alpha) := \left\{ \beta \in \mathbb{R}; \lim_{n \rightarrow \infty} \|q_n \beta\| = 0 \right\},$$

is uncountable.

Proof. Let $[a_0; a_1, \dots, a_n, \dots]$ be the regular continued fraction expansion of α . Then there exists an infinite subsequence $(n_l)_{l \geq 1}$ of positive integers such that for all $l \geq 1$, $a_{n_l+1} > 2^l$. Due to the well-known fact that

$$T_n \in (1/(a_{n+1} + 1), 1/a_{n+1}) \quad \text{for all } n \geq 0,$$

we have, for all $l \geq 1$, $T_{n_l} < 1/a_{n_l+1} < 2^{-l}$. From (1), respectively (3), we moreover have, for all $l \geq 1$, $V_{n_l+1} < 1/a_{n_l+1} < 2^{-l}$ and $\theta_{n_l} < T_{n_l} < 2^{-l}$. Define the real number β by

$$\beta := \sum_{l=1}^{\infty} \varepsilon_l (q_{n_l} \alpha - p_{n_l}), \quad \text{where } \varepsilon_l \in \{0, 1\}, \quad l \geq 1.$$

Notice that β is well defined and the series absolutely converges. Moreover, different sequences $(\varepsilon_l)_l$ lead to different β . In fact, from usual recurrent relations satisfying by sequences $(q_n)_n$ and $(p_n)_n$, we get $\sum_{k=1}^{\infty} a_{m+2k+1} \|q_{m+2k} \alpha\| = \|q_{m+1} \alpha\|$. Hence for $\eta_l \in \{0, 1, -1\}$, $l \geq k$ and $\eta_k = 1$, one has

$$\sum_{l=k}^{\infty} \eta_l (q_{n_l} \alpha - p_{n_l}) \geq \|q_{n_k} \alpha\| - \sum_{l=k+1}^{\infty} \|q_{n_l} \alpha\| > 0.$$

We claim that $\beta \in E(\alpha)$, from which the theorem follows.

Let m and n be two positive integers. Put $\Delta(m, n) := (p_m q_n - p_n q_m)$. Then

$$q_n (q_m \alpha - p_m) + \Delta(n, m) = q_m (q_n \alpha - p_n).$$

Now let n be a positive integer, $n > n_3$. Then there exists a unique positive integer $k = k(n) \geq 3$ such that $n_k < n \leq n_{k+1}$. In order to show that $\beta \in E(\alpha)$ it is now sufficient to show that we have $\|q_n \beta\| < 3/2^k$, since k and n go to infinity simultaneously.

Suppose that $l \leq k$. Then $n_l < n$. Put $C_l := \Delta(n, n_l)$. Applying the above with $m = n_l$ and putting

$$\beta(k) := \sum_{l \leq k} \varepsilon_l (q_{n_l} \alpha - p_{n_l}), \quad S_k := \sum_{l \leq k} \varepsilon_l C_l$$

we get

$$|q_n \beta(k) - S_k| \leq \sum_{l \leq k} \varepsilon_l q_{n_l} \|q_n \alpha\| \leq \sum_{l \leq k} \varepsilon_l \frac{q_{n_l}}{q_n}.$$

Since $n_l \leq n_k < n$ and

$$\begin{aligned} \frac{q_{n_l}}{q_n} &= \frac{q_{n_l}}{q_{n_l+1}} \frac{q_{n_l+1}}{q_{n_l+2}} \dots \frac{q_{n-1}}{q_n} \\ &\leq V_{n_l+1} V_{n_l+1+1} \dots V_{n_k+1} < 2^{-l} 2^{-l-1} \dots 2^{-k} \end{aligned}$$

we find that

$$0 \leq \sum_{l \leq k} \varepsilon_l \frac{q_{n_l}}{q_n} \leq \sum_{l \leq k} 2^{-l} 2^{-l-1} \dots 2^{-k} < \frac{1}{2^{k-1}}.$$

Hence

$$(9) \quad |q_n \beta(k) - S_k| < 2^{1-k}.$$

Now we estimate the sum $q_n \sum_{l \geq k+1} \varepsilon_l \|q_{n_l} \alpha\|$. For $l \geq k+1$ one has $q_n \leq q_{n_l}$, and therefore

$$(10) \quad 0 \leq q_n \sum_{l \geq k+1} \varepsilon_l \|q_{n_l} \alpha\| \leq \sum_{l \geq k+1} q_{n_l} \|q_{n_l} \alpha\| = \sum_{l \geq k+1} \theta_{n_l} < \sum_{l \geq k+1} \frac{1}{2^l} = \frac{1}{2^k}.$$

From (10) we now have

$$(11) \quad \left| q_n \sum_{l \geq k+1} \varepsilon_l (q_{n_l} \alpha - p_{n_l}) \right| < 2^{-k}.$$

Using (9) and (11) we obtain $|q_n \beta - S_k| < 3/2^k$. Since $S_k \in \mathbb{Z}$ we have $\|q_n \beta\| < 3/2^k$ for all $n > n_3$. $\square$

Remark 4. Let α be the real number given by $[0; 2, 4, \dots, 2^n, \dots]$. Put $n_l := l$, for $l \geq 0$, then $a_{n_{l+1}} = a_{l+1} = 2^{l+1}$. Let $(\varepsilon_l)_{l \geq 1}$ be an arbitrary sequence of 0's and 1's, and let β be defined as in the above proof. Then we have $\beta \in E(\alpha)$ but β is not an element of $\alpha\mathbb{Z} + \mathbb{Z}$ if $\varepsilon_l = 1$ infinitely often. Moreover, by the choice of α , we have $\limsup_{n \rightarrow \infty} (\|q_n \beta\| - \frac{1}{2} \theta_n) = 0$ (see Remark 3).

Now as a direct consequence of Theorem 3 we have the following new characterization:

Proposition. Let α be an irrational number and let the group $E(\alpha)$ be defined as in Theorem 3. Then α has bounded partial quotients if and only if $E(\alpha) = \alpha\mathbb{Z} + \mathbb{Z}$.

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